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Symbolic Model Checking and Bounded Model Checking

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SYMBOLIC MODEL CHECKING

Tackling the state-space explosion problem

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- The exploration of such a a huge state space may be prohibitive even for an algorithm running in linear time in the size of the model.
- the size of system that could be verified by explicit model checkers was restricted to $\approx 10^6$ states. Solution: Symbolic Model Checking

State-space explosion problem

Citation for the 2007 Turing Award

Although the 1981 paper demonstrated that the model checking was possible in principle, its application to practical systems was severely limited. The most pressing limitation was the number of states to search. Early model checkers required explicitly computing every possible configuration of values the program might assume. For example, if a program counts the millimeters of rain at a weather station each day of the week, it will need 7 storage locations. Each location will have to be big enough to hold the largest rain level expected in a single day. If the highest rain level in a day is 1 meter, this simple program will have 10^{21} possible states, slightly less than the number of stars in the observable universe. Early model checkers would have to verify that the required property was true for every one of those states.

https://amturing.acm.org/award_winners/clarke_1167964.cfm

Symbolic model checking has been proposed by McMillan (1993).

Reference:

Kenneth L McMillan (1993). "Symbolic model checking". In: *Symbolic Model Checking*. Springer, pp. 25–60

It replaces fixed-point computations over individual states by manipulations of definitions of state sets. It allows an exhaustive implicit enumeration of a huge number of states

Main definition: *symbolic* transition system.

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The corresponding symbolic Kripke structure is the tuple $(\overline{x}, f_I, f_T, \{f_{p_1}, \ldots, f_{p_k}\})$.

- we will write simply $\mathcal{M} = (S, I, T, L)$, meaning a symbolic transition system
- a path (or trace) $\pi = m_0, m_1, \ldots$ is an infinite sequence of assignment to the state variables such that:
  - $m_0 \models I(\overline{x})$;
  - $m_i, m'_{i+1} \models T(\overline{x}, \overline{x}')$ holds, for all $i \geq 0$.

  where $\overline{x}' := \{x'_0, \ldots, x'_{n-1}\}$.

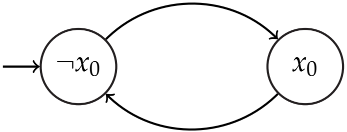
Symbolic Transition Systems

Three main techniques have been proposed:

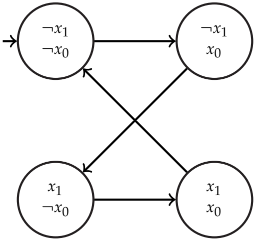
- partial order reduction
- BDD-based symbolic model checking
  - kind of *compressed truth tables*
- SAT-based symbolic model checking, aka *Bounded Model Checking*.

They allowed for the verification of systems with $> 10^{120}$ states.

- substantially larger than the number of atoms in the observable universe (around 10^{80})




```
MODULE main
VAR
  x0 : boolean;
INIT
  !x0;
TRANS
  x0 <-> next(!x0);
```




```
MODULE main
VAR
  x0 : boolean;
  x1 : boolean;
INIT
  !x0 & !x1;
TRANS
  (next(x0) <-> !x0)
    &
  (next(x1) <-> ((x0 & !x1) | (!x0
    & x1)));
```

```
while true do
  if x < 200 then
    x := x + 1
od
```

```
while true do
  if x > 0 then
    x := x-1
od
```

```
while true do
  if x = 200 then
    x := 0
od
```

```
MODULE main
VAR
  x    : 0 .. 200;
INIT
  x = 199;
TRANS
  (x<200 & next(x)=x+1) |
  (x>0 & next(x)=x+(-1)) |
  (x=200 & next(x)=0);
```

An example of Symbolic Model Checking

- Let $S = \{0,1\}^3$ and let the `target` be defined by the function $x_0 \wedge x_1 \wedge \neg x_2$. Let R be defined by:

$$(\bigwedge_{i=0}^{2}(x_i \rightarrow x'_i)) \wedge (x_0 \leftrightarrow x'_0 \vee x_1 \leftrightarrow x'_1)$$

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$\equiv \neg x_2$ fixpoint: $[EF(\texttt{target})]$

Key step:

- proceed from a set of states T to the set of its predecessors:

$$\{s \in S \mid \exists (s, s') \in R, s' \in T\}$$

- proceed from a formula $\beta_T(\overline{x})$ to:

$$\beta'(\overline{x}) = \exists \overline{y}(f_R(\overline{x}, \overline{y}) \wedge \beta_T(\overline{y}))$$

Problem:

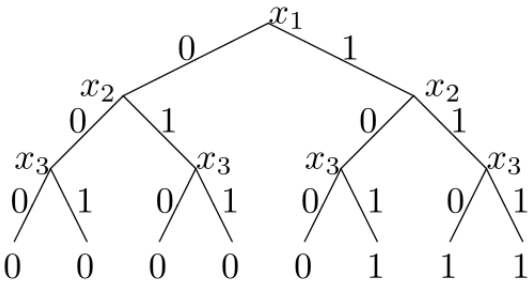
what is a good normal form for the representation of Boolean functions which allows efficient application of $\neg$, $\wedge$, $\vee$, and $\exists$?

