Torsione alla de Saint Venant di travi a parete sottile

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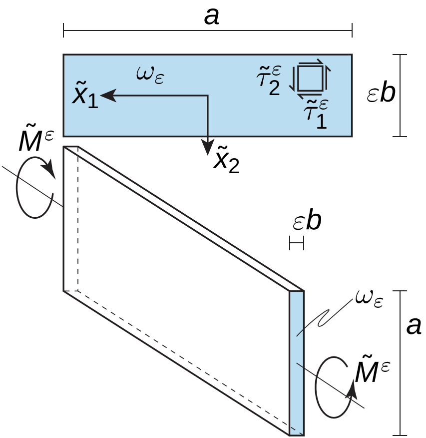
Astrazione e realizzazione

temi di carattere interdisciplinare tra fisica, matematica e ingegneria
Udine, 21 Giugno 2007



μ: shear modulus
α: angle of twist per unit length

Differential formulation

$\tilde{\psi}^\varepsilon$ stress function

$$(*) \quad \begin{cases} \triangle \tilde{\psi}^\varepsilon & = -2 \text{ on } \omega_\varepsilon \\ \tilde{\psi}^\varepsilon & = 0 \text{ on } \partial\omega_\varepsilon \end{cases}$$

Twisting moment

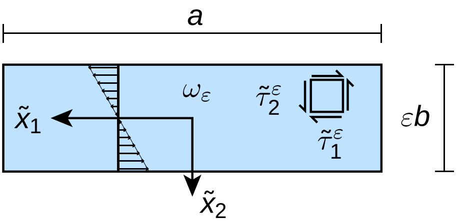
$\tilde{\psi}^\varepsilon_m$: solution of $(*)$

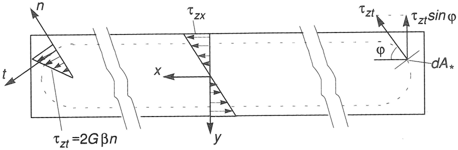
$$\tilde{\boldsymbol{\tau}}^\varepsilon = \mu\,\alpha\,\boldsymbol{e}_3 \times \nabla\tilde{\psi}^\varepsilon_m$$

$$\tilde{M}^\varepsilon = 2\,\mu\,\alpha \int_{\tilde{\omega}_\varepsilon} \tilde{\psi}^\varepsilon d\tilde{a}$$

$$\tilde{M}^\varepsilon = \int_{\tilde{\omega}_\varepsilon} \tilde{\boldsymbol{x}} \times \tilde{\boldsymbol{\tau}}^\varepsilon d\tilde{a}$$

Approximate solution for thin structures





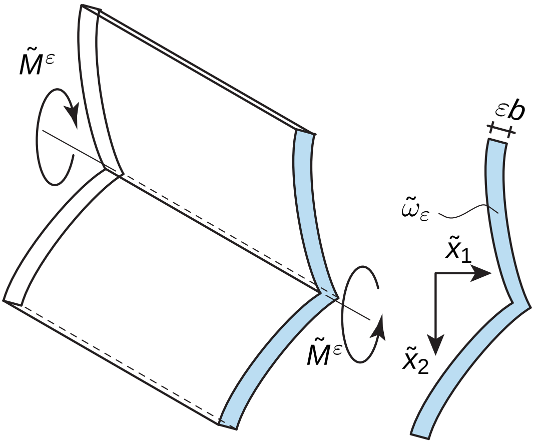
$$\tilde{\psi}_m^\varepsilon \simeq \frac{(\varepsilon b)^2}{4} - \tilde{x}_2^2$$

$$\tilde{\tau}_1^\varepsilon = \mu\alpha \frac{\partial \tilde{\psi}_m^\varepsilon}{\partial \tilde{x}_2} = -2\,\mu\,\alpha\,\tilde{x}_2$$

$$\tilde{M}^\varepsilon = 2\,\mu\,\alpha \int_{\tilde{\omega}_\varepsilon} \tilde{\psi}^\varepsilon d\tilde{a} = \tfrac{1}{3}\,\mu\,\alpha\,a\,(\varepsilon b)^3$$

$$\tilde{M}^\varepsilon \neq -\int_{\tilde{\omega}_\varepsilon} \tilde{x}_2\,\tilde{\tau}_1^\varepsilon\,d\tilde{a} = \tfrac{1}{6}\,\mu\,\alpha\,a\,(\varepsilon b)^3$$

