Conway Games, coalgebraically*

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Abstract. Using *coalgebraic methods*, we extend *Conway's original theory of games* to include *infinite games* (*hypergames*). We take the view that a play which goes on forever is a *draw*, and hence rather than focussing on winning strategies, we focus on *non-losing strategies*. Infinite games are a fruitful metaphor for non-terminating processes, *Conway's sum* of games being similar to *shuffling*. Hypergames have a rather interesting theory, already in the case of generalized *Nim*. The theory of hypergames generalizes Conway's theory rather smoothly, but significantly. We indicate a number of intriguing directions for future work. We briefly compare infinite games with other notions of games used in computer science.
Keywords: Conway games, coalgebraic games, non-losing strategies.

1 Introduction

We focus on *combinatorial games*, that is no chance 2-player games, the two players being conventionally called *Left* (L) and *Right* (R). Such games have *positions*, and in any position there are rules which restrict L to move to any of certain positions, called the *Left positions*, while R may similarly move only to certain positions, called the *Right positions*. L and R move in turn, and the game is of *perfect knowledge*, *i.e.* all positions are public to both players. The game ends when one of the players has no move, the other player being the winner. Many games played on boards are combinatorial games, *e.g. Nim*, *Domineering*, *Go*, *Chess*. Games, like Nim, where for every position both players have the same set of moves, are called *impartial*. More general games, like Domineering, Go, Chess, where L and R may have different sets of moves are called *partizan*.

Combinatorial Game Theory started at the beginning of 1900 with the study of the famous impartial game Nim. In the 1930s, Sprague and Grundy generalized the results on Nim to all impartial finite (*i.e.* terminating) games, [Gru39,Spra35]. In the 1960s, Berlekamp, Conway, Guy introduced the theory of partizan games, which first appeared in the book "On Numbers and Games" [Con01]. In [Con01], the theory of games is connected to the theory of *surreal numbers*.

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However, in [Con01], the author focusses only on *finite, i.e. terminating games*. Infinite games are neglected as ill-formed or trivial games, not interesting for "busy men", and their discussion is essentially confined to a single chapter. Infinity (or *loopy*) games have been later considered in [BCG82], Chapters 11-12. However, in Chapter 12 the authors focus on well-behaved classes of impartial games, which can be dealt with a generalization of the Grundy-Sprague theory, due to Smith [Smi66]. In Chapter 11, a theory for the special class of partizan *fixed loopy games* is developed; a game is *fixed* if infinite plays are winning either for L or R player. On the contrary, in the present paper we develop a general *coalgebraic account* of infinite games, taking the different (but sometimes more natural) view that an infinite play is a *draw*. We call such games *hypergames*.

Infinite games are extremely useful in various fields, such as Mathematical Logic and Computer Science. The importance of games for Computer Science comes from the fact that they capture in a natural way the notion of *interaction*. Infinite games model in a faithful way *reactive processes* (operating systems, controllers, communication protocols, etc.), that are characterised by their *non-terminating* behaviour and perpetual *interaction* with their environment.

The coalgebraic account of games developed in this paper is very natural and it paves the way to a smooth and insightful treatment of infinite games. It allows us to consider games up-to bisimilarity, and to generalize operations and relations on them as congruences up-to bisimilarities. Moreover, the coalgebraic setting makes explicit the common nature between processes and games. For hypergames the notion of *winning strategy* has to be replaced by that of *non-losing strategy*, since we take non terminating plays to be *draws*. Hypergames can be naturally defined as a *final coalgebra* of *non-wellfounded sets* (*hypersets*), which are the sets of a universe of Zermelo-Fraenkel satisfying an Antifoundation Axiom, see [FH83,Acz88]. Our theory of hypergames generalizes the original theory on finite games of [Con01] rather smoothly, but significantly. Our main results amount to a Determinacy and a Characterization Theorem of non-losing strategies on hypergames. The latter requires (a non-trivial) generalization of Conway's partial order relation on games to hypergames. Once hypergames are defined as a final coalgebra, operations on games, such as *disjunctive sum*, can be naturally extended to hypergames, by defining them as *final morphisms* into the coalgebra of hypergames. We will also discuss the class of impartial hypergames. In particular, we will extend the theory of Grundy-Sprague and Smith, based on the canonical *Nim games*, by introducing suitable *canonical* ∞*-hypergames*.

Finally, we will briefly compare our hypergames with other games arising in Combinatorial Game Theory and in Computer Science.

Summary. In Section 2, we recall Conway's theory of finite games and winning strategies. In Secton 3, we introduce hypergames as a final coalgebra, and we develop the theory of hypergames and non-losing strategies, which extends Conway's theory. In Section 4, we study in particular the theory of impartial hypergames. Comparison with related games and directions for future work appear in Section 5.

2 The Theory of Conway Games

We recall that Conway games are 2-player games, the two players are called *Left* (L) and *Right* (R). Such games have *positions*, and in any position p there are rules which restrict Left to move to any of certain positions, called the *Left positions* of p, while Right may similarly move only to certain positions, called the *Right positions* of p. Since we are interested only in the abstract structure of games, we can regard any position p as being completely determined by its Left and Right options, and we shall use the notation $p = (P^L, P^R)$, where P^L, P^R denote sets of positions. Games are identified with their initial positions. Left and Right move in turn, and the game ends when one of the two players does not have any option. Conway considers only terminating (inductively defined) games. These can be viewed as an *initial algebra* of a suitable functor, which we define *e.g.* on the category Class* of classes of (possibly non-wellfounded) sets and functional classes.

Definition 1 (Conway Games). *The set of* Conway Games $\mathcal{G}$ *is inductively defined by*

- *the empty game* $(\{\}, \{\}) \in \mathcal{G}$;
- *if* $P, P' \subseteq \mathcal{G}$, *then* $(P, P') \in \mathcal{G}$.

Equivalently, $\mathcal{G}$ *is the carrier of the* initial algebra $(\mathcal{G}, id)$ *of the functor* $F : \mathit{Class}^* \to \mathit{Class}^*$, *defined by* $F(X) = \mathcal{P}(X) \times \mathcal{P}(X)$ *(with usual definition on morphisms).*
Games will be denoted by small letters, e.g. p, *with* $p = (P^L, P^R)$ *and* p^L, p^R *generic elements of* P^L, P^R. *We denote by* Pos_p *the set of* positions *hereditarily reachable from* p.

