A simple variational derivation of slender rods theory

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We present an asymptotic analysis of the three-dimensional problem for a thin linearly elastic cantilever $\Omega_\varepsilon = \varepsilon\omega \times (0, \ell)$ as ε goes to zero. By assuming ω simply connected and under suitable assumptions on the given loads, we show that the 3D problem converges in a variational sense to the classical dimensional models for extension, flexure and torsion of slender rods.

Keywords: slender rods; thin beams; linear elasticity; Γ-convergence.

1. Introduction

Structures with one or two dimensions much smaller than the remaining are very often encountered in engineering problems. The peculiar geometry of these thin structures suggests a lower, two or one, dimensional modelling. Classically, these lower dimensional models are based on some a-priori assumptions inspired by the smallness of certain dimensions. In the seventies new techniques which make circumvent the use of any a-priori assumption have been developed. The French school tuned a method based on a rigorous asymptotic expansion, while the Italian school followed the inspiration of E. De Giorgi[3] and adopted the use of Γ-convergence theory. Since then Γ-limits of energy functionals have been successfully applied to derive one or two-dimensional models of a variety of thin structures starting from linear as well as non linear three-dimensional elasticity.

In 1994, Anzellotti, Baldo and Percivale[1] derived variational models for linearly elastic homogeneous and isotropic rods and plates by using Γ-asymptotic developments (see also Percivale[10]). They deduce the mechanical behavior of the beam by calculating two different Γ-limits, one for the extensional problem and one for the flexural and torsional problems. The two Γ-limits are originated by different scalings of the energy functionals and correspond to terms of different order in the asymptotic development.

In this paper, by suitably scaling the axial component of the displacement in the three-dimensional energies and using a technique developed in Freddi, Morassi and Paroni,[4,5] we obtain, in an easier way, the extensional, flexural and torsional problems for a linearly elastic homogeneous and isotropic slender rod with only one Γ-limit.

Notation. Unless otherwise stated, we use the Einstein summation convention and we index vector and tensor components as follows: Greek indices α, β and γ take values in the set $\{1,2\}$ and Latin indices i, j, h in the set $\{1,2,3\}$. The component k of a vector v will be denoted either with $(v)_k$ or v_k and an analogous notation will be used to denote tensor components. $\mathcal{E}_{\alpha\beta}$ denotes the Ricci's symbol, that is $\mathcal{E}_{11} = \mathcal{E}_{22} = 0$, $\mathcal{E}_{12} = 1$ and $\mathcal{E}_{21} = -1$. $L^2(A;B)$ and $H^s(A;B)$ are the standard Lebesgue and Sobolev spaces of functions defined on A and taking values in B, with the usual norms $\|\cdot\|_{L^2(A;B)}$ and $\|\cdot\|_{H^s(A;B)}$, respectively. When $B = \mathbb{R}$ or when the right set B is clear from the context, we will simply write $L^2(A)$ or $H^s(A)$, sometimes even in the notation used for norms. Convergence in the norm will be denoted by $\to$ while weak convergence is denoted with $\rightharpoonup$.

With a little but harmless abuse of notation, we use to call "sequences" even those families indicized by a continuous parameter ε which, throughout the whole paper, will be assumed to belong to the interval $(0,1]$.

2. The 3-Dimensional problem

Let $\omega \subset \mathbb{R}^2$ be a simply connected, bounded, open set with a Lipschitz boundary. We consider a three-dimensional body $\Omega_\varepsilon \subset \mathbb{R}^3$, where $\Omega_\varepsilon := \omega_\varepsilon \times (0,\ell)$, $\omega_\varepsilon := \varepsilon\omega$, $\varepsilon \in (0,1]$ and $\ell > 0$. For any $x_3 \in (0,\ell)$ we further set $S_\varepsilon(x_3) := \omega_\varepsilon \times \{x_3\}$. Henceforth we shall refer to Ω_ε as the reference configuration of the body and denote by

$$Eu(x) := \mathrm{sym}(Du(x)) := \frac{Du(x) + Du^T(x)}{2},$$

the strain of $u : \Omega_\varepsilon \to \mathbb{R}^3$. The material is assumed to be homogeneous and isotropic, so that $\mathbb{C}A = 2\mu A + \lambda(\mathrm{tr}A)I$ for every symmetric matrix A. I denotes the identity matrix of order 3. We assume $\mu > 0$ and $\lambda \geq 0$ so to have, for every symmetric tensor A,

$$\mathbb{C}A \cdot A \geq \mu|A|^2, \tag{1}$$

where $\cdot$ denotes the scalar product. Define

$$H^1_\#(\Omega_\varepsilon;\mathbb{R}^3) := \{u \in H^1(\Omega_\varepsilon;\mathbb{R}^3) : u = 0 \text{ on } S_\varepsilon(0)\}.$$

Due to the coercivity condition (1) and the strict convexity of the integrand, the energy functionals

$$\mathcal{F}_\varepsilon(u) := \frac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}Eu \cdot Eu\, dx - \int_{\Omega_\varepsilon} b^\varepsilon \cdot u\, dx$$

admit for every $\varepsilon > 0$ a unique minimizer among all competing displacements $u \in H^1_\#(\Omega_\varepsilon;\mathbb{R}^3)$.