Objectives and method



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Variational formulation

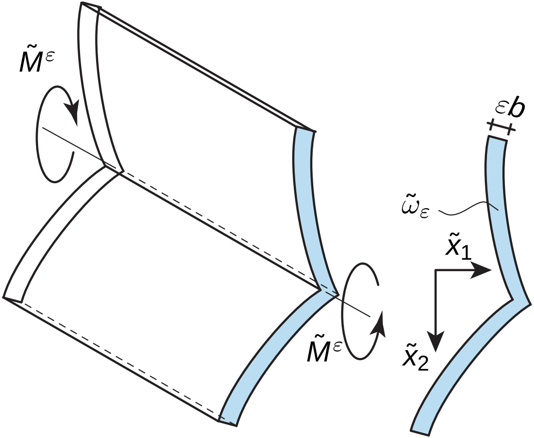
$\tilde{\psi}^\varepsilon_m$ solution of $(*)$

$$\tilde{\mathcal{F}}^\varepsilon\left(\tilde{\psi}^\varepsilon_m\right) = \min_{\tilde{\eta}\in H^1_0(\tilde{\omega}_\varepsilon)} \tilde{\mathcal{F}}^\varepsilon(\tilde{\eta})$$

$$= \min_{\tilde{\eta}\in H^1_0(\tilde{\omega}_\varepsilon)} \int_{\tilde{\omega}_\varepsilon} (\nabla\tilde{\eta})^2 - 4\tilde{\eta}\, d\tilde{a}$$

Objective: solution in terms of stresses and torsional stiffness as $\varepsilon \to 0$

Method: Γ-convergence based approach.



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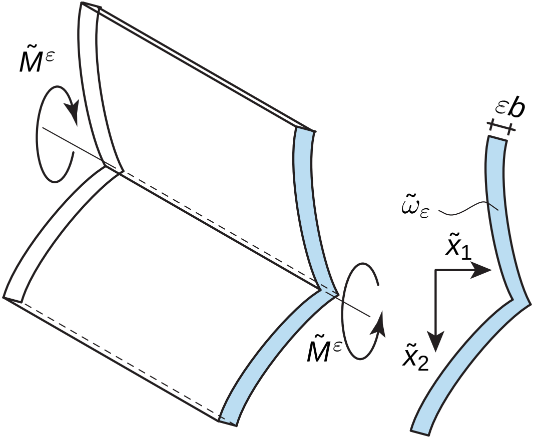

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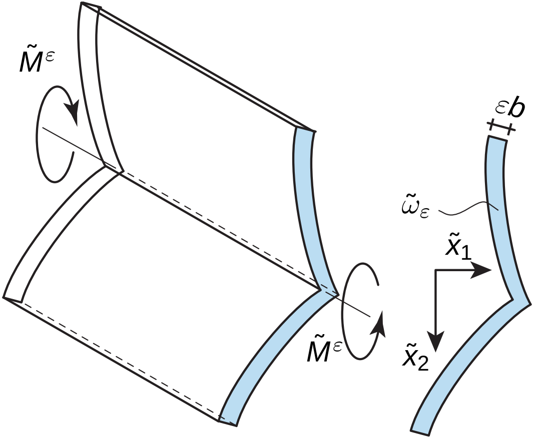