Some simple games. The simplest game is the empty one, *i.e.* $(\{\}, \{\})$, which will be denoted by 0. Then we define the games $1 = (\{0\}, \{\})$, $-1 = (\{\}, \{0\})$, $* = (\{0\}, \{0\})$.

Winning strategies. In the game 0, the player who starts will lose (independently whether he plays L or R), since there are no options. Thus the second player (II) has a winning strategy. In the game 1 there is a winning strategy for L, since, if L plays first, then L has a move to 0, and R has no further move; otherwise, if R plays first, then he loses, since he has no moves. Symmetrically, -1 has a winning strategy for R. Finally, the game $*$ has a winning strategy for the first player (I), since he has a move to 0, which is losing for the next player.

Formally, we first define *(finite) plays* over a game p as alternating sequences of moves on the game, starting from the initial position. One might think that the following definitions are a little involved, but this is necessarily so if we want to "dot all our i's and cross all our t's".

Definition 2 (Finite Plays). *Let* $p = (P^L, P^R)$ *be a game. The set of* finite plays *over* p, $FPlay_p$, *is defined by:*

$$\pi = p_1^{K_1} \dots p_n^{K_n} \in FPlay_p, \ \textit{for } n \geq 0, \quad \textit{iff}$$

– $K_1, \ldots, K_n \in \{L, R\}$;
– $(K_1 = L \ \wedge \ p_1^{K_1} \in P^L) \ \vee \ (K_1 = R \ \wedge \ p_1^{K_1} \in P^R)$
– $\forall i.\ 1 \leq i < n.\ (p_i^{K_i} = (P_i^L, P_i^R) \wedge p_{i+1}^{K_{i+1}} \in P_i^{\overline{K}_i})$, *where* $\overline{K} = \begin{cases} L & \textit{if } K = R \\ R & \textit{if } K = L. \end{cases}$

We denote by $FPlay_p^{LI}$ *the set of plays starting with a Left move and ending with a Right move,* i.e. $\{p_1^{K_1} \ldots p_n^{K_n} \in FPlay_p \mid K_1 = L \ \wedge \ K_n = R,\ n \geq 0\}$*, and by* $FPlay_p^{LII}$ *the set of plays starting with a Right move and ending with a Right move,* i.e. $\{p_1^{K_1} \ldots p_n^{K_n} \in FPlay_p \mid K_1 = R \ \wedge \ K_n = R,\ n \geq 1\}$*. Similarly, we define* $FPlay_p^{RI}$ *and* $FPlay_p^{RII}$*.*

Only *finite* plays can arise on a Conway game.

Winning strategies for a given player can be formalized as functions on plays ending with a move of the opponent player, telling which is the next move of the given player:

Definition 3 (Winning Strategies). *Let* $p = (P^L, P^R)$ *be a game.*
– *A* winning strategy on p for LI, i.e. *for Left as I player, is a partial function* $f : FPlay_p^{LI} \to Pos_p$*, such that:*

- $\forall \pi \in FPlay_p^{LI}.\ f(\pi) = p_{n+1}^L \implies \pi p_{n+1}^L \in FPlay_p$;
- f *is defined on the empty play* ϵ*, denoted by* $f(\epsilon)\downarrow$;
- $\forall \pi \in FPlay_p^{LI}.\ (f(\pi) = p_{n+1}^L \ \wedge \ p_{n+1}^L = (P_{n+1}^L, P_{n+1}^R) \implies \forall p_{n+2}^R \in P_{n+1}^R.\ f(\pi p_{n+1}^L p_{n+2}^R)\downarrow)$.

– *A* winning strategy on p for LII, i.e. *for Left as II player, is a partial function* $f : FPlay_p^{LII} \to Pos_p$*, such that:*

- $\forall \pi \in FPlay_p^{LII}.\ f(\pi) = p_{n+1}^L \implies \pi p_{n+1}^L \in FPlay_p$;
- *for all* $p^R \in P^R$, $f(p^R)\downarrow$;
- $\forall \pi \in FPlay_p^{LII}.\ (f(\pi) = p_{n+1}^L \ \wedge \ p_{n+1}^L = (P_{n+1}^L, P_{n+1}^R) \implies \forall p_{n+2}^R \in P_{n+1}^R.\ f(\pi p_{n+1}^L p_{n+2}^R)\downarrow)$.

– Winning strategies on p for RI and RII *are defined similarly, as partial functions* $f : FPlay_p^{RI} \to Pos_p$ *and* $f : FPlay_p^{RII} \to Pos_p$*, respectively.*
– *A* winning strategy on p for L *is a partial function* $f_L : FPlay_p^{LI} \cup FPlay_p^{LII} \to Pos_p$ *such that* $f_L = f_{LI} \cup f_{LII}$*, for* $f_{LI},\ f_{LII}$ *winning strategies for LI and LII.*
– *A* winning strategy on p for R *is a partial function* $f_R : FPlay_p^{RI} \cup FPlay_p^{RII} \to Pos_p$ *such that* $f_R = f_{RI} \cup f_{RII}$*, for* $f_{RI},\ f_{RII}$ *winning strategies for RI and RII.*
– *A* winning strategy on p for I *is a partial function* $f_I : FPlay_p^{LI} \cup FPlay_p^{RI} \to Pos_p$ *such that* $f_I = f_{LI} \cup f_{RI}$*, for* $f_{LI},\ f_{RI}$ *winning strategies for LI and RI.*
– *A* winning strategy on p for II *is a partial function* $f_{II} : FPlay_p^{LII} \cup FPlay_p^{RII} \to Pos_p$ *such that* $f_{II} = f_{LII} \cup f_{RII}$*, for* $f_{LII},\ f_{RII}$ *winning strategies for LII and RII.*

By induction on games, one can prove that any game has exactly one winner (see [Con01] for more details):

Theorem 1 (Determinacy, [Con01]). *Any game has a winning strategy either for L or for R or for I or for II.*

In [Con01], a relation $\gtrsim$ on games is introduced, inducing a partial order (which is a total order on the subclass of games corresponding to numbers). Such relation allows to characterize games with a winning strategy for L, R, I or II (see Theorem 2 below).

