Department of Mathematics, Computer Science and Physics, University of Udine

The Safety Fragment of Temporal Logics on Infinite Sequences

Lesson 5

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THE SAFETY FRAGMENT

OF ω-REGULAR LANGUAGES

In this part, we will mainly deal with language of *infinite words* and with logics interpreted over *infinite words*.

Informal definitions:

Safety properties express the fact that "something bad never happens".

E.g.: a deadlock or a simultaneous access to a critical section.

Any violation of a safety property is irremediable.

E.g.: once a deadlock occured, we don't have any hope to do better.

Any violation of a safety property has a finite witness.

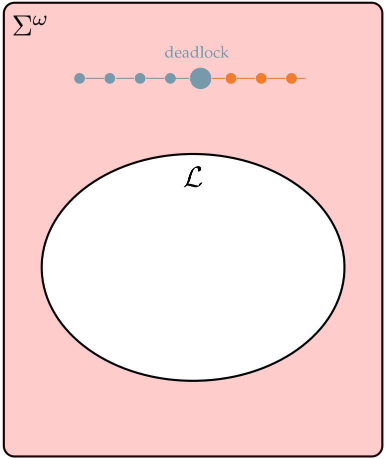
Notation:

- For any $i \in \mathbb{N}$, $\sigma_{[0,i]}$ is the prefix of σ up to position i.
- for any $\sigma \in \Sigma^*$ and for any $\sigma' \in \Sigma^\omega$, $\sigma \cdot \sigma'$ is the *concatenation* of σ' to the end of σ.

Definition (Safety Property)

$\mathcal{L} \subseteq \Sigma^\omega$ is a *safety property* iff, for all $\sigma \notin \mathcal{L}$, there exists an position $i \in \mathbb{N}$ such that $\sigma_{[0,i]} \cdot \sigma' \notin \mathcal{L}$, for all $\sigma' \in \Sigma^\omega$.

$\sigma_{[0,i]}$ is called the *bad prefix* of σ.



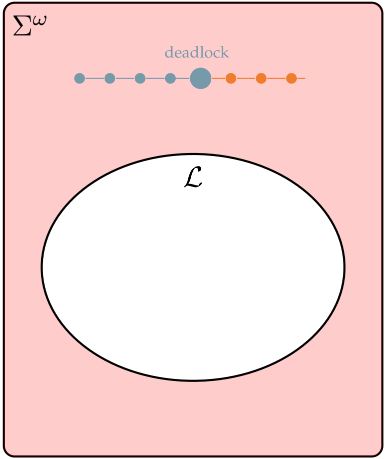
Examples

- $b \cdot (a)^\omega$ is a safety language.
- *"The set of infinite words in which each 'a' is followed by some 'b' "* is <u>not</u> a safety language.

- We denote with $\texttt{bad}(\mathcal{L})$ the set of bad prefixes of $\mathcal{L}$.
- For any safety language $\mathcal{L}$, it holds that:

$$\overline{\mathcal{L}} = \texttt{bad}(\mathcal{L}) \cdot \Sigma^\omega$$

where $\overline{\mathcal{L}}$ is the *complement* of $\mathcal{L}$.



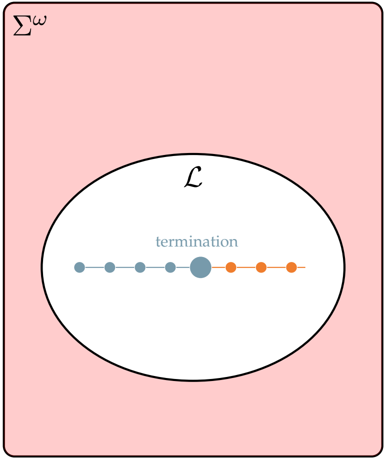
Definition (Cosafety Property)

$\mathcal{L} \subseteq \Sigma^\omega$ is a *cosafety property* iff for all $\sigma \in \mathcal{L}$, there exists an position $i \in \mathbb{N}$ such that $\sigma_{[0,i]} \cdot \sigma' \in \mathcal{L}$, for all $\sigma' \in \Sigma^\omega$.