3. The rescaled problem

To state our results it is convenient to stretch the domain Ω_ε along the transverse directions x_1 and x_2 in a way that the transformed domain does not depend on ε. Let us therefore set $\Omega := \Omega_1$, $S(x_3) := S_1(x_3)$ and let $p_\varepsilon : \Omega \to \Omega_\varepsilon$ be defined by $p_\varepsilon(y) = p_\varepsilon(y_1, y_2, y_3) = (\varepsilon y_1, \varepsilon y_2, y_3)$. Let us consider the following 3×3 matrix

$$H^\varepsilon v := (\frac{D_1 v}{\varepsilon}, \frac{D_2 v}{\varepsilon}, D_3 v),$$

where $D_i v$ denotes the column vector of the partial derivatives of v with respect to y_i. We will use moreover the following notation

$$E^\varepsilon v := \mathrm{sym}(H^\varepsilon v), \quad W^\varepsilon v := \mathrm{skw}(H^\varepsilon v)$$

and also denote by $Wv := W^1 v$ the skew symmetric part of the gradient. Let $H^1_\#(\Omega;\mathbb{R}^3) := \{v \in H^1(\Omega;\mathbb{R}^3) : v = 0 \text{ on } S(0)\}$ and consider the rescaled energy $F_\varepsilon : H^1_\#(\Omega;\mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$F_\varepsilon(v) := \frac{1}{\varepsilon^2}\mathcal{F}_\varepsilon(v \circ p_\varepsilon^{-1}) = I_\varepsilon(v) - \int_\Omega b^\varepsilon \circ p_\varepsilon \cdot v\, dy,$$

where $I_\varepsilon(v) := \frac{1}{2}\int_\Omega \mathbb{C}E^\varepsilon v \cdot E^\varepsilon v\, dy$. We further suppose the loads to have the following form

$$\begin{gathered} b_1^\varepsilon \circ p_\varepsilon(y) = \varepsilon^2 b_1(y) - \varepsilon\frac{m(y_3)}{I_O}y_2, \quad b_2^\varepsilon \circ p_\varepsilon(y) = \varepsilon^2 b_2(y) + \varepsilon\frac{m(y_3)}{I_O}y_1, \\ b_3^\varepsilon \circ p_\varepsilon(y) = \varepsilon b_3(y), \end{gathered} \tag{2}$$

with $b = (b_1, b_2, b_3) \in L^2(\Omega;\mathbb{R}^3)$, $m \in L^2(0,\ell)$ and $I_O := \int_\omega (y_1^2 + y_2^2)\, dy_1\, dy_2$ is the polar moment of inertia of the section ω. With the loads given by (2), the energy $F_\varepsilon(v)$ can be rewritten as

$$F_\varepsilon(v) = I_\varepsilon(v) - \varepsilon^2 \int_\Omega b \cdot \big(v_1, v_2, \frac{v_3}{\varepsilon}\big)\, dy - \varepsilon^2 \int_0^\ell m\, \vartheta^\varepsilon(v)\, dy_3, \tag{3}$$

where we have set

$$\vartheta^\varepsilon(v)(y_3) := \frac{1}{I_O} \int_\omega \Big(\frac{y_1}{\varepsilon} v_2(y_1, y_2, y_3) - \frac{y_2}{\varepsilon} v_1(y_1, y_2, y_3)\Big)\, dy_1 dy_2. \tag{4}$$

We note that if $v \in L^2(\Omega;\mathbb{R}^3)$ then $\vartheta^\varepsilon(v) \in L^2(0,\ell)$. A similar statement holds if we replace L^2 with H^1.

4. Compactness lemmata

To prove the compactness of the displacements we need the following scaled Korn inequality.

Theorem 4.1. *There exists a positive constant K such that*

$$\int_\Omega \Big(\big|(u_1, u_2, \frac{u_3}{\varepsilon})\big|^2 + |H^\varepsilon u|^2\Big) dy \le \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2\, dy,$$

for every $u \in H^1_\#(\Omega;\mathbb{R}^3)$ and every $\varepsilon \in (0,1]$.

Proof. The inequality $\int_\Omega |H^\varepsilon u|^2\, dy \le (K/\varepsilon^2) \int_\Omega |E^\varepsilon u|^2\, dy$ simply follows by rescaling the Korn's inequality of Anzellotti, Baldo and Percivale,[1] Theorem A.1. To show that $\int_\Omega |(u_1, u_2, u_3/\varepsilon)|^2\, dy \le (K/\varepsilon^2) \int_\Omega |E^\varepsilon u|^2\, dy$, it suffices to set $v := (u_1, u_2, u_3/\varepsilon)$, notice that $|E^\varepsilon u| \ge \varepsilon |Ev|$ and apply the standard Korn inequality to v on Ω (see, for instance, Oleinik, Shamaev and Yosifian,[9] Theorem 2.7). □