Definition 4. *Let $x = (X^L, X^R)$, $y = (Y^L, Y^R)$ be games. We define, by induction on games:*

$$x \gtrsim y \quad \textit{iff} \quad \forall x^R \in X^R.\ (y \not\gtrsim x^R) \ \wedge \ \forall y^L \in Y^L.\ (y^L \not\gtrsim x) \ .$$

Furthermore, we define:
– $x > y$ *iff* $x \gtrsim y \ \wedge \ y \not\gtrsim x$
– $x \sim y$ *iff* $x \gtrsim y \ \wedge \ y \gtrsim x$
– $x||y$ *(x fuzzy y) iff* $x \not\gtrsim y \ \wedge \ y \not\gtrsim x$

Notice that $\not\gtrsim$ does not coincide with $<$, *e.g.* $* = (\{0\}, \{0\})$ is such that $* \not\gtrsim 0$ holds, but $* \gtrsim 0$ does not hold.

As one may expect, $1 > 0 > -1$, while for the game $*$ (which is not a number), we have $*||0$.

The following important theorem gives the connection between games and numbers, and it allows to characterize games according to winning strategies:

Theorem 2 (Characterization, [Con01]). *Let x be a game. Then*

$x > 0$	*(x is positive)*	*iff*	*x has a winning strategy for L.*
$x < 0$	*(x is negative)*	*iff*	*x has a winning strategy for R.*
$x \sim 0$	*(x is zero)*	*iff*	*x has a winning strategy for II.*
$x\|\|0$	*(x is fuzzy)*	*iff*	*x has a winning strategy for I.*

Generalizations of Theorems 1 and 2 to infinite games will be discussed in Section 3.

3 The Theory of Hypergames

Here we extend the class of games originally considered by Conway, by introducing *hypergames*, where plays can be unlimited. For such games the notion of winning strategy has to be replaced by that of *non-losing strategy*, since we take non terminating plays to be draws. In this section, we develop the theory of hypergames, which generalizes the one for finite games. Special care requires the generalization of the Characterization Theorem 2.

Hypergames can be naturally defined as a *final coalgebra* on non-wellfounded sets:

Definition 5 (Hypergames). *The set of* Hypergames $\mathcal{H}$ *is the carrier of the* final coalgebra $(\mathcal{H}, id)$ *of the functor $F : Class^* \rightarrow Class^*$, defined by $F(X) = \mathcal{P}(X) \times \mathcal{P}(X)$ (with usual definition on morphisms).*

Defining hypergames as a final coalgebra, we immediately get a *Coinduction Principle* for reasoning on infinite games:

Lemma 1. *A* F-bisimulation on the coalgebra $(\mathcal{H}, id)$ *is a symmetric relation* $\mathcal{R}$ *on hypergames such that, for any* $x = (X^L, X^R),\ y = (Y^L, Y^R)$*,*

$$x\mathcal{R}y \implies (\forall x^L \in X^L.\exists y^L \in Y^L.x^L\mathcal{R}y^L) \ \wedge \ (\forall x^R \in X^R.\exists y^R \in Y^R.x^R\mathcal{R}y^R) \ .$$

Coinduction Principle. *Let us call a* F*-bisimulation on* $(\mathcal{H}, id)$ *a* hyperbisimulation*. The following principle holds:*

$$\frac{\mathcal{R} \text{ hyperbisimulation} \quad x\mathcal{R}y}{x = y}$$

All important notions and constructions on games turn out to be invariant w.r.t. hyperbisimilarity, in particular hyperbisimilar games will have the same outcome. Moreover, the coalgebraic representation of games naturally induces a minimal representative for each bisimilarity equivalence class.

Some simple hypergames. Let us consider the following pair of simple hypergames: $a = (\{b\}, \{\})$ and $b = (\{\}, \{a\})$. If L plays as II on a, then she immediately wins since R has no move. If L plays as I, then she moves to b, then R moves to a and so on, an infinite play is generated. This is a draw. Hence L has a non-losing strategy on a. Simmetrically, b has a non-losing strategy for R.

Now let us consider the hypergame $c = (\{c\}, \{c\})$. On this game, any player (L, R, I, II) has a non-losing strategy; namely there is only the non-terminating play consisting of infinite c's.

It is remarkable that the formal definition of non-losing strategy is precisely the same as that of winning strategy, *i.e.* a function on finite plays (see Definition 3). This shows that the definition of non-losing strategy is the natural generalization to hypergames of the notion of winning strategy.

The main difference in the theory of hypergames with respect to the theory of games is that on a hypergame we can have non-losing strategies for various players at the same time, as in the case of the game c above.

To prove Theorem 3 below, which is the counterpart of Theorem 1 of Section 2, we use the following lemma, that follows from the definition of non-losing strategy:

Lemma 2. *Let* p *be a hypergame.*

- *If L as I player does not have a non-losing strategy on* p*, then R as II player has a non-losing strategy on* p*.*
- *If L as II player does not have a non-losing strategy on* p*, then R as I player has a non-losing strategy on* p*.*
- *Symmetrically for R.*

Theorem 3 (Determinacy). *Any hypergame has a non-losing strategy at least for one of the players L, R, I, II.*

Proof. Assume by contradiction that p has no non-losing strategies for L, R, I, II. Then in particular p has no non-losing strategy for LI or for LII. Assume the first case holds (the latter can be dealt with similarly). Then, by Lemma 2, p has a non-losing strategy for RII. Hence, by hypothesis there is no non-losing strategy for RI. But then, by Lemma 2, there is a non-losing strategy for LII. Therefore, by definition, there is a non-losing strategy for II. Contradiction. □

Theorem 3 above can be sharpened, by considering when the non-losing strategy f is in particular a winning strategy, *i.e.* it only generates terminating plays. First, we state the following lemma:

Lemma 3. *Let p be a hypergame.*
- *If L as I player has a winning strategy on p, then R as II player does not have a non-losing strategy on p.*
- *If L as II player has a winning strategy on p, then R as I player does not have a non-losing strategy on p.*
- *Symmetrically for R.*
- *If L as I player has a non-losing strategy on p, but no winning strategies, then R as II player has a non-losing strategy on p.*
- *If L as II player has a non-losing strategy on p, but no winning strategies, then R as I player has a non-losing strategy on p.*
- *Symmetrically for R.*

Theorem 4. *Let p be a hypergame. Then either case 1 or case 2 arises:*
1. *There exists a winning strategy for exactly one of the players L, R, I, II, and there are no non-losing strategies for the other players.*
2. *There are no winning strategies, but there is a non-losing strategy for L or R or I or II. Furthermore:*
   (a) *if there is a non-losing strategy for L or R, then there is a non-losing strategy for at least one of the players I or II;*
   (b) *if there is a non-losing strategy for I or II, then there is a non-losing strategy for at least one of the players L or R;*
   (c) *if there are non-losing strategies for both L and R, then there are non-losing strategies also for both I and II;*
   (d) *if there are non-losing strategies for both I and II, then there are non-losing strategies also for both L and R.*

Proof. 1) If L has a winning strategy, then by Lemma 3 both RI and RII have no non-losing strategies. Hence neither R nor I nor II have non-losing strategies.
2a) Assume that there is a non-losing strategy for L, but no winning strategies. Then there is a non-losing strategy but no winning strategies for LI or for LII. Then, assume *w.l.o.g.* that there is a non-losing strategy but no winning strategies for LI, by Lemma 3 there is a non-losing strategy also for RII. Therefore, since there are non-losing strategies for LII and RII, then there is a non-losing strategy for II.
2c) If there are non-losing strategies both for L and R, then we there are non-losing strategies for LI, LII, RI, RII, thus there are non-losing strategies also for I and II.
The remaining items are proved similarly. □

According to Theorem 4 above, the space of hypergames can be decomposed as in Figure 1. For example, the game $c = (\{c\}, \{c\})$ belongs to the center of the space, while the games $a = (\{b\}, \{\})$ and $b = (\{\}, \{a\})$ belong to the sectors marked with L, II and R, II, respectively.

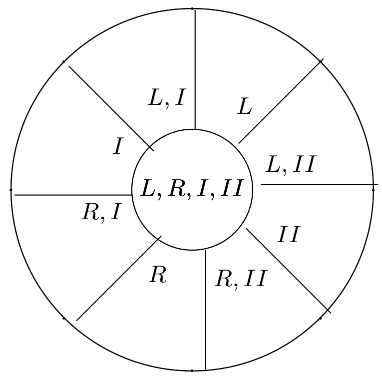

Fig. 1. The space of hypergames.

3.1 Characterization Theorem for non-losing Strategies

The generalization to hypergames of Theorem 2 of Section 2 is quite subtle, because it requires to extend the relation $\gtrsim$ to hypergames, and this needs particular care. We would like to define such relation *by coinduction*, as the greatest fixpoint of a monotone operator on relations, however the operator which is naturally induced by the definition of $\gtrsim$ on games (see Definition 4) is *not* monotone. This problem can be overcome as follows.

Observe that the relation $\gtrsim$ on games is defined in terms of the relation $\not\gtrsim$. Vice versa $\not\gtrsim$ is defined in terms of $\gtrsim$. Therefore, on hypergames the idea is to define both relations at the same time, as the greatest fixpoint of the following operator on pairs of relations:

Definition 6. *Let $\Phi : \mathcal{P}(\mathcal{H} \times \mathcal{H}) \times \mathcal{P}(\mathcal{H} \times \mathcal{H}) \longrightarrow \mathcal{P}(\mathcal{H} \times \mathcal{H}) \times \mathcal{P}(\mathcal{H} \times \mathcal{H})$ be the operator defined by:*

$$\Phi(\mathcal{R}_1, \mathcal{R}_2) = (\{(x, y) \mid \forall x^R . y \mathcal{R}_2 x^R \ \wedge \ \forall y^L . y^L \mathcal{R}_2 x\}, \\ \{(x, y) \mid \exists x^R . y \mathcal{R}_1 x^R \ \vee \ \exists y^L . y^L \mathcal{R}_1 x\})$$

The above operator turns out to be monotone componentwise. Thus we can define:

Definition 7. *Let the pair $(\gtrsim, \not\gtrsim)$ be the greatest fixpoint of Φ.*
Furthermore, we define:
- $x > y$ *iff* $x \gtrsim y \ \wedge \ y \not\gtrsim x$
- $x \sim y$ *iff* $x \gtrsim y \ \wedge \ y \gtrsim x$
- $x || y$ *iff* $x \not\gtrsim y \ \wedge \ y \not\gtrsim x$

As an immediate consequence of Tarski's Theorem, the above definition of the pair of relations ($\gtrsim$, $\not\gtrsim$) as the greatest fixpoint of Φ gives us *Coinduction Principles*, which will be extremely useful:

Coinduction Principles. *We call* Φ-bisimulation *a pair of relations* $(\mathcal{R}_1, \mathcal{R}_2)$ *such that* $(\mathcal{R}_1, \mathcal{R}_2) \subseteq \Phi(\mathcal{R}_1, \mathcal{R}_2)$*. The following principles hold:*

$$\frac{(\mathcal{R}_1, \mathcal{R}_2)\ \Phi\text{-}bisimulation \quad x\mathcal{R}_1 y}{x \gtrsim y} \qquad \frac{(\mathcal{R}_1, \mathcal{R}_2)\ \Phi\text{-}bisimulation \quad x\mathcal{R}_2 y}{x \not\gtrsim y}$$

Notice that the pair of relations ($\gtrsim$, $\not\gtrsim$) on hypergames extends the corresponding pair on games, the latter being the least fixpoint of Φ.