$\sigma_{[0,i]}$ is called the *good prefix* of σ.

Property:

$\mathcal{L}$ is a cosafety property iff $\overline{\mathcal{L}}$ is a safety property.

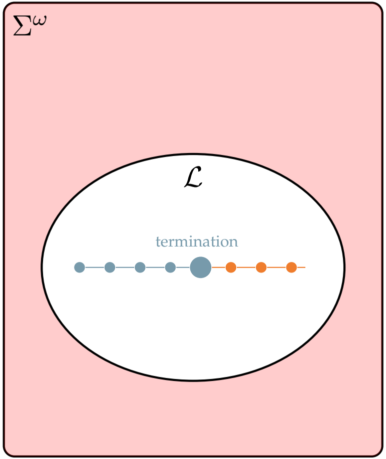


Examples

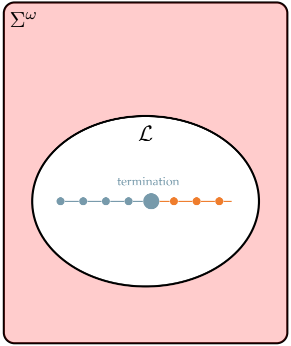
- *"The set of infinite words in which there is an 'a' that is followed by some 'b' "* is a cosafety language.
- *"The set of infinite words in which each 'a' is followed by some 'b' "* is not a cosafety language.

- We denote with $\mathsf{good}(\mathcal{L})$ the set of good prefixes of $\mathcal{L}$.
- For any cosafety language $\mathcal{L}$, it holds that:

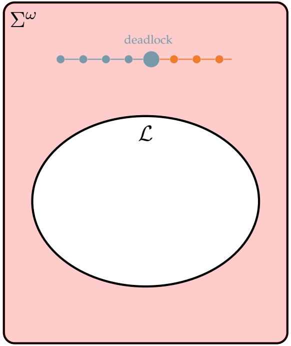
$$\mathcal{L} = \mathsf{good}(\mathcal{L}) \cdot \Sigma^\omega$$



We denote with coSAFETY the set of all cosafety ω-regular languages.



We denote with SAFETY the set of all safety ω-regular languages.



We denote with coSAFETY the set of all cosafety ω-regular languages.

ω-Regular Expressions

coSAFETY is characterized by the following type of ω-regular expressions:

$$K \cdot \Sigma^\omega$$

where $K \in \mathsf{REG}$.

We denote with SAFETY the set of all safety ω-regular languages.

ω-Regular Expressions

SAFETY is characterized by the following type of ω-regular expressions:

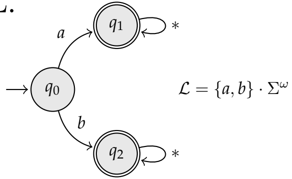
$$\overline{K \cdot \Sigma^\omega}$$

where $K \in \mathsf{REG}$.

We denote with coSAFETY the set of all cosafety ω-regular languages.

Automata

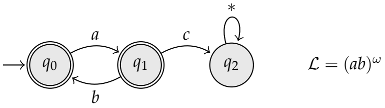
coSAFETY is characterized by the following type of automata: *terminal deterministic Büchi automata* (tDBA, for short), that is DBAs in which each final state has self-loop labeled with each letter in Σ.



We denote with SAFETY the set of all safety ω-regular languages.

Automata

SAFETY is characterized by the following type of automata: *deterministic safety automata* (DSA, for short). Accepting condition: visit *only* final states.



We denote with coSAFETY the set of all cosafety ω-regular languages.

S1S

To the best of our knowledge, no characterizations of coSAFETY in terms of S1S have been studied.

We denote with SAFETY the set of all safety ω-regular languages.

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Temporal Logics

To the best of our knowledge, no characterizations of coSAFETY in terms of temporal logics have been studied.