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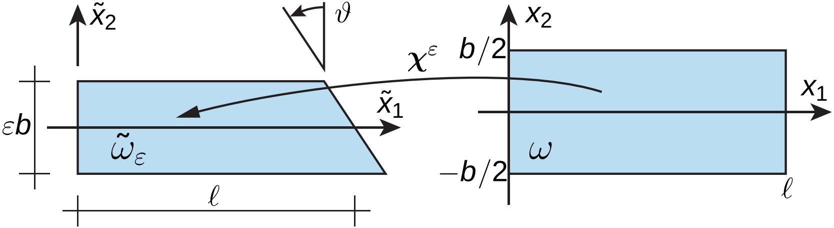
The trapezoidal domain



Coordinate transformation

$$\chi^\varepsilon(\boldsymbol{x}) = \left(x_1 - \varepsilon\frac{x_1 x_2}{\rho}, \varepsilon x_2\right) \quad ; \quad \frac{1}{\rho} := \frac{\tan\vartheta}{\ell}$$

Covariant and contravariant bases

$$\boldsymbol{g}_1^\varepsilon = \chi^\varepsilon{}_{,1} = \left(1 - \varepsilon\frac{x_2}{\rho}, 0\right) \quad \boldsymbol{g}_2^\varepsilon = \chi^\varepsilon{}_{,2} = \varepsilon\left(-\frac{x_1}{\rho}, 1\right)$$

$$\boldsymbol{g}_\varepsilon^1 = \frac{1}{1-\varepsilon x_2/\rho}\left(1, \frac{x_1}{\rho}\right) \quad \boldsymbol{g}_\varepsilon^2 = \left(0, \frac{1}{\varepsilon}\right)$$

$$g^{11} = \frac{1}{(1-\varepsilon x_2/\rho)^2}\left(1 + \left(\frac{x_1}{\rho}\right)^2\right) \quad g^{12} = \frac{x_1/\rho}{\varepsilon(1-\varepsilon x_2/\rho)} \quad g^{22} = \frac{1}{\varepsilon^2}$$

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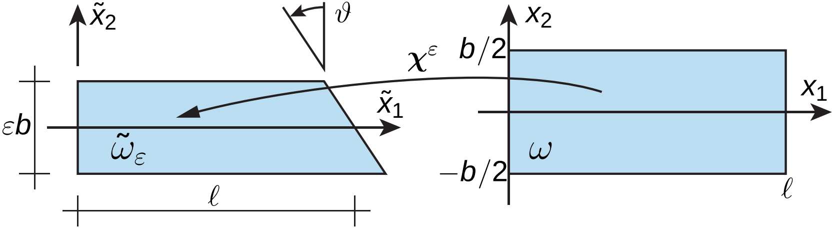

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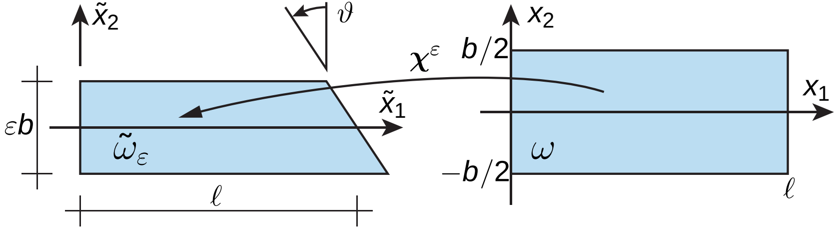

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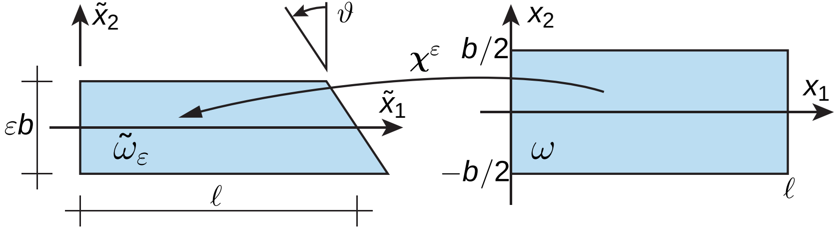