Moreover, somewhat surprisingly at a first sight, notice that the relations $\gtrsim$ and $\not\gtrsim$ are *not* disjoint. *E.g.* the game $c = (\{c\}, \{c\})$ is such that both $c \gtrsim 0$ and $c \not\gtrsim 0$ (and also $0 \gtrsim c$ and $0 \not\gtrsim c$) hold. However, this is perfectly consistent in the hypergame scenario, since it is in accordance with the fact that some hypergames have non-losing strategies for more than one player. Namely, we have:

Theorem 5 (Characterization). *Let x be a hypergame. Then*

$x > 0$	*(x is positive)*	*iff*	*x has a non-losing strategy for L.*
$x < 0$	*(x is negative)*	*iff*	*x has a non-losing strategy for R.*
$x \sim 0$	*(x is zero)*	*iff*	*x has a non-losing strategy for II.*
$x \vert\vert 0$	*(x is fuzzy)*	*iff*	*x has a non-losing strategy for I.*

Proof. ($\Rightarrow$) Assume $x > 0$, *i.e.* $x \gtrsim 0$ and $0 \not\gtrsim x$. We show how to build a non-losing strategy for L. We have to build non-losing strategies both for LI and LII. For LII: since $x \gtrsim 0$, then, by definition, $\forall x^R . 0 \not\gtrsim x^R$, *i.e.*, for any R move x^R, $0 \not\gtrsim x^R$. Let $x^R = (X^{RL}, X^{RR})$, then $\exists x^{RL} \in X^{RL} . x^{RL} \gtrsim 0$, that is there exists a L move x^{RL} such that $x^{RL} \gtrsim 0$. Thus we can apply again the two steps above, going on forever or stopping when R cannot move. For LI: since $0 \not\gtrsim x$, then $\exists x^L . x^L \gtrsim 0$. Thus by the previous case there is a non-losing strategy for LII on x^L.
The other cases are dealt with similarly.
($\Leftarrow$) We proceed by coinduction, by showing all the four cases at the same time. Let
$\mathcal{R}_1 = \{(x, 0) \mid x$ has a non-losing strategy for $LII\} \cup$
$\{(0, x) \mid x$ has a non-losing strategy for $RII\}$,
$\mathcal{R}_2 = \{(x, 0) \mid x$ has a non-losing strategy for $RI\} \cup$
$\{(0, x) \mid x$ has a non-losing strategy for $LI\}$.
We prove that $(\mathcal{R}_1, \mathcal{R}_2)$ is a Φ-bisimulation. There are various cases to discuss. We only show one case, the others being similar. We prove that, if $x\mathcal{R}_1 0$ and x has a non-losing strategy for LII, then $\forall x^R . 0\mathcal{R}_2 x^R$. If LII has a non-losing strategy on x, then, by definition, for all x^R there is a non-losing strategy for LI, hence $\forall x^R . 0\mathcal{R}_2 x^R$. □

The following table summarizes the Characterization Theorem:

Non-losing strategies	Relations w.r.t. 0	
L	$x > 0$	$x \gtrsim 0 \ \wedge \ 0 \not\gtrsim x$
R	$x < 0$	$x \not\gtrsim 0 \ \wedge \ 0 \gtrsim x$
II	$x \sim 0$	$x \gtrsim 0 \ \wedge \ 0 \gtrsim x$
I	$x \| 0$	$x \not\gtrsim 0 \ \wedge \ 0 \not\gtrsim x$

Properties of $\gtrsim$. The following proposition, which can be proved by coinduction, generalizes to hypergames the corresponding results of [Con01]:

Proposition 1. *For all hypergames x, we have*

$$x \not\gtrsim x^R \ \wedge \ x^L \not\gtrsim x \ \wedge \ x \gtrsim x \ \wedge \ x \sim x \ .$$

However, contrary to what happens on games, the relation $\gtrsim$ is *not* a partial order on hypergames (and $\sim$ is *not* an equivalence), since $\gtrsim$ fails to be transitive.

Counterexample. Let $b = (\{b\}, \{b\})$ and $a = (\{a\}, \{0\})$. Then $b \gtrsim 0$, since b has non-losing strategies for all the players. Moreover, one can show that $a \gtrsim b$, by coinduction, by considering the relations $\mathcal{R}_1 = \{(a, b)\} \cup \{(0, b)\}$ and $\mathcal{R}_2 = \{(b, a)\} \cup \{(b, 0)\}$. Thus we have $a \gtrsim b \ \wedge \ b \gtrsim 0$. However, one can easily check that $a \gtrsim 0$ does not hold.

The problem is that the "pivot" b in the above counterexample allows unlimited plays. Namely, if we restrict ourselves to "well-behaved" pivots, then we recover transitivity, *i.e.*:

Lemma 4. *Let x, y, z be hypergames such that y is "well-behaved",* i.e. *y has no unlimited plays. If $x \gtrsim y \ \wedge \ y \gtrsim z$, then $x \gtrsim z$.*

Proof. (Sketch) One can proceed by coinduction, by showing that the relations $\mathcal{R}_1 = \{(x, z) | \exists y$ well-behaved. $x \gtrsim y \wedge y \gtrsim z\}$ and $\mathcal{R}_2 = \{(z, x) | \exists y$ well-behaved. $x \gtrsim y \wedge z \not\gtrsim y\} \cup \{(z, x) | \exists y$ well-behaved. $y \not\gtrsim x \wedge y \gtrsim z\}$ form a Φ-bisimulation. The difficult part is to prove that $\mathcal{R}_2$ is included in the second component of $\Phi(\mathcal{R}_1, \mathcal{R}_2)$. Here is where we need the hypothesis that b is well-behaved. □

3.2 Sum and Negation of Hypergames

There are various ways in which we can play several different (hyper)games at once. One way consists, at each step, in allowing the next player to select any of the component games and making any legal move on that game, the other games remaining unchanged. The following player can either choose to move in the same component or in a different one. This kind of compound games can be formalized through the *(disjunctive) sum*, [Con01]. The following coinductive definition extends to hypergames the definition of Conway sum:

Definition 8 (Hypergame Sum). *The* sum on hypergames *is given by the the final morphism $+ : (\mathcal{H} \times \mathcal{H}, \alpha_+) \longrightarrow (\mathcal{H}, id)$, where the coalgebra morphism $\alpha_+ : \mathcal{H} \times \mathcal{H} \longrightarrow F(\mathcal{H} \times \mathcal{H})$ is defined by*

$$\alpha_+(x, y) = (\{(x^L, y) \mid x^L \in X^L\} \cup \{(x, y^L) \mid y^L \in Y^L\}, \\ \{(x^R, y) \mid x^R \in X^R\} \cup \{(x, y^R) \mid y^R \in Y^R\}) \ .$$

That is $+$ is such that:
$$x + y = (\{x^L + y \mid x^L \in X^L\} \cup \{x + y^L \mid y^L \in Y^L\},$$
$$\{x^R + y \mid x^R \in X^R\} \cup \{x + y^R \mid y^R \in Y^R\}) \ .$$

The definition of hypergame sum resembles that of *shuffling* on processes. In fact it coincides with interleaving, when impartial games are considered.