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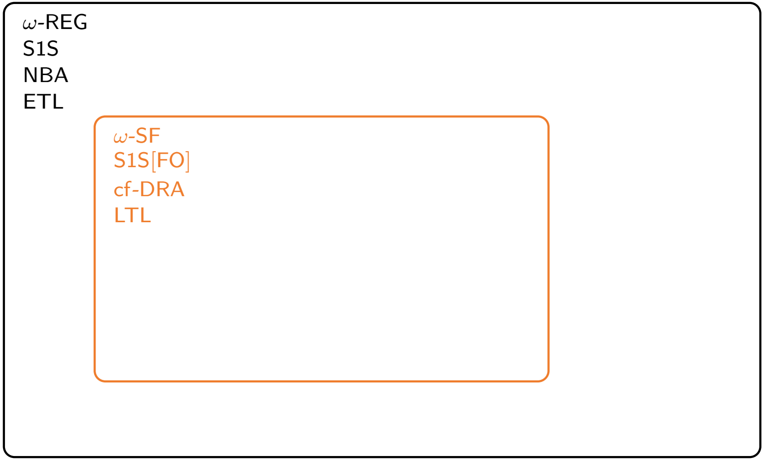
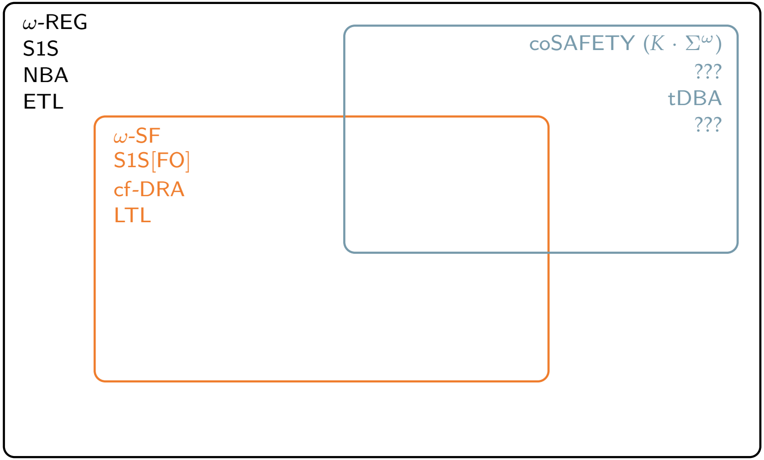
Temporal Logics

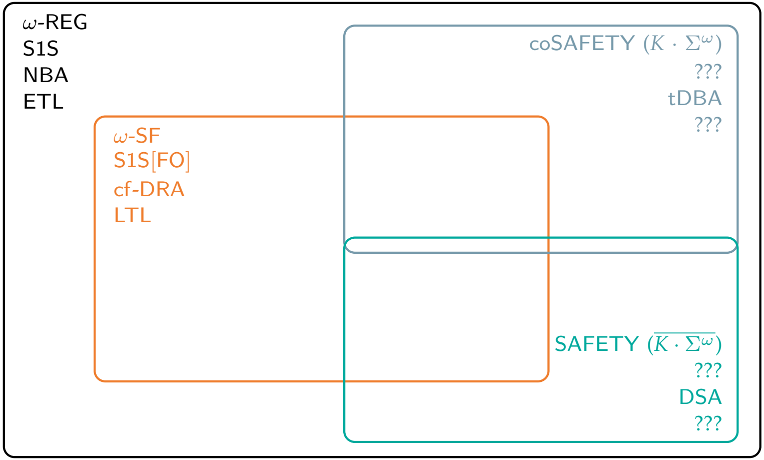
To the best of our knowledge, no characterizations of SAFETY in terms of temporal logics have been studied.



ω-REG
S1S
NBA
ETL
ω-SF
S1S[FO]
cf-DRA
LTL

Set-theoretic view of (co)safety ω-languages

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LTL
coSAFETY $(K \cdot \Sigma^\omega)$
???
tDBA
???
SAFETY $(\overline{K \cdot \Sigma^\omega})$
???
DSA
???