Let $H_{BN}(\Omega;\mathbb{R}^3) := \{v \in H^1_\#(\Omega;\mathbb{R}^3) : (Ev)_{i\alpha} = 0 \text{ for } i = 1,2,3,\ \alpha = 1,2\}$ be the space of Bernoulli-Navier displacements on Ω. It can be characterized also as follows (see Le Dret,[8] Section 4.1)

$$\begin{aligned} H_{BN}(\Omega;\mathbb{R}^3) = \{v \in H^1_\#(\Omega;\mathbb{R}^3) : \exists\, \xi_\alpha \in H^2_\#(0,\ell), \exists\, \xi_3 \in H^1_\#(0,\ell) \\ \text{such that } v_\alpha(y) = \xi_\alpha(y_3),\ v_3(y) = \xi_3(y_3) - y_\alpha \xi'_\alpha(y_3)\}. \end{aligned} \tag{5}$$

In the remaining part of this section we assume u^ε to be a sequence of functions in $H^1_\#(\Omega;\mathbb{R}^3)$ such that

$$\|E^\varepsilon u^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \le C\varepsilon, \tag{6}$$

for some constant C and for every $\varepsilon \in (0,1]$.

Theorem 4.2. *Let* (6) *hold. Then, for any sequence of positive numbers* ε_n *converging to 0 there exist a subsequence (not relabelled) and a couple of functions* $v \in H_{BN}(\Omega;\mathbb{R}^3)$ *and* $\vartheta \in L^2(\Omega)$ *such that (as* $n \to +\infty$*)*

$$(u_1^{\varepsilon_n}, u_2^{\varepsilon_n}, \frac{u_3^{\varepsilon_n}}{\varepsilon_n}) \rightharpoonup v \text{ in } H^1(\Omega;\mathbb{R}^3), \tag{7}$$

$$W^{\varepsilon_n} u^{\varepsilon_n} \rightharpoonup \begin{pmatrix} 0 & -\vartheta & D_3 v_1 \\ \vartheta & 0 & D_3 v_2 \\ -D_3 v_1 & -D_3 v_2 & 0 \end{pmatrix} \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}). \tag{8}$$

Proof. It is convenient to set $v^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon/\varepsilon)$. Since $|E^\varepsilon u^\varepsilon| \geq \varepsilon |E v^\varepsilon|$, by (6), Ev^ε is uniformly bounded in $L^2(\Omega;\mathbb{R}^{3\times 3})$ and by Korn's inequality v^ε is uniformly bounded in $H^1(\Omega;\mathbb{R}^3)$. It then exists a $v \in H^1_\#(\Omega;\mathbb{R}^3)$ and a subsequence of ε_n such that $v^{\varepsilon_n} \rightharpoonup v$ in $H^1(\Omega;\mathbb{R}^3)$. Again, it is easy to check that $|(E^\varepsilon u^\varepsilon)_{i\alpha}| \geq |(Ev^\varepsilon)_{i\alpha}|$, thus, using (6) we deduce that $C\varepsilon \geq \|(Ev^\varepsilon)_{i\alpha}\|_{L^2(\Omega)}$ and consequently, as $n \to \infty$, $(Ev)_{i\alpha} = 0$ for $i = 1,2,3$ and $\alpha = 1,2$. Hence $v \in H_{BN}(\Omega;\mathbb{R}^3)$.

Using (6) and Theorem 4.1 we obtain that the sequence $H^{\varepsilon_n} u^{\varepsilon_n}$ is bounded in $L^2(\Omega;\mathbb{R}^{3\times 3})$ so that, up to subsequences, it weakly converges in $L^2(\Omega;\mathbb{R}^{3\times 3})$ to a matrix $H \in L^2(\Omega;\mathbb{R}^{3\times 3})$. Since, from (6), $E^{\varepsilon_n} u^{\varepsilon_n} \to 0$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$, we have $W^{\varepsilon_n} u^{\varepsilon_n} \rightharpoonup H$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$. In particular, H is, almost everywhere, a skew-symmetric matrix. Since $(H^\varepsilon u^\varepsilon)_{13} = u^\varepsilon_{1,3} = v^\varepsilon_{1,3}$ and $(H^\varepsilon u^\varepsilon)_{23} = u^\varepsilon_{2,3} = v^\varepsilon_{2,3}$, we deduce that $(H)_{13} = v_{1,3}$ and $(H)_{23} = v_{2,3}$. We conclude the proof by denoting $(H)_{12} := -\vartheta$. □