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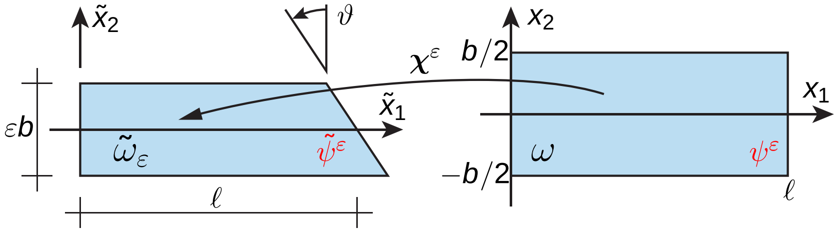
Definition of the objective functional



$\tilde{\psi}^\varepsilon : \tilde{\omega}_\varepsilon \to \mathbb{R}$ $\quad$ $\psi^\varepsilon : \omega \to \mathbb{R}$ $\quad$ $\psi^\varepsilon = \tilde{\psi}^\varepsilon \circ \boldsymbol{\chi}^\varepsilon$

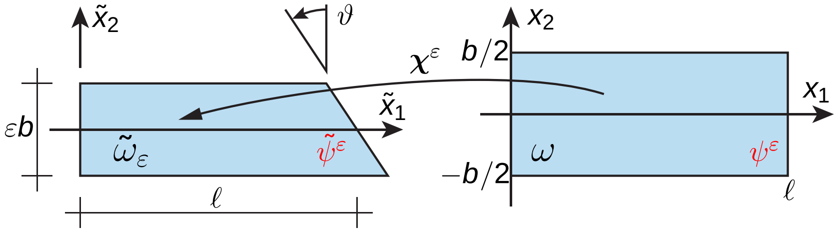
$$\psi^\varepsilon{}_{,\alpha} = \frac{\partial \psi^\varepsilon}{\partial x_\alpha} = \frac{\partial \tilde{\psi}^\varepsilon}{\partial \tilde{x}_\beta} \circ \boldsymbol{\chi}^\varepsilon \frac{\partial \chi^\varepsilon{}_\beta}{\partial x_\alpha} = \nabla \tilde{\psi}^\varepsilon \circ \boldsymbol{\chi}^\varepsilon \cdot \boldsymbol{g}^\varepsilon_\alpha$$

$$\left(\nabla \tilde{\psi}^\varepsilon\right) \circ \boldsymbol{\chi}^\varepsilon = \left(\nabla \tilde{\psi}^\varepsilon \circ \boldsymbol{\chi}^\varepsilon\right) \cdot \boldsymbol{g}^\varepsilon_\alpha \, \boldsymbol{g}^\alpha_\varepsilon = \psi^\varepsilon{}_{,\alpha} \, \boldsymbol{g}^\alpha_\varepsilon$$

$$\tilde{\mathcal{F}}^\varepsilon\left(\tilde{\psi}^\varepsilon\right) = \int_{\tilde{\omega}_\varepsilon} \left|\nabla \tilde{\psi}^\varepsilon\right|^2 - 4\tilde{\psi}^\varepsilon \, d\tilde{a}$$

$$\mathcal{F}^\varepsilon\left(\psi^\varepsilon\right) := \int_\omega \left(\left|\psi^\varepsilon{}_{,\alpha} \, \boldsymbol{g}^\alpha_\varepsilon\right|^2 - 4\psi^\varepsilon\right) \sqrt{g_\varepsilon} da$$

$$\sqrt{g_\varepsilon} = \varepsilon \left|1 - \varepsilon \frac{x_2}{\rho}\right|$$