Informal definitions:

> *In a liveness property, no partial execution is irremediable.*

E.g.: *"each request is eventually followed by a grant"* is a liveness property.

Definition (Liveness Property)

$\mathcal{L} \subseteq \Sigma^\omega$ is a *liveness property* iff, for all $\sigma \in \Sigma^*$, there exists a $\sigma' \in \Sigma^\omega$ such that $\sigma \cdot \sigma' \in \mathcal{L}$.

Examples:

- *"The set of infinite words in which each 'a' is followed by some 'b' "* is a liveness language.
- $b \cdot (a)^\omega$ is not a liveness language.

Theorem (Alpern & Schneider (1987))

Each ω-regular property is the intersection of a safety property and a liveness property.

Reference:

Bowen Alpern and Fred B. Schneider (1987). "Recognizing Safety and Liveness". In: *Distributed Comput.* 2.3, pp. 117–126. DOI: 10.1007/BF01782772. URL: https://doi.org/10.1007/BF01782772

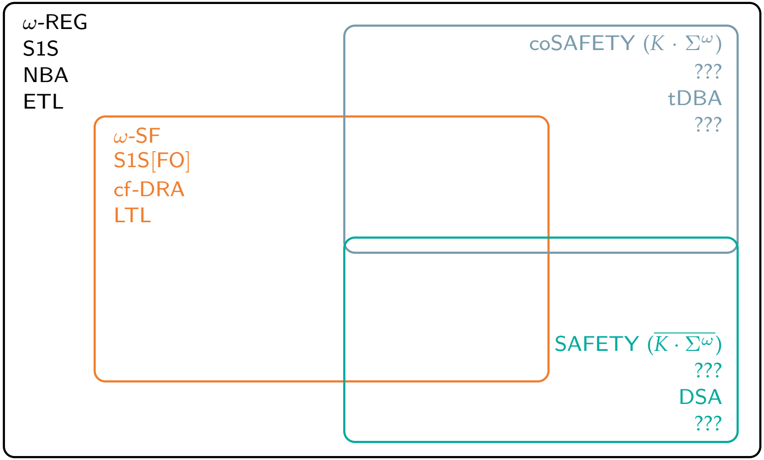
Theorem (Alpern & Schneider (1987))

Each ω-regular property is the intersection of a safety property and a liveness property.

This decomposition can be performed effectively:

Given a NBA $\mathcal{A}$, there is an algorithm to build two NBA $\mathcal{A}_s$ and $\mathcal{A}_l$ such that:

- $\mathcal{L}(\mathcal{A}_s)$ is safety;
- $\mathcal{L}(\mathcal{A}_l)$ is liveness;
- $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_s) \cap \mathcal{L}(\mathcal{A}_l)$.



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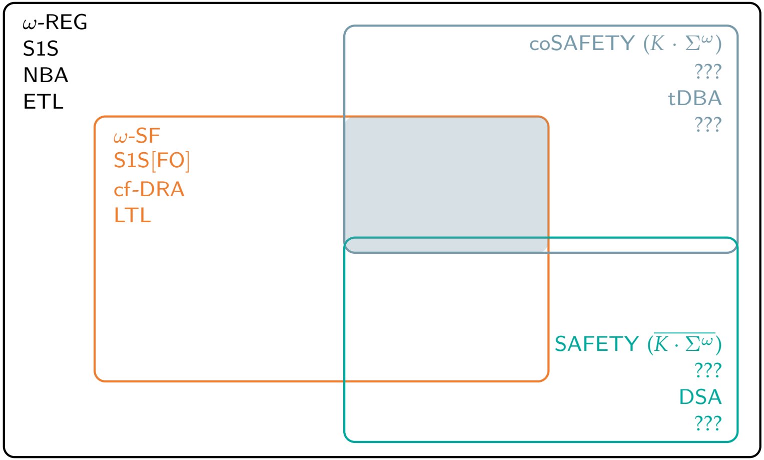
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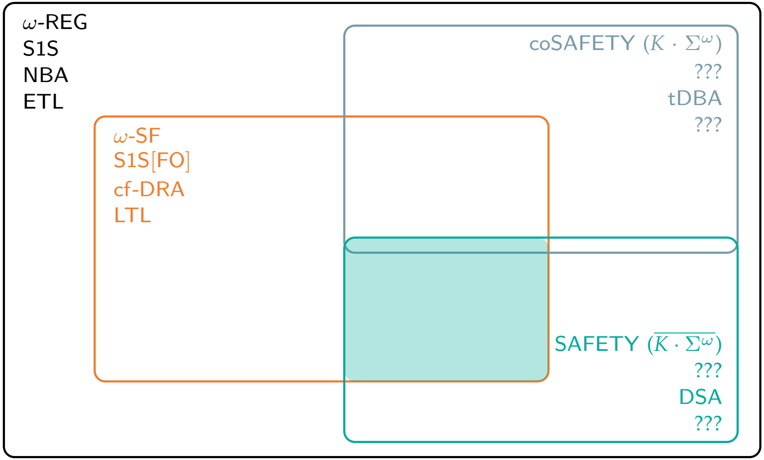
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THE SAFETY FRAGMENT OF LTL