For concrete examples of sum games see Section 4, where generalized Nim and Traffic Jam games are discussed.

Another operation on games, which admits an immediate coinductive extension to hypergames, is *negation*, where the roles of L and R are exchanged:

Definition 9 (Hypergame Negation). *The* negation *of a hypergame is given by the final morphism* $- : (\mathcal{H}, \alpha_-) \longrightarrow (\mathcal{H}, id)$*, where the coalgebra morphism* $\alpha_- : \mathcal{H} \longrightarrow F(\mathcal{H})$ *is defined by*
$\alpha_-(x) = (\{-x^R \mid x^R \in X^R\},\ \{-x^L \mid x^L \in X^L\})\ .$

That is $-$ is such that: $-x = (\{-x^R \mid x^R \in X^R\},\ \{-x^L \mid x^L \in X^L\})\ .$

In particular, if x has a non-losing strategy for LI (LII), then $-x$ has a non-losing strategy for RI (RII), and symmetrically. Taking seriously L and R players and not fixing a priori L or R to play first, makes the definition of $-$ very natural.

In the following propositions, we summarize some interesting results on sum and negation, that can be extended to hypergames:

Proposition 2.
i) $x - x \sim 0$.
ii) $x \gtrsim 0 \ \wedge\ y \gtrsim 0 \implies x + y \gtrsim 0$.
iii) If $y \sim 0$*, then* $x + y$ *has the same outcome as* x*.*
iv) If $y - z \sim 0$*, then the games* $x + y$ *and* $x + z$ *have the same outcome.*

The proofs of the items in the above proposition are similar to the ones provided in [Con01], pag.76, based on the construction of winning/non-losing strategies.

Proposition 3.
i) $x > y$ *iff* $x - y$ *has a non-losing strategy for L.*
ii) $x < y$ *iff* $x - y$ *has a non-losing strategy for R.*
iii) $x \sim y$ *iff* $y - x$ *has a non-losing strategy for II.*
iv) $x || y$ *iff* $y - x$ *has a non-losing strategy for I.*

The implications ($\Rightarrow$) in the above proposition are proved by building non-losing strategies, using the definitions of $\gtrsim$ and $\lesssim$, while the converse implications are proved using the Φ-coinduction principle.

4 The Theory of Impartial Hypergames

In this section, we focus on *impartial hypergames*, where, at each position, L and R have the same moves. Such hypergames can be simply represented by

$x = X$, where X is the set of moves (for L or R). Coalgebraically, this amounts to say that impartial hypergames are a final coalgebra of the powerset functor. In this section, we first recall the Grundy-Sprague theory for dealing with finite impartial games, then we discuss the theory of impartial hypergames, using Smith generalization of Grundy-Sprague results. In particular, we show how to provide a more complete account of such a theory, by introducing a class of *canonical hypergames*, extending the *Nim numbers*. This can only be given in the hypergame setting. We illustrate our results on an example.

4.1 The Grundy-Sprague Theory

Central to the theory of Grundy-Sprague, [Gru39,Spra35], is *Nim*, a classical impartial game, which is played with a number of heaps of matchsticks. The legal move is to strictly decrease the number of matchsticks in any heap (and throw away the removed sticks). A player unable to move because no sticks remain is the loser.

The Nim game with one heap of size n can be represented as the Conway game $*n$, defined (inductively) by

$$*n = \{*0, *1, \dots, *(n-1)\} \ .$$

Namely, with a heap of size n, the options of the next player consist in moving to a heap of size $0, 1, \dots, n-1$. The number n is called the *Grundy number* of the game. Clearly, if $n = 0$, the II player wins, otherwise player I has a winning strategy, moving to $*0$.

Nim games $*n$ are called *nimbers*, to distinguish them from the games n representing numbers, which have a different definition, see [Con01] for more details. Nimbers correspond to von Neumann finite numerals in Set Theory.

Nim games are central in game theory, since there is a classical result (by Grundy and Sprague, independently, [Gru39,Spra35]) showing that any impartial game "behaves" as a Nim game, or, using Conway terminology, is $\sim$-equivalent to a single-heap Nim game (see [Con01], Chapter 11). The algorithm for discovering the Nim game (or the Grundy number) corresponding to a given impartial game x proceeds inductively as follows. Assume that the Grundy numbers of the options of x are $n_0, n_1, \dots$, then the Grundy number of x is the *minimal excludent* (*mex*) of $n_0, n_1, \dots$ The mex of a list of numbers $n_0, n_1, \dots$ is the least natural number which does not appear among $n_0, n_1, \dots$ Then, having the Grundy number of (the positions of) a game, we know the winning strategy for that game.

Sums of impartial games. Here we explain how, using the above theory and the sum on Nim numbers, one can easily deal with compound impartial hypergames.

An example of a compound impartial game is the Nim game with more heaps. Using sum, the Nim game with two heaps of sizes n_1, n_2 can be represented as the Conway game $*n_1 + *n_2$. By the general result by Grundy-Sprague on impartial games, such game is also equivalent to a Nim game with a single heap, and thus there is a Grundy number n such that $*n \sim *n_1 + *n_2$. The sum of Nim numbers is particularly easy to compute and, as we will see, it is useful for analyzing the sum of generic impartial games. Thus, it deserves a special definition; following [Con01], we define the *Nim sum* $+_2$ by: $n_1 +_2 n_2 = n$, where n is the Nim number corresponding to the sum game $*n_1 + *n_2$. The Nim sum is quite easy to calculate, since one can show that it amounts to binary sum without carries. *E.g.* $1 +_2 3 = 2$, since $01 \oplus 11 = 10$, where $\oplus$ is binary sum.