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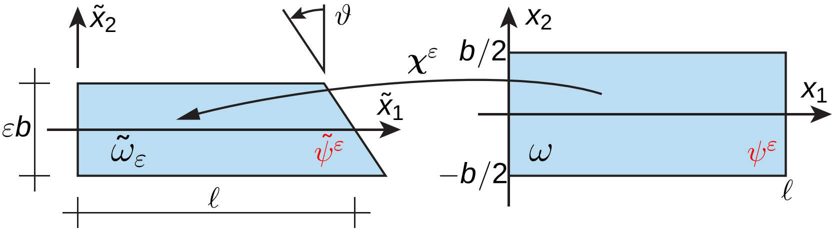

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Lemmas

Lemma (Coercivity)

$$\exists\, c > 0 \text{ s.t. } \int_\omega |\psi^\varepsilon{}_{,\alpha} \boldsymbol{g}^\alpha_\varepsilon|^2 \, da \geq c \int_\omega \left(\psi^\varepsilon{}_{,1}{}^2 + \left(\frac{\psi^\varepsilon{}_{,2}}{\varepsilon} \right)^2 \right) da$$

Proof

Lemma (Boundedness)

$$W \equiv L^2\left((0,\ell); H^1_0\left(-\tfrac{b}{2}, \tfrac{b}{2}\right)\right), \ \{\psi^\varepsilon\} \subset H^1_0(\omega) \text{ s.t. } \sup_\varepsilon \frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} < +\infty$$

$$\Rightarrow \sup_\varepsilon \left\| \frac{\psi^\varepsilon{}_{,1}}{\varepsilon} \right\|_{L^2(\omega)} < +\infty \ , \ \sup_\varepsilon \left\| \frac{\psi^\varepsilon}{\varepsilon^2} \right\|_W < +\infty$$

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Lemma (Compactness)

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$\exists\, \psi \in W$ *and a subsequence of* $\{\psi^\varepsilon\}$*, not re-labeled, s.t.*

$$\frac{\psi^\varepsilon{}_{,1}}{\varepsilon} \xrightarrow{L^2(\omega)} 0 \quad , \quad \frac{\psi^\varepsilon}{\varepsilon^2} \xrightarrow{W} \psi$$

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▸ Proof

The limit functional

$$W \equiv L^2\left((0,\ell);H_0^1\left(-\frac{b}{2},\frac{b}{2}\right)\right)$$

$$\mathcal{F}^0 : W \to \mathbb{R}; \qquad \mathcal{F}^0(\psi) = \int_\omega (\psi_{,2})^2 - 4\psi\, da$$

Theorem (Γ– convergence)

Lim inf $\frac{\psi^\varepsilon{}_{,1}}{\varepsilon} \rightharpoonup 0$ *in* $L^2(\omega)$ *and* $\frac{\psi^\varepsilon}{\varepsilon^2} \rightharpoonup \psi$ *in* W

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Proof

Recovery $\forall\, \psi \in W, \exists\, \psi^\varepsilon \in H_0^1(\omega)$ *s. t.*

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Theorem (Γ– convergence)

Lim inf $\frac{\psi^\varepsilon{}_{,1}}{\varepsilon} \rightharpoonup 0 \text{ in } L^2(\omega)$ *and* $\frac{\psi^\varepsilon}{\varepsilon^2} \rightharpoonup \psi \text{ in } W$

$$\Rightarrow \liminf_{\varepsilon\longrightarrow 0} \frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} \geq \mathcal{F}^0(\psi)$$

▸ Proof

Recovery $\forall\, \psi \in W,\ \exists\, \psi^\varepsilon \in H_0^1(\omega)$ s. t.