AND ITS THEORETICAL FEATURES

The cosafety fragment of LTL

Definition

The *cosafety fragment of* LTL is the set of languages in this set:

$$[\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY}$$

We will see four characterizations in terms of:

- regular expressions
- first-order logic
- automata
- temporal logic

Definition

The *cosafety fragment of* LTL is the set of languages in this set:

$$[\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY}$$

ω-regular expressions

$\mathsf{SF} \cdot \Sigma^\omega = \{K \cdot \Sigma^\omega \mid K \in \mathsf{SF}\}$

- the "SF " part corresponds to LTL
- the "$\cdot\Sigma^\omega$" part corresponds to being a cosafety fragment

Ina Schiering and Wolfgang Thomas (1996). "Counter-free automata, first-order logic, and star-free expressions extended by prefix oracles". In: *Developments in Language Theory, II (Magdeburg, 1995), Worl Sci. Publishing, River Edge, NJ*, pp. 166–175

Definition

The *cosafety fragment of* LTL is the set of languages in this set:

$$[\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY}$$

First-order logic

We define coSafety-FO as the fragment of S1S[FO] in which quantifiers are bounded as follows:

- $\exists y \,.\, (x < y \wedge \ldots)$
- $\forall y \,.\, (x < y < z \rightarrow \ldots)$

Alessandro Cimatti et al. (2022). "A first-order logic characterisation of safety and co-safety languages". In: *Foundations of Software Science and Computation Structures - 25th International Conference, FOSSACS 2022, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2022, Munich, Germany, April 2-7, 2022, Proceedings*. Ed. by Patricia Bouyer and Lutz Schröder. Vol. 13242. Lecture Notes in Computer Science. Springer, pp. 244–263. DOI: 10.1007/978-3-030-99253-8_13. URL: https://doi.org/10.1007/978-3-030-99253-8%5C_13

Definition

The *cosafety fragment of* LTL is the set of languages in this set:

$$[\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY}$$

First-order logic

Example

$\phi(x) := \exists y \,.\, (x < y \wedge P(y) \wedge \forall z \,.\, (x < z < y \rightarrow Q(z)))$

The cosafety fragment of LTL

Definition

The *cosafety fragment of* LTL is the set of languages in this set:

$$[\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY}$$

First-order logic

- the "first-order" part corresponds to LTL
- the "bounded quantifiers" part corresponds to being a cosafety fragment