Let $\wp$ denote the projection of $L^2(\omega;\mathbb{R}^2)$ on the subspace

$$\mathcal{R}_2 = \{ r \in L^2(\omega;\mathbb{R}^2) : \exists\, \varphi \in \mathbb{R},\, c \in \mathbb{R}^2 : r_1(y) = -y_2\varphi + c_1,\, r_2(y) = y_1\varphi + c_2 \}$$

of the infinitesimal rigid displacements on ω. It is easy to see that $\mathcal{R}_2$ is a closed subspace of $H^1(\omega;\mathbb{R}^2)$ (see also Freddi, Morassi and Paroni[4]). Moreover, if $w \in L^2(\omega;\mathbb{R}^2)$ we have that

$$(\wp w)_\alpha = \mathcal{E}_{\beta\alpha} y_\beta \Big(\frac{1}{I_O} \int_\omega \mathcal{E}_{\gamma\delta} y_\gamma w_\delta \, dy_1 dy_2 \Big) + \frac{1}{|\omega|} \int_\omega w_\alpha \, dy_1 dy_2, \tag{9}$$

where $\mathcal{E}_{\alpha\beta}$ denote the Ricci's symbol. The two-dimensional Korn's inequality then writes as

$$\|w - \wp w\|^2_{H^1(\omega;\mathbb{R}^2)} \leq C \|Ew\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} \tag{10}$$

for all $w \in H^1(\omega;\mathbb{R}^2)$.

Lemma 4.1. *Under assumption* (6) *and the notation of Theorem 4.2 and of* (4) *we have*

1. $\|\vartheta^\varepsilon(u^\varepsilon)+(W^\varepsilon u^\varepsilon)_{12}\|_{L^2(\Omega)}\le C\varepsilon$ *for every* $\varepsilon\in(0,1]$*;*
2. $\vartheta^\varepsilon(u^\varepsilon)\rightharpoonup\vartheta$ *in* $L^2(\Omega)$*; therefore* ϑ *does not depend on* y_1 *and* y_2*;*
3. $\vartheta\in H^1_\#(\Omega)$.

Proof. It is convenient to set $w^\varepsilon:=(u^\varepsilon_1/\varepsilon,u^\varepsilon_2/\varepsilon,u^\varepsilon_3/\varepsilon^2)$. Then for almost $y_3\in(0,\ell)$ and any $\varepsilon\in(0,1]$ we consider the projection of the first two components of $w^\varepsilon(\cdot,y_3)$. From (9) and recalling (4) we have

$$(\wp w^\varepsilon)_\alpha=\mathcal{E}_{\beta\alpha}y_\beta\vartheta^\varepsilon(u^\varepsilon)+\frac{1}{|\omega|}\int_\omega w^\varepsilon_\alpha\,dy_1dy_2.$$

Since $(Ew^\varepsilon)_{11}=(E^\varepsilon u^\varepsilon)_{11}$, $(Ew^\varepsilon)_{12}=(E^\varepsilon u^\varepsilon)_{12}$ and $(Ew^\varepsilon)_{22}=(E^\varepsilon u^\varepsilon)_{22}$, we get $\|(Ew^\varepsilon)_{\alpha\beta}\|_{L^2(\Omega;\mathbb{R}^{2\times2})}=\|(E^\varepsilon u^\varepsilon)_{\alpha\beta}\|_{L^2(\Omega;\mathbb{R}^{2\times2})}$ for $\alpha,\beta=1,2$. Then, integrating (10) on $(0,\ell)$ and taking into account (6), we deduce that

$$\int_0^\ell\|w^\varepsilon-\wp w^\varepsilon\|_{H^1(\omega;\mathbb{R}^2)}dy_3\le C\|E^\varepsilon u^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times3})}\le C\varepsilon$$

and then $\|D_\alpha(w^\varepsilon_\beta-\wp w^\varepsilon_\beta)\|_{L^2(\Omega;\mathbb{R})}\to0$ for $\alpha,\beta=1,2$. Since $(W\wp w^\varepsilon)_{12}=-\vartheta^\varepsilon(u^\varepsilon)$ and $(Ww^\varepsilon)_{12}=(W^\varepsilon u^\varepsilon)_{12}$, we obtain, from the identity

$$\vartheta^\varepsilon(u^\varepsilon)=-(W\wp w^\varepsilon)_{12}=(W(w^\varepsilon-\wp w^\varepsilon))_{12}-(W^\varepsilon u^\varepsilon)_{12},$$

the first claim of the Lemma.

Using (8), for $\varepsilon\to0$, we obtain that $\vartheta^\varepsilon(u^\varepsilon)\rightharpoonup\vartheta$ in $L^2(\Omega)$. From the fact that $\vartheta^\varepsilon(u^\varepsilon)$ does not depend on y_1 and y_2, the same holds for ϑ.

Setting $w^\varepsilon:=(u^\varepsilon_1/\varepsilon,u^\varepsilon_2/\varepsilon,u^\varepsilon_3/\varepsilon^2)$, the proof of part *3* proceeds along the same lines of that of Lemma 4.6 of Freddi, Morassi and Paroni.[4] □

Lemma 4.2. *Under the same assumption and with the notation of Theorem 4.2 the following identities hold in* $L^2(\Omega)$

$$E_{33}=D_3v_3,\tag{11}$$

$$-D_2E_{13}+D_1E_{23}=D_3\vartheta,\tag{12}$$

where, up to subsequences, E_{33}*,* E_{13} *and* E_{23} *are, respectively, the limits of* $(E^\varepsilon u^\varepsilon)_{33}/\varepsilon$*,* $(E^\varepsilon u^\varepsilon)_{13}/\varepsilon$ *and* $(E^\varepsilon u^\varepsilon)_{23}/\varepsilon$ *in the weak convergence of* $L^2(\Omega)$.