In general, in order to analyze the sum of impartial games, one can proceed as follows. Using the Grundy-Sprague algorithm, one can compute the Nim numbers corresponding to the compound games. Then, Nim-summing such numbers one gets the Nim number corresponding to the starting game. If the result is 0, there is a winning strategy for the II player, otherwise there is a winning strategy for the I player, who can move to a position of Nim sum 0.

4.2 The Smith Theory in the Hypergame Setting

In [Con01], Chapter 11, the author briefly analyzes infinite impartial games, even if they escape his inductive definition. These games are represented as finite or infinite, cyclic graphs, having a node for each position of the game, and a direct edge from p to q when it is legal to move from p to q. Thus they exactly correspond to non-wellfounded sets, or impartial hypergames, in our setting. Theorem 3 specializes to impartial hypergames as follows:

Theorem 6.
Any impartial hypergame has non-losing strategies either for I player or for II player or for both.

Smith [Smi66] extended the Grundy-Sprague theory on impartial games to cover infinite games (see [Con01], pag. 133-135). In particular, Smith provides an algorithm (which works for a large class of cyclic graphs) for marking the nodes of the game graph with naturals (ordinals if the graph is infinite) plus some infinity symbols. This generalizes the Grundy-Sprague inductive algorithm, based on the *mex*, for computing the Grundy number of an impartial game. From Smith's marking one can then immediately discover whether a given position is winning for I, for II or it is a draw.

Smith's Marking of the Game Graph, [Smi66]. A position p in the graph will be marked with the number n if the following conditions hold. Firstly, n must be the *mex* (minimal excludent) of all numbers that already appear as marks of any of the options of p. Secondly, each of the positions immediately following p which has not been marked with some number less than n must already have an option marked by n. We continue in this way until it is impossible to mark any further node with any ordinal number, and then attach the symbol ∞ to any remaining node (which we call *unmarked*). Finally, the label of a position marked as n is n, while the label of an unmarked position is the symbol ∞ followed by the labels of all marked options as subscripts, see for example the graph of Fig. 2.

Now, the following result holds:

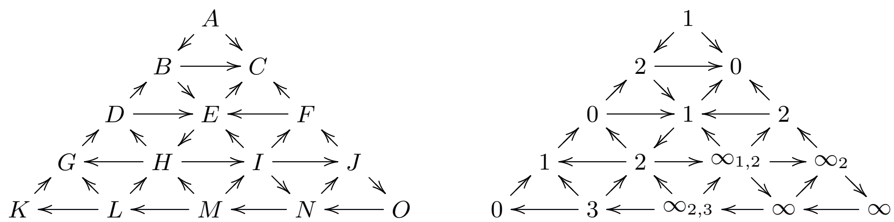

Fig. 2. The graph of an impartial hypergame, and Smith's marking.

Theorem 7 (see [Con01], pag. 134). *A position marked as n is a II player win if and only if n is 0, otherwise it is a I player win. A position marked by ∞_K, where K is a set of naturals, is a I player win if and only if $0 \in K$, otherwise it is a draw.*

The above theorem can be proved by induction on n. The idea underlying such technique is that a node marked by n behaves as the Nim game $*n$. This can be viewed as a "canonical game" $\sim$-corresponding to the given node. However, the theory, as it is presented in the literature, is not completely satisfactory for ∞ nodes, since ∞ symbols do not correspond to "canonical infinite games". In the sequel, we show how to do this in our setting of hypergames.

Let us consider a position p marked with ∞_K. We claim that such node behaves as the (canonical) hypergame

$$*\infty_K = \{*\infty\} \cup \{*k \mid k \in K\} ,$$

where $*\infty = \{*\infty\}$.

Namely, one can show that:

Theorem 8. *If x is the canonical hypergame associated to a position p in a graph, then*
$x||0$ *iff* *x has a non-losing strategy for I.*
$x \sim 0$ *iff* *x has a non-losing strategy for II.*

Proof. For positions marked by n, the thesis follows immediately from Theorem 7. Then let p be a position marked by ∞_K. Using Theorem 7, we only need to prove that:
(a) the hypergame $*\infty_K$ has subscript 0 iff $*\infty_K||0$ but not $*\infty_K \sim 0$;
(b) the hypergame $*\infty_K$ has no subscript 0 iff $*\infty_K||0$ and $*\infty_K \sim 0$.
($a \Rightarrow$) First of all, notice that $*\infty$ is such that $*\infty \gtrsim 0$ and $0 \gtrsim *\infty$. Assume $0 \in K$. Then $*\infty_K \not\gtrsim 0$, since $0 \gtrsim *\infty$. Similarly $0 \not\gtrsim *\infty_K$, since $*\infty \gtrsim 0$. Hence $*\infty_K||0$. Moreover, neither $*\infty_K \gtrsim 0$ nor $0 \gtrsim *\infty_K$ hold, since $0 \not\gtrsim 0$ does not hold.
($a \Leftarrow$) Assume $*\infty_K||0$, but not $*\infty_K \sim 0$. Assume by contradiction that 0 is not subscript of ∞_K. Then, since for all $k \in K$. $*k \not\gtrsim 0 \ \wedge \ 0 \not\gtrsim *k$, we have $*\infty_K \sim 0$. Contradiction.
($b \Rightarrow$) Assume $0 \notin K$. Then $*\infty_K||0$, since $*\infty \gtrsim 0$ and $0 \gtrsim *\infty$. Moreover

$*\infty_K \sim 0$, since, for all elements $x \in *\infty_K$, $x \not\gtrsim 0$ and $0 \not\gtrsim x$.
($b \Rightarrow$) Assume $*\infty_K || 0$ and $*\infty_K \sim 0$. If by contradiction $0 \in K$, then $*\infty_K \sim 0$ does not hold, since $0 \not\gtrsim 0$ does not hold. □

Thus, generalizing to impartial hypergames, Grundy-Sprague result on impartial games, we have:

Theorem 9. *Any impartial hypergame behaves either like a Nim game or like a hypergame of the shape $*\infty_K$.*

In the following, we show that our canonical hypergames are well-behaved also w.r.t. sum.