- $\frac{\psi^\varepsilon{}_{,1}}{\varepsilon} \rightharpoonup 0 \text{ in } L^2(\omega)$
- $\frac{\psi^\varepsilon}{\varepsilon^2} \rightharpoonup \psi \text{ in } W$
- $\limsup_{\varepsilon\longrightarrow 0} \frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} \leq \mathcal{F}^0(\psi)$

▸ Proof

Solution of the limit functional

$$W \equiv L^2\left((0,\ell)\,; H_0^1\left(-\frac{b}{2},\frac{b}{2}\right)\right)$$

$$\mathcal{F}^\varepsilon\left(\psi_m^\varepsilon\right) = \min_{\eta\in H_0^1(\omega)} \mathcal{F}^\varepsilon(\eta)$$

$$\mathcal{F}^0\left(\psi_m\right) = \min_{\eta\in W} \mathcal{F}^0(\eta)$$

Property of the minimizers

Γ–convergence theorem $\Rightarrow$ $\dfrac{\psi_m^\varepsilon}{\varepsilon^2} \xrightarrow{W} \psi_m$

Computing ψ_m

$$\lim_{s\to 0}\frac{\mathcal{F}^0(\psi_m+s\,\eta)-\mathcal{F}^0(\psi_m)}{s} = 0 \Rightarrow \int_\omega 2\,\psi_{m,2}\,\eta_{,2} - 4\,\eta\,da = 0 \quad \forall\eta\in W$$

$$\begin{cases}\psi_{m,22} = -2 \\ \psi_m\left(\cdot,-\frac{b}{2}\right) = 0 \quad \psi_m\left(\cdot,\frac{b}{2}\right) = 0\end{cases}$$

$$\psi_m = -\left(x_2^2 - \frac{b^2}{4}\right)$$

Solution of the limit functional

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Property of the minimizers

$$\Gamma\text{–convergence theorem} \Rightarrow \qquad \frac{\psi_m^\varepsilon}{\varepsilon^2} \overset{W}{\rightharpoonup} \psi_m$$

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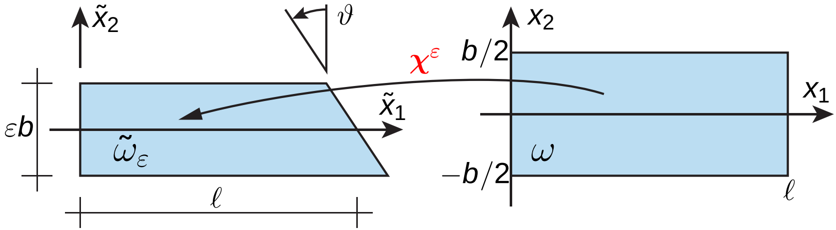
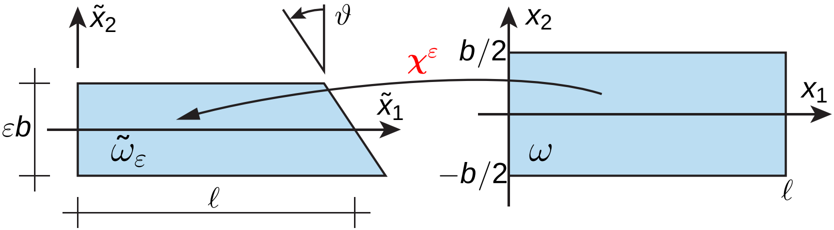


$$\tilde{\boldsymbol{\tau}}^\varepsilon = \mu\,\alpha\,\boldsymbol{e}_3 \times \nabla\left(\frac{\tilde{\psi}_m^\varepsilon}{\varepsilon^2}\right)$$

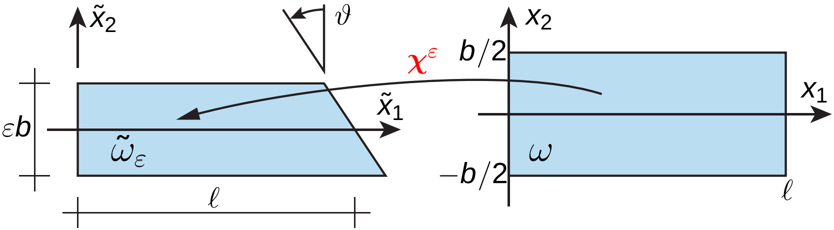
$\boldsymbol{\tau}^\varepsilon$?