Proof. To prove (11) it suffices to notice that $(E^\varepsilon u^\varepsilon)_{33}/\varepsilon=D_3(u^\varepsilon_3/\varepsilon)$ and apply (7). Let's prove (12). From (6) we deduce that, up to subsequences, $(E^\varepsilon u^\varepsilon)_{13}/\varepsilon\rightharpoonup E_{13}$ and $(E^\varepsilon u^\varepsilon)_{23}/\varepsilon\rightharpoonup E_{23}$ in $L^2(\Omega)$. To characterize $E_{13},E_{23}\in L^2(\Omega)$ note that

$$D_3(W^\varepsilon u^\varepsilon)_{12}=D_2\Big(\frac{(E^\varepsilon u^\varepsilon)_{13}}{\varepsilon}\Big)-D_1\Big(\frac{(E^\varepsilon u^\varepsilon)_{23}}{\varepsilon}\Big),$$

in the sense of distributions. Hence for $\psi \in C_c^\infty(\Omega)$ we obtain

$$\int_\Omega (W^\varepsilon u^\varepsilon)_{12} D_3\psi\, dy = \int_\Omega \frac{(E^\varepsilon u^\varepsilon)_{13}}{\varepsilon} D_2\psi\, dy - \int_\Omega \frac{(E^\varepsilon u^\varepsilon)_{23}}{\varepsilon} D_1\psi\, dy.$$

Passing to the limit in the previous equality we find

$$\int_\Omega -\vartheta D_3\psi\, dy = \int_\Omega E_{13} D_2\psi\, dy - \int_\Omega E_{23} D_1\psi\, dy.$$

Thus $D_3\vartheta = -D_2 E_{13} + D_1 E_{23}$ in the sense of distributions, hence in $L^2(\Omega)$ since $\vartheta \in H^1_\#(\Omega)$. □

5. The limit energy

Let us consider the usual De Saint Venant - Kirchhoff energy density

$$f(A) = \frac{1}{2}\mathbb{C}A \cdot A = \mu|A|^2 + \frac{\lambda}{2}|\mathrm{tr}A|^2$$

and define $f_0(\alpha,\beta) := \min\{f(A) : A \in \mathrm{Sym},\ A_{13}^2 + A_{23}^2 = \alpha^2,\ A_{33} = \beta\}$.

A simple computation shows that

$$f_0(\alpha,\beta) := 2\mu\alpha^2 + \frac{E}{2}\beta^2, \tag{13}$$

where $E = \mu(2\mu + 3\lambda)/(\mu+\lambda)$ is the Young modulus.

Lemma 5.1. *Let u^ε be a sequence of functions in the space $H^1_\#(\Omega;\mathbb{R}^3)$. If $\sup_\varepsilon \big(F_\varepsilon(u^\varepsilon)/\varepsilon^2\big) < +\infty$, then (6) holds for some constant $C > 0$.*

Proof. Setting $v^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon/\varepsilon)$, the proof proceeds exactly along the same lines of that of Lemma 5.1 of Freddi, Morassi and Paroni.[4] □

The above Lemma 5.1 and Lemma 4.2 imply that the family of functionals $(1/\varepsilon^2)F_\varepsilon$ is coercive with respect to the weak convergence of the sequence $q_\varepsilon(u^\varepsilon) := \Big(u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon/\varepsilon, (W^\varepsilon u^\varepsilon)_{12}\Big)$ in the space $H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R})$, uniformly with respect to ε. Hence, for any sequence u^ε which is bounded in energy, that is $(1/\varepsilon^2)F_\varepsilon \le C$ for a suitable constant $C > 0$, and satisfies the boundary conditions, that is $u^\varepsilon = 0$ on $S(0)$, the corresponding sequence $q_\varepsilon(u^\varepsilon)$ is weakly relatively compact in $H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R})$.

Now we introduce some auxiliary functions defined on ω. The so-called *Prandtl stress function* is defined as the unique solution ψ of

$$\begin{cases} \triangle\psi = -2 \\ \psi \in H^1_0(\omega). \end{cases} \tag{14}$$

Since ω is simply connected, then it remains defined, up to constants, the *torsion function* φ defined by

$$\begin{cases} D_1\psi = -D_2\varphi - y_1 \\ D_2\psi = D_1\varphi - y_2. \end{cases} \tag{15}$$

It's easy to see that

$$\begin{cases} \triangle\varphi = 0 \quad \text{in } \omega \\ D\varphi \cdot n = -y_1 n_2 + y_2 n_1 \quad \text{on } \partial\omega, \end{cases} \tag{16}$$

where $n = n(\sigma)$ is the normal unit vector to $\partial\omega$ at the point σ.