Traffic Jams and Generalized Sums. Following [Con01], we consider a concrete hypergame to illustrate how compound impartial hypergames are handled using Smith's generalized marking algorithm and the extension to ∞-nodes of the Nim sum. Let us consider the following concrete game, corresponding to the game graph in Fig. 2. We can think of the graph as the map of a fictitious country, where nodes correspond to towns, and edges represent motorways between them. The initial position of the game corresponds to the town where a vehicle is initially placed. Each player has to move such vehicle to a next town along the motorway. If this is not possible, then the player loses. Theorem 7 tells us which player has a non-losing strategy in any position. Now there is a natural generalization of the above traffic game, where more than one vehicle is considered. We assume that each town is big enough to accommodate all vehicles at once, if needed. At each step, the current player chooses a vehicle to move. Such game corresponds to the sum of the hypergames with single vehicles. In order to compute non-losing strategies for the sum game, one can use the *generalized Nim sum*, which amounts to the Nim sum extended to ∞-nodes as follows:

$$n +_2 \infty_K = \infty_K +_2 n = \infty_{\{k+_2 n \ \mid \ k \in K\}} \qquad \infty_K +_2 \infty_H = \infty \ .$$

Thus for example, if we have vehicles at positions H and I in Fig. 2, then the game is winning for I player, since $2 +_2 \infty_{1,2} = \infty_{2+_2 1, 2+_2 2} = \infty_{3,0}$. While a game with vehicles in I and J is a draw, since $\infty_{1,2} +_2 \infty_2 = \infty$.

On the other hand, having assigned canonical hypergames to the nodes of the graph, one could use hypergame sum (as defined in Definition 8) for summing them. Hence the question naturally arises whether canonical hypergames are well-behaved w.r.t. sum. The answer is positive, since one can prove that the hypergame sum behaves as the canonical hypergame corresponding to the result of the extended Nim sum, *i.e.*:

Proposition 4. *Let $*\infty_K, *\infty_H$ be hypergames. Then*
i) the hypergame $*\infty_K + *n$ behaves as the hypergame $*\infty_{\{k+_2 n \ \mid \ k \in K\}}$;
ii) the hypergame $*\infty_K + *\infty_H$ behaves as the hypergame $*\infty$.

Proof. (Sketch) Both items i) and ii) are proved using Theorem 5, by showing that the sum game has a non-losing strategy for L, R, I, II iff the corresponding game has one. □

5 Comparison with Related Work and Directions for Future Work

Loopy games. The theory of general loopy games, where infinite plays can be either winning for L,R or draws is very difficult. For instance, already for the case of fixed games (where no draws are admitted), determinacy fails if the Axiom of Choice is assumed. In [BCG82], Chapter 11, fixed loopy games are studied. A $\geq$ relation is introduced, which is proved to be transitive and it allows to approximate the behavior of a loopy game, possibly with finite games. But this technique works only if certain fixpoints exist. Such theory has been later further developed and revisited in other works, see *e.g.* [San02,San02a].

On the contrary, our theory allows to deal with the class of games where infinite plays are draws in a quite general and comprehensive way. We plan to investigate more general classes of (possibly mixed) loopy games, where infinite plays can be considered as winning or draws.

Games and automata. The notion of hypergame that we have investigated in this paper is related to the notion of infinite game considered in the automata theoretic approach, originating in work of Church, Büchi, McNaughton and Rabin (see *e.g.* [Tho02]). In this approach, games are defined by the graphs of positions. L and R have different options, in general, but L is always taken as first player. Only games with infinite plays are considered. These games are fixed, according to the above definition. Winning strategies are connected with automata, and also the problem of a (efficient) computation of such strategies is considered. However, recently, non-losing strategies have been considered also in this setting, *e.g.* in the context of model checking for the μ-calculus, see [GLLS07].

Games for semantics of logics and programming languages. Game Semantics was introduced in the early 90's in the construction of the first fully complete model of Classical Multiplicative Linear Logic [AJ94], and of the first syntax-independent fully abstract model of PCF, by Abramsky-Jagadeesan-Malacaria, Hyland-Ong, and Nickau, independently. Game Semantics has been used for modeling a variety of programming languages and logical systems, and more recently for applications in computer-assisted verification and program analysis, [AGMO03]. In Game Semantics, 2-players games are considered, which are in some way related to Conway games, despite the rather different presentation. For more details see [AJ94]. The main difference between the Game Semantics approach and our approach lies in the fact that, in Game Semantics, infinite plays are always considered as winning for one of the two players, as in the case of Conway's fixed games.

Traced categories of games. In [Joy77], Joyal showed how Conway (finite) games and winning strategies can be endowed with a structure of a traced category. This work admits an extension to loopy games, when these are fixed. However, when draws are considered, Joyal's categorical construction apparently does not work, since we lose closure under composition (this is related to the fact that our relation $\gtrsim$ is not transitive). As future work, we plan to study traced categories for general infinite games, and to investigate the trace operation in a coalgebraic setting.

Games and coalgebras. In [BM96], a simple coalgebraic notion of game is introduced and utilized. It is folklore that bisimilarity can be defined as a 2-player game, where one player tries to prove bisimilarity, while the other tries to disprove it. This game turns out to be a fixed game in the sense of [BCG82], where infinite plays are winning for the player who tries to prove bisimilarity.

Conumbers. Conway's numbers [Con01] amount to Conway's games x such that no member of X^L is $\gtrsim$ any member of X^R, and all positions of a number are numbers. Thus, once we have defined hypergames and the relations $\gtrsim$, $\not\gtrsim$, we can define the subclass of *conumbers*, together with suitable operations extending those on numbers. It would be interesting to investigate the properties of such a class of hypergames. An intriguing point is whether it is possible to define a partial order, since, as seen in this paper, the relation $\gtrsim$ is not transitive on hypergames.

Compound games. In this paper, we have considered the (disjunctive) sum for building compound games. However, there are several different ways of combining games, which are analyzed in [Con01], Chapter 14, for the case of finite games. It would be interesting to extend to hypergames such theory on compound games.

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