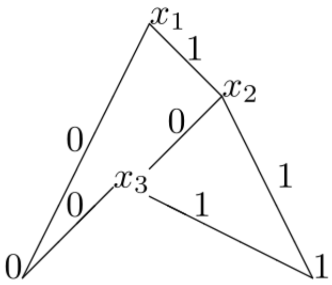
Ordered Binary Decision Diagrams (OBDDs)

Ordered Binary Decision Diagrams (OBDDs) are based on work by Akers (1978) and Bryant (1986). They are reduced versions of decision trees for Boolean functions.

Example: $x_1 \wedge (x_2 \vee x_3)$

Decision tree vs. OBDD





Basic Results on OBDDs

1. Using the two reduction rules:
   - identify isomorphic "subgraphs"
   - for $x \in \{0, 1\}$, replace the paths $x0$ and $x1$ by the arc x, whenever $x0$ and $x1$ lead to the same value (cf. previous example)
   - one obtains a canonical (unique) OBDD for a given Boolean function and a given order of variables
2. The operations $\neg, \wedge, \vee, \exists$ can be performed efficiently on OBDDs

Partial order reduction can be used to reduce the size of the state space making use of the following observation:

computations that differ in the ordering of independently executed events are usually indistinguishable by the specification and thus they can be considered equivalent

It suffices to check a reduced state space, which contains (at least) one representative computation for each class of equivalent computations.

Advanced Techniques

- Bounded model checking: SAT solvers + symbolic model checking + bounded models
- Satisfiability Modulo Theories (SMT solvers): satisfiability of logical formulas with respect to one or more background theories formulated in first-order logic with equality (integers, real numbers, ..)
- K-Liveness: a simple but effective technique for LTL verification; it checks the absence of lasso-shaped counterexamples by trying to prove that bad states are visited at most k times, for increasing values of k (liveness checking as a sequence of safety checks)
- IC3: a very successful SAT-based model checking algorithm based on induction

BOUNDED MODEL CHECKING

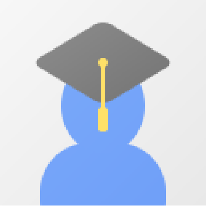
Reference:

Armin Biere et al. (1999). "Symbolic model checking without BDDs". In: *International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS)*. Springer, pp. 193–207

(a) A. Biere

(b) A. Cimatti

(c) E. Clarke



(d) Y. Zhu

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- why not in DNF?

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- there are several efficient algorithms for solving SAT (*e.g.*, DPLL, CDCL...) along with many heuristics (*e.g.*, 2 watching literals, glue clauses...)
- some numbers:
  - $> 100'000$ variables;
  - $> 1'000'000$ clauses;

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$$\mathcal{M}, s \not\models \phi$$

- the universal problem $\mathcal{M}, s \models A\psi$ is reduced to the existential problem $\mathcal{M}, s \models E\phi$, where $\phi := \neg\psi$;

Bounded Model Checking (BMC) solves the problem $\mathcal{M}, s \models E\phi$ by proceeding incrementally:

- we start with $k = 0$;
- check if there exists and execution π of $\mathcal{M}$ of length k that satisfies ϕ; encode this problem into a SAT instance and call a SAT-solver;
- if so, we have found a counterexample to ψ; if not, $k{+}{+}$.

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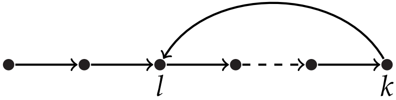
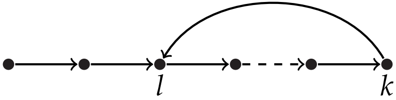
Crucial observation:

- a finite trace can still represent an infinite state sequence, if it contains a loop-back.



k-loop, aka Lasso-Shaped Models



Definition (k-loop)

A path π is a (k, l)-loop, with $l \leq k$, if $T(\pi(k), \pi(l))$ holds and $\pi = u \cdot v^\omega$, where:

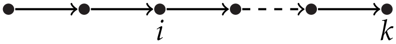
- $u = \pi(1) \dots \pi(l-1)$;
- $v = \pi(l) \dots \pi(k)$.