Theorem 5.1. *Let ψ be the Prandtl stress function defined above and let $F : H^1_\#(\Omega;\mathbb{R}^3) \times H^1_\#(\Omega;\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ be defined by*

$$F(v,\vartheta) := \int_\Omega f_0\Big(\frac{|D\psi|\, D_3\vartheta}{2}, D_3 v_3\Big)\, dy - \int_\Omega b\cdot v\, dy - \int_0^\ell m\vartheta\, dy_3 \tag{17}$$

if $v \in H_{BN}(\Omega;\mathbb{R}^3)$, and $+\infty$ otherwise. As $\varepsilon \to 0$, the sequence of functionals $(1/\varepsilon^2)F_\varepsilon$ defined in (3) *and* (4) *Γ-converges to the functional F, in the following sense:*

(1) (liminf inequality) for every sequence of positive numbers ε_k converging to 0 and for every sequence $\{u^k\} \subset H^1_\#(\Omega;\mathbb{R}^3)$ such that

$$(u_1^k, u_2^k, \frac{u_3^k}{\varepsilon_k}) \rightharpoonup v \text{ in } H^1(\Omega;\mathbb{R}^3), \qquad (W^{\varepsilon_k}u^k)_{12} \rightharpoonup -\vartheta \text{ in } L^2(\Omega),$$

we have

$$\liminf_{k\to+\infty} \frac{F_{\varepsilon_k}(u^k)}{\varepsilon_k^2} \geq F(v,\vartheta);$$

(2) (recovery sequence) for every sequence of positive numbers ε_k converging to 0 and for every $(v,\vartheta) \in H^1_\#(\Omega;\mathbb{R}^3) \times H^1_\#(\Omega;\mathbb{R})$ there exists a sequence $\{u^k\} \subset H^1_\#(\Omega;\mathbb{R}^3)$ such that

$$(u_1^k, u_2^k, \frac{u_3^k}{\varepsilon_k}) \rightharpoonup v \text{ in } H^1(\Omega;\mathbb{R}^3), \qquad (W^{\varepsilon_k}u^k)_{12} \rightharpoonup -\vartheta \text{ in } L^2(\Omega),$$

and

$$\limsup_{k\to+\infty} \frac{F_{\varepsilon_k}(u^k)}{\varepsilon_k^2} \leq F(v,\vartheta).$$

Proof. Let us prove the liminf inequality. Without loss of generality we may suppose that

$$\liminf_{k\to+\infty} \frac{F_{\varepsilon_k}(u^k)}{\varepsilon_k^2} = \lim_{k\to+\infty} \frac{F_{\varepsilon_k}(u^k)}{\varepsilon_k^2} < +\infty.$$

Then Lemma 5.1 applies to the sequence $(1/\varepsilon_k^2)F_{\varepsilon_k}(u^k)$. Hence assumption (6) is fulfilled and the results of Section 4, namely Lemma 4.1 and Lemma 4.2, hold true.

Looking at the expressions (3) and (4) of the functional F_ε, and setting $L_\varepsilon := F_\varepsilon - I_\varepsilon$ the work done by loads, using Lemma 4.1 and the convergence assumptions on the sequence (u^k) we can see that

$$\frac{L_{\varepsilon_k}(u^k)}{\varepsilon_k^2} = \int_\Omega b\cdot(u_1^k, u_2^k, \frac{u_3^k}{\varepsilon_k})\, dy + \int_0^\ell m\vartheta^{\varepsilon_k}(u^k)\, dy_3 \to \int_\Omega b\cdot v\, dy + \int_0^\ell m\vartheta\, dy_3.$$

Thus we have only to prove that

$$\liminf_{k\to+\infty} \frac{I_{\varepsilon_k}(u^k)}{\varepsilon_k^2} \geq \int_\Omega f_0\big(\frac{|D\psi|\, D_3\vartheta}{2}, D_3v_3\big)\, dy. \tag{18}$$

By definition of f and f_0 and using (13), we observe that

$$\frac{1}{2}\mathbb{C}A\cdot A \geq 2\mu(A_{13}^2 + A_{23}^2) + \frac{E}{2}A_{33}^2.$$

Then we get

$$\frac{I_{\varepsilon_k}(u^k)}{\varepsilon_k^2} \geq 2\mu \int_\Omega \big(\frac{(E^{\varepsilon_k}u^k)_{13}^2}{\varepsilon_k^2} + \frac{(E^{\varepsilon_k}u^k)_{23}^2}{\varepsilon_k^2}\big)\, dy + \frac{E}{2}\int_\Omega \frac{|(E^{\varepsilon_k}u^k)_{33}|^2}{\varepsilon_k^2}\, dy.$$

Using Lemma 4.1 and Lemma 4.2 then we have

$$\liminf_{k\to+\infty} \frac{I_{\varepsilon_k}(u^k)}{\varepsilon_k^2} \geq 2\mu \int_\Omega (E_{13}^2 + E_{23}^2)\, dy + \frac{E}{2}\int_\Omega |D_3v_3|^2\, dy. \tag{19}$$

From equation (12), i.e. $-D_2E_{13} + D_1E_{23} = D_3\vartheta$, which we can rewrite as

$$D_2\big(E_{13} + \frac{y_2}{2}D_3\vartheta\big) = D_1\big(E_{23} - \frac{y_1}{2}D_3\vartheta\big) \ \text{ in } \mathcal{D}'(\Omega),$$

and the weak version of Poincaré's Lemma (see Girault and Raviart,[7] Theorem 2.9) we can find a function $\widehat{\varphi} \in L^2((0,\ell); H_m^1(\omega))$ such that

$$\begin{cases} E_{13} = D_1\widehat{\varphi} - \dfrac{y_2}{2}D_3\vartheta \\ E_{23} = D_2\widehat{\varphi} + \dfrac{y_1}{2}D_3\vartheta, \end{cases}$$

where $H_m^1(\omega) := \{v \in H^1(\omega) : \fint_\omega v = 0\}$. Thus

$$\int_\Omega (E_{13}^2 + E_{23}^2)\, dy \geq \inf_{\widehat{\varphi}} \int_\Omega |D_1\widehat{\varphi} - \frac{y_2}{2}D_3\vartheta|^2 + |D_2\widehat{\varphi} + \frac{y_1}{2}D_3\vartheta|^2\, dy, \tag{20}$$

where the infimum is taken over all functions $\widehat{\varphi}$ in $L^2((0,\ell);H^1_m(\omega))$. Furthermore, we now show that the infimum is achieved and we characterize a minimizer $\widehat{\varphi}$. First, by using Green's identities and the fact that ϑ depends only on y_3, we have

$$\int_\Omega \big(|D_1\widehat{\varphi} - \frac{y_2}{2}D_3\vartheta|^2 + |D_2\widehat{\varphi} + \frac{y_1}{2}D_3\vartheta|^2\big)\,dy =$$
$$= \int_\Omega \big[|D_\alpha\widehat{\varphi}|^2 + \frac{1}{4}(y_2^2+y_1^2)|D_3\vartheta|^2 + \operatorname{div}(-y_2\widehat{\varphi}, y_1\widehat{\varphi})D_3\vartheta\big]\,dy$$
$$= \int_\Omega |D_\alpha\widehat{\varphi}|^2\,dy + \frac{I_O}{4}\int_0^\ell |D_3\vartheta|^2\,dy_3 + \int_0^\ell D_3\vartheta \int_{\partial\omega}(-y_2n_1+y_1n_2)\widehat{\varphi}\,ds\,dy_3$$

where D_α denotes the gradient with respect to y_1, y_2 and $n=(n_1,n_2)$ is the normal unit vector to $\partial\omega$. Let us define

$$E(\widehat{\varphi}) := \int_\Omega |D_\alpha\widehat{\varphi}|^2\,dy + \int_0^\ell D_3\vartheta\int_{\partial\omega}(-y_2n_1+y_1n_2)\widehat{\varphi}\,ds\,dy_3.$$

The existence of a minimizer of $E(\widehat{\varphi})$ in the Hilbert space $L^2((0,\ell);H^1_m(\omega))$ follows from a standard application of the direct method of the Calculus of Variations. Let now $\widehat{\varphi}$ be a minimizer. Then it follows, by taking appropriate variations, that

$$\begin{cases} \operatorname{div}D_\alpha\widehat{\varphi} = \triangle\widehat{\varphi} = 0 & \text{in } \omega \\ D_\alpha\widehat{\varphi}\cdot n = \dfrac{D_3\vartheta}{2}(-y_1n_2+y_2n_1) & \text{on } \partial\omega \end{cases}$$

for almost every $y_3\in(0,\ell)$. We note that $D_\alpha\widehat{\varphi}$ depends linearly by $D_3\vartheta$ on $\partial\omega$. If $\varphi\in H^1_m(\omega)$ is the solution of (16) having zero mean value, then

$$\widehat{\varphi} = \frac{1}{2}\varphi D_3\vartheta. \tag{21}$$

By putting together (15), (19), (20) and (21) we obtain the liminf inequality

$$\liminf_{k\to+\infty}\frac{I_{\varepsilon_k}(u^k)}{\varepsilon_k^2} \ge \int_\Omega \big(\frac{E}{2}|D_3v_3|^2 + \frac{\mu}{2}|D\psi|^2|D_3\vartheta|^2\big)\,dy$$

that is (18).

Let us now find a recovery sequence. Let $F(v,\vartheta)<+\infty$, otherwise there is nothing to prove. Then $v\in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta\in H^1_\#(\Omega;\mathbb{R})$.

We first assume further that v and ϑ are smooth and equal to zero near by $y_3=0$. By (5) there exists ξ smooth and equal to zero near by $y_3=0$ such that $v_\alpha(y)=\xi_\alpha(y_3)$, and $v_3(y)=\xi_3(y_3)-y_\alpha\xi'_\alpha(y_3)$. Let $u^{0,\varepsilon}$ be the sequence defined by