Definition

The *cosafety fragment of* LTL is the set of languages in this set:

$$[\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY}$$

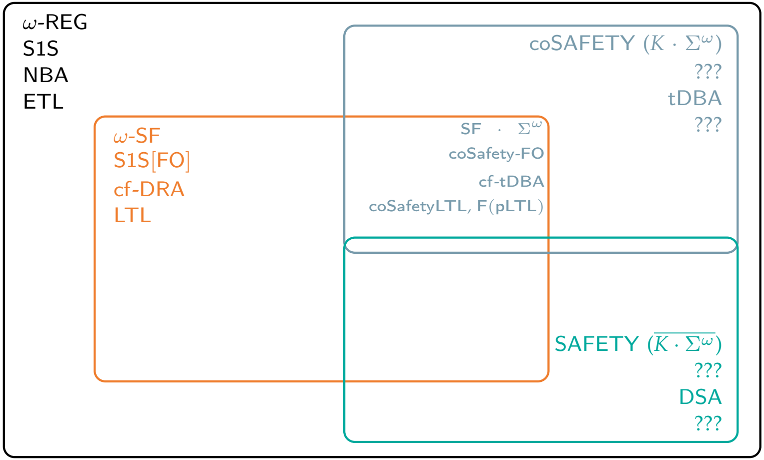
Automata

cf-tDBA = counter-free terminal DBA

- the "counter-free" part corresponds to LTL
- the "terminal" part corresponds to being a cosafety fragment

Ina Schiering and Wolfgang Thomas (1996). "Counter-free automata, first-order logic, and star-free expressions extended by prefix oracles". In: *Developments in Language Theory, II (Magdeburg, 1995), Worl Sci. Publishing, River Edge, NJ*, pp. 166–175

Set-theoretic view of the (co)safety fragment of LTL



The cosafety fragment of LTL

We say that a temporal logic $\mathbb{L}$ is *cosafety* iff, for any $\phi \in \mathbb{L}$, $\mathcal{L}(\phi)$ is *cosafety*.

coSafetyLTL

Definition

$\phi := p \mid \neg p \mid \phi \vee \phi \mid \phi \wedge \phi \mid \mathsf{X}\phi \mid \mathsf{F}\phi \mid \phi \, \mathsf{U} \, \phi$

Example:

$p \, \mathsf{U} \, q$

F(pLTL)

Definition

$\phi := \mathsf{F}(\alpha)$, where $\alpha \in \mathsf{pLTL}$, that is α is a pure-past LTL formula.

Example:

$\mathsf{F}(q \wedge \widetilde{\mathsf{Y}}\mathsf{H}p)$

F(pLTL) is the canonical form of coSafetyLTL.

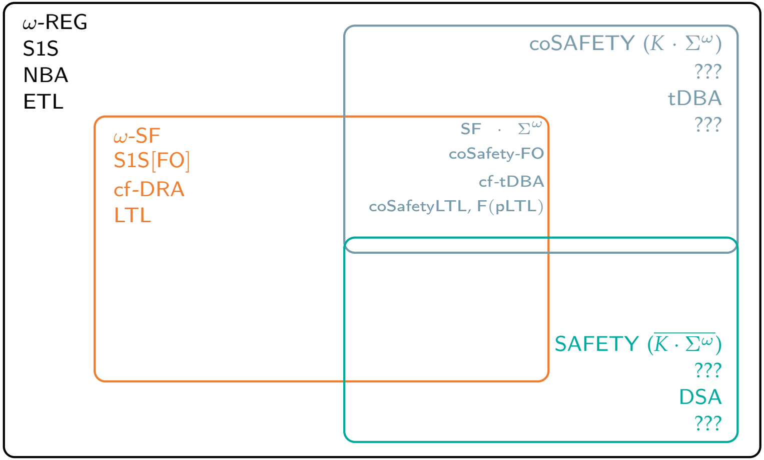
The cosafety fragment of LTL

Theorem

- coSafetyLTL *and* F(pLTL) *are expressively equivalent.*
- coSafetyLTL *and* F(pLTL) *are expressively complete w.r.t.* $[\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY}$.

Reference:

Edward Y. Chang, Zohar Manna, and Amir Pnueli (1992). "Characterization of Temporal Property Classes". In: *Proceedings of the 19th International Colloquium on Automata, Languages and Programming*. Ed. by Werner Kuich. Vol. 623. Lecture Notes in Computer Science. Springer, pp. 474–486. DOI: 10.1007/3-540-55719-9_97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REFERENCES

Bibliography I

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Bibliography II

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