We call π a k-loop if there exists $l \leq k$ for which π is a (k, l)-loop.

Given a finite trace π of the system $\mathcal{M}$, BMC distinguishes between two cases:

- either π contains a loop-back (π is lasso-shaped):
  - $\Rightarrow$ apply standard LTL semantics to check if $\pi \models \phi$;
- or π is loop-free:
  - $\Rightarrow$ apply bounded semantics
  - $\Rightarrow$ if a path is a model of ϕ under bounded semantics then any extension of the path is a model of ϕ under standard semantics (conservative semantics)

Bounded Semantics for LTL

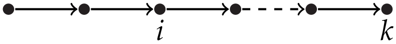


If π is not a k-loop, we introduce bounded semantics for LTL.

Definition (Bounded semantics for LTL)

Let $k \geq 0$ and π a path that is not a k-loop. An LTL formula ϕ is valid along π with bound k, written $\pi \models_k^0 \phi$, iff:

- $\pi \models_k^i p$ iff $p \in L(\pi(i))$
- $\pi \models_k^i \neg p$ iff $p \notin L(\pi(i))$



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- $\pi \models_k^i \phi_1 \vee \phi_2$ iff $\pi \models_k^i \phi_1$ or $\pi \models_k^i \phi_2$
- $\pi \models_k^i \phi_1 \wedge \phi_2$ iff $\pi \models_k^i \phi_1$ and $\pi \models_k^i \phi_2$

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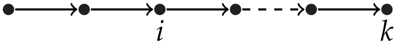

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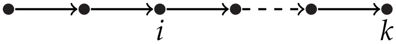
- $\pi \models_k^i \mathsf{X}\phi_1$ iff $i < k$ and $\pi \models_k^{i+1} \phi_1$
- $\pi \models_k^i \phi_1 \,\mathsf{U}\, \phi_2$ iff $\exists i \leq j \leq k$ such that $\pi \models_k^j \phi_2$ and $\forall i \leq n < j$ it holds that $\pi \models_k^n \phi_1$



If π is not a k-loop, we introduce bounded semantics for LTL.

Definition (Bounded semantics for LTL)

Let $k \geq 0$ and π a path that is not a k-loop. An LTL formula ϕ is valid along π with bound k, written $\pi \models_k^0 \phi$, iff:

- $\pi \models_k^i \mathsf{G}\phi_1$ iff ???
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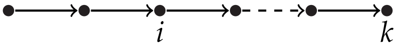

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- $\pi \models_k^i \mathsf{G}\phi_1$ is always false
- $\pi \models_k^i \mathsf{F}\phi_1$ iff $\exists i \leq j \leq k$ such that $\pi \models_k^j \phi_1$

Now we see how to reduce BMC to SAT.

- the first thing to do is to define a Boolean formula that encodes all the paths of $\mathcal{M}$ of length k.

SAT-based encoding of BMC

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Definition (Unfolding of the Transition Relation)

For a Kripke structure $\mathcal{M}$ and $k \geq 0$, we define:

$$[\![\mathcal{M}]\!]_k := I(\overline{x}^0) \wedge \bigwedge_{i=0}^{k-1} T(\overline{x}^i, \overline{x}^{i+1})$$

N.B.: For each $i \geq 0$, with $\overline{x}^i$ we represent the i^{th}-stepped version of the set of variables $\overline{x}$. For example, $\overline{x}^1 := \overline{x}'$.

Encoding of the LTL formula

So far, we have seen how to encode paths of length k of the model $\mathcal{M}$.

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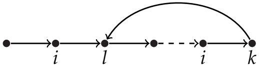
- now we see how to encode the right-hand side.

We have seen that BMC distinguishes between lasso-shaped (k-loop) and loop-free paths:

- we start with the encoding in case of k-loops.

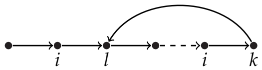


Definition (Loop Encoding)

Let $l \leq k$. We define:

$$_{l}L_k := T(\overline{x}^k, \overline{x}^l) \qquad L_k := \bigvee_{l=0}^{k} {}_{l}L_k$$

Encoding of a loop



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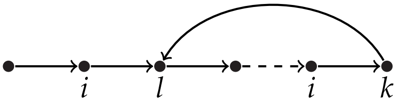
Definition (Successor in a Loop)

Let $l, i \leq k$ and π be a (k, l)-loop. We define the successor $succ(i)$ of i in π as:

- $succ(i) := i + 1$ if $i < k$;
- $succ(i) := l$ if $i = k$.