$$
\begin{aligned}
u_1^{0,\varepsilon} &= \xi_1 - \varepsilon y_2\vartheta + \varepsilon^2\frac{\nu}{2}\big(- y_2^2\xi_1'' + y_1^2\xi_1'' + 2y_1y_2\xi_2''\big) - \varepsilon^2\nu y_1\xi_3' \\
u_2^{0,\varepsilon} &= \xi_2 + \varepsilon y_1\vartheta + \varepsilon^2\frac{\nu}{2}\big(- y_1^2\xi_2'' + y_2^2\xi_2'' + 2y_1y_2\xi_1''\big) - \varepsilon^2\nu y_2\xi_3' \\
u_3^{0,\varepsilon} &= \varepsilon(\xi_3 - y_1\xi_1' - y_2\xi_2') + \varepsilon^2\varphi D_3\vartheta
\end{aligned}
$$

where $\nu = \lambda/2(\lambda+\mu)$ is the Poisson's coefficient, and φ is the torsion function with zero mean value. We have that $u^{0,\varepsilon}$ is equal to zero in $y_3 = 0$ and it is easily checked that, as $\varepsilon \to 0$,

$$
\begin{aligned}
&\frac{(E^\varepsilon u^{0,\varepsilon})_{11}}{\varepsilon} \to -\nu D_3v_3, && \frac{(E^\varepsilon u^{0,\varepsilon})_{22}}{\varepsilon} \to -\nu D_3v_3 \\
&\frac{(E^\varepsilon u^{0,\varepsilon})_{33}}{\varepsilon} \to D_3v_3, && \frac{(E^\varepsilon u^{0,\varepsilon})_{12}}{\varepsilon} \to 0 \\
&\frac{(E^\varepsilon u^{0,\varepsilon})_{13}}{\varepsilon} \to \frac{1}{2}(D_1\varphi - y_2)D_3\vartheta, && \frac{(E^\varepsilon u^{0,\varepsilon})_{23}}{\varepsilon} \to \frac{1}{2}(D_2\varphi + y_1)D_3\vartheta,
\end{aligned}
$$

and $(W^\varepsilon u^{0,\varepsilon})_{12} \to -\vartheta$, in $L^2(\Omega)$. Therefore, performing computations, we obtain that

$$
\frac{I_\varepsilon(u^{0,\varepsilon})}{\varepsilon^2} \to \int_\Omega \big(\frac{E}{2}|D_3v_3|^2 + \frac{\mu}{2}|D\psi|^2|D_3\vartheta|^2\big)\,dy.
$$

It is also easy to check that the following estimates are satisfied

$$
\begin{aligned}
&\Big|\frac{1}{\varepsilon^2}F_\varepsilon(u^{0,\varepsilon}) - F(v,\vartheta)\Big| \le \varepsilon C(v,\vartheta), \\
&\|(W^\varepsilon u^{0,\varepsilon})_{12} + \vartheta\|_{L^2(\Omega)} \le \varepsilon C(v,\vartheta), \\
&\Big\|\Big(u_1^{0,\varepsilon}, u_2^{0,\varepsilon}, \frac{u_3^{0,\varepsilon}}{\varepsilon}\Big) - v\Big\|_{H^1(\Omega)} \le \varepsilon C(v,\vartheta),
\end{aligned}
$$

where $C(v,\vartheta)$ depends only on v and ϑ. Hence, in this case, (u^{0,ε_k}) is a recovery sequence. In the general case, i.e. $v \in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta \in H^1_{\#}(\Omega;\mathbb{R})$, a standard diagonal argument concludes the proof. □

6. Convergence of minima and minimizers

For every $\varepsilon \in (0,1]$ let us denote by $\tilde{u}^\varepsilon$ the solution of the following minimization problem

$$
\min\big\{F_\varepsilon(u) : u \in H^1(\Omega;\mathbb{R}^3),\ u = 0 \text{ on } S(0)\big\}.
$$

The existence of the solution can be proved by the direct method of the Calculus of Variations and the uniqueness follows by the strict convexity of the functional F_ε.

Corollary 6.1. *The following minimization problem for the* Γ*-limit functional* F *defined in* (17)

$$\min\left\{F(v,\vartheta) : v \in H_{BN}(\Omega;\mathbb{R}^3),\ \vartheta \in H^1(0,\ell),\ v = 0 \text{ on } S_\varepsilon(0),\ \vartheta(0) = 0\right\}$$

admits a unique solution $(\tilde{v}, \tilde{\vartheta})$*. Moreover, as* $\varepsilon \to 0$*,*

1. $(\tilde{u}_1^\varepsilon, \tilde{u}_2^\varepsilon, \tilde{u}_3^\varepsilon/\varepsilon) \rightharpoonup \tilde{v}$ *in* $H^1(\Omega;\mathbb{R}^3)$*;*
2. $(W^\varepsilon \tilde{u}^\varepsilon)_{12} \rightharpoonup -\tilde{\vartheta}$ *in* $L^2(\Omega)$*;*
3. $(1/\varepsilon^2)F_\varepsilon(\tilde{u}^\varepsilon)$ *converges to* $F(\tilde{v}, \tilde{\vartheta})$*.*

Proof. Property *3* and the weak convergence in *1* and *2* follow from the Γ-convergence Theorem 5.1, the uniform coercivity of the sequence $(1/\varepsilon^2)F_\varepsilon$ and the variational property of Γ-convergence (see for instance Dal Maso[2] or Freddi and Paroni,[6] Proposition 3.4). □

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