$$\tilde{\boldsymbol{\tau}}^\varepsilon = \mu\,\alpha\,\boldsymbol{e}_3 \times \nabla\left(\frac{\tilde{\psi}_m^\varepsilon}{\varepsilon^2}\right) \qquad \boldsymbol{\tau}^\varepsilon?$$

$$\int_{\tilde{\omega}_\varepsilon} \tilde{\boldsymbol{\tau}}^\varepsilon \cdot \nabla\tilde{\eta}\, d\tilde{a} = \int_\omega \boldsymbol{\tau}^\varepsilon \cdot \nabla\eta\, da$$

$$\tilde{\eta} = \eta \circ \boldsymbol{\chi}^{\varepsilon^{-1}} = \boldsymbol{\chi}_\sharp^\varepsilon\,\eta \qquad \eta \in H^1(\omega)$$



$$\tilde{\boldsymbol{\tau}}^\varepsilon = \mu\,\alpha\,\boldsymbol{e}_3 \times \nabla\left(\frac{\tilde{\psi}^\varepsilon_m}{\varepsilon^2}\right)$$

$\boldsymbol{\tau}^\varepsilon$?

$$\left\langle \tilde{\boldsymbol{\tau}}^\varepsilon, \nabla\left(\boldsymbol{\chi}^\varepsilon_\sharp\,\eta\right)\right\rangle = \left\langle \boldsymbol{\chi}^\sharp_\varepsilon\,\tilde{\boldsymbol{\tau}}^\varepsilon, \nabla\eta\right\rangle$$

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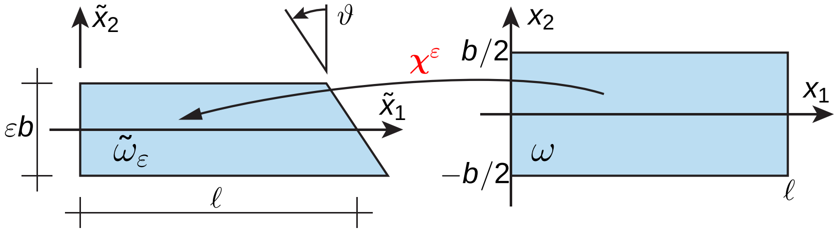


$$\tilde{\boldsymbol{\tau}}^\varepsilon = \mu\,\alpha\,\boldsymbol{e}_3 \times \nabla\left(\frac{\tilde{\psi}^\varepsilon_m}{\varepsilon^2}\right) \qquad \boldsymbol{\tau}^\varepsilon ?$$

$$\left\langle \tilde{\boldsymbol{\tau}}^\varepsilon, \nabla\left(\chi^\varepsilon_\sharp\,\eta\right)\right\rangle = \left\langle \chi^\sharp_\varepsilon\,\tilde{\boldsymbol{\tau}}^\varepsilon, \nabla\eta\right\rangle$$

$$\tilde{\eta} = \eta \circ \chi^{\varepsilon^{-1}} = \chi^\varepsilon_\sharp\,\eta \qquad \eta \in H^1(\omega)$$

$$\boldsymbol{\tau}^\varepsilon = \chi^\sharp_\varepsilon\,\tilde{\boldsymbol{\tau}}^\varepsilon = \sqrt{g_\varepsilon}\left(\nabla\chi^{\varepsilon^{-1}}\,\tilde{\boldsymbol{\tau}}^\varepsilon\right)\circ\chi^\varepsilon = \mu\,\alpha\left(-\frac{\psi^\varepsilon_{m,2}}{\varepsilon^2}\boldsymbol{e}_1 + \frac{\psi^\varepsilon_{m,1}}{\varepsilon^2}\boldsymbol{e}_2\right)$$

Limit tractions on rescaled domain

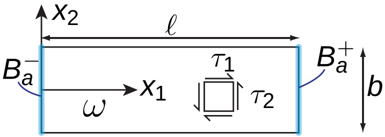
Tractions parallel to mean line

$$\tau_1^\varepsilon = \mu\,\alpha\,\frac{\psi^\varepsilon_{m,2}}{\varepsilon^2} \xrightharpoonup{L^2(\mathbb{R}^2)} \mu\alpha\,\psi_{m,2} := \tau_1\ ; \qquad \tau_1 = -2\mu\alpha\,x_2 \text{ in } \omega$$

Tractions normal to mean line

$$\tau_2^\varepsilon = -\mu\,\alpha\,\frac{\psi^\varepsilon_{m,1}}{\varepsilon^2} \rightharpoonup -\mu\alpha\,\psi_{m,1} := \tau_2 \text{ in } H^{-1}\left(\mathbb{R}, L^2(\mathbb{R})\right)$$

$$\tau_2 = \mu\,\alpha\,\psi_m\left(\mathcal{H}^1 \lfloor B_a^+ - \mathcal{H}^1 \lfloor B_a^-\right) \in H^{-1}\left(\mathbb{R}; H_0^1(\mathbb{R})\right)$$



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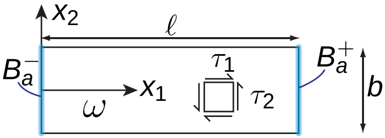
$$\tau_2 = \mu\,\alpha\,\psi_m\,\left(\mathcal{H}^1 LB_a^+ - \mathcal{H}^1 LB_a^-\right) \in H^{-1}\left(\mathbb{R}; H^1_0\left(\mathbb{R}\right)\right)$$



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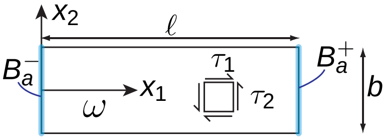
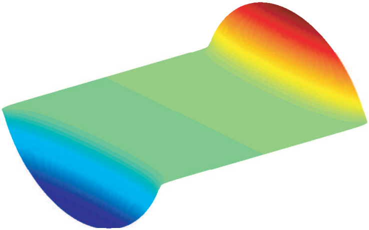
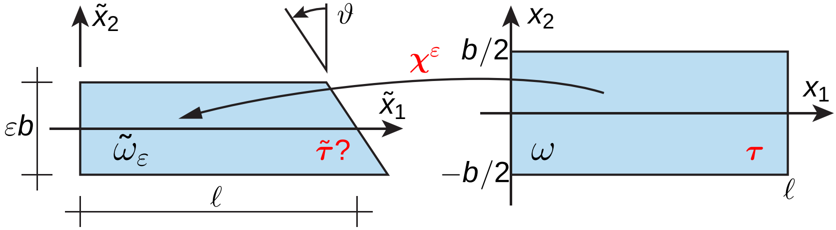






$$\boldsymbol{\tau} = -2\mu\alpha x_2 \boldsymbol{e}_1 + \mu\,\alpha\,\psi_m\,\left(\mathcal{H}^1 LB_a^+ - \mathcal{H}^1 LB_a^-\right)\boldsymbol{e}_2$$

$$\tilde{\eta} \in H^1\left(\tilde{\omega}_\varepsilon\right) \qquad \eta = \boldsymbol{\chi}_\varepsilon^\sharp\,\eta = \tilde{\eta}\circ\boldsymbol{\chi}^\varepsilon$$

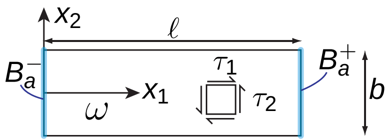
$$\left\langle \boldsymbol{\chi}_\sharp^\varepsilon\,\boldsymbol{\tau}, \nabla\tilde{\eta}\right\rangle = \left\langle \boldsymbol{\tau}, \nabla\left(\boldsymbol{\chi}_\varepsilon^\sharp\,\tilde{\eta}\right)\right\rangle$$