Encoding in case of Loop



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Let ϕ be an LTL formula and $l, i, k \geq 0$ such that $l, i \leq k$. We define ${}_l[\![\phi]\!]_k^i$ recursively as follows:

- ${}_l[\![p]\!]_k^i := p(\overline{x}^i)$
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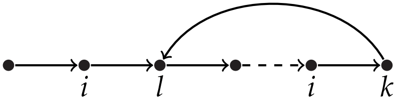

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- ${}_l[\![\phi_1 \vee \phi_2]\!]_k^i := {}_l[\![\phi_1]\!]_k^i \vee {}_l[\![\phi_2]\!]_k^i$
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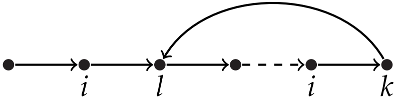

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- ${}_l[\![\mathsf{X}\phi_1]\!]_k^i := {}_l[\![\phi_1]\!]_k^{succ(i)}$
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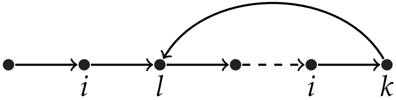

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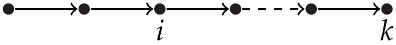
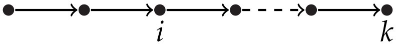


Definition (Encoding of an LTL formula for a loop-free path)

Let ϕ be an LTL formula and $i, k \geq 0$. We define $[\![\phi]\!]_k^i$ recursively as follows:

- $[\![\phi]\!]_k^{k+1} := \bot$

Encoding in case of NO Loops



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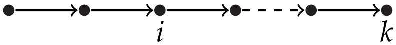

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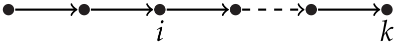

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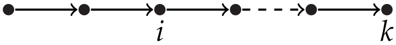

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Overall encoding

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Let ϕ be an LTL formula, $\mathcal{M}$ be a Kripke structure and $k \geq 0$:

$$[\![M, \phi]\!]_k := \underbrace{[\![\mathcal{M}]\!]_k}_{\substack{\text{encoding of} \\ \text{the machine}}} \wedge \left(\underbrace{\left(\neg L_k \wedge [\![\phi]\!]_k^0\right)}_{\substack{\text{loop-free} \\ \text{models}}} \vee \underbrace{\bigvee_{l=0}^{k} \left({}_lL_k \wedge {}_l[\![\phi]\!]_k^0\right)}_{\substack{\text{lasso-shaped} \\ \text{models}}} \right)$$

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Theorem (Soundness)

$[\![\mathcal{M}, \phi]\!]_k$ is satisfiable iff $\mathcal{M} \models_k E\phi$.

Completeness

Algorithm:

- start with $k = 0$
- call a SAT-solver on $[\![\mathcal{M}, \phi]\!]_k$
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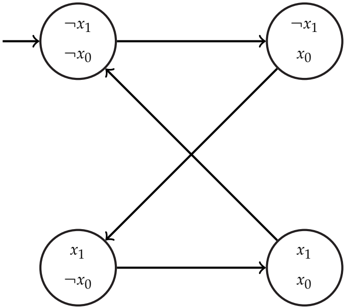
What happens if $\mathcal{M} \not\models \phi$?

- the procedure does not terminate
- in order to be complete, BMC needs to compute the recurrence diameter: very costly
- BMC is mainly used as a bug finder, rather than as a prover.

Example



- $\phi_1 \coloneqq \mathsf{GF}(x_0) \wedge x_1)$ ✓
- $\phi_2 \coloneqq \mathsf{FG}(\neg x_0 \wedge \neg x_1)$ ✗

Solving LTL-SAT with BMC

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$$[\![\mathcal{M}]\!]_k \wedge \left((\neg L_k \wedge [\![\phi]\!]_k^0) \vee \bigvee_{l=0}^{k} ({}_l L_k \wedge {}_l[\![\phi]\!]_k^0) \right)$$

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- satisfiability checking

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BLACK

- we developed this tool based on the idea of *bounded satisfiability checking*
- BLACK = Bounded Ltl sAtisfiability ChecKer
- `https://www.black-sat.org/en/stable/`
- Examples

Reference:

Luca Geatti, Nicola Gigante, and Angelo Montanari (2021). "BLACK: A Fast, Flexible and Reliable LTL Satisfiability Checker". In: *Proceedings of the 3rd Workshop on Artificial Intelligence and Formal Verification, Logic, Automata, and Synthesis*. Ed. by Dario Della Monica, Gian Luca Pozzato, and Enrico Scala. Vol. 2987. CEUR Workshop Proceedings. CEUR-WS.org, pp. 7–12. URL: `http://ceur-ws.org/Vol-2987/paper2.pdf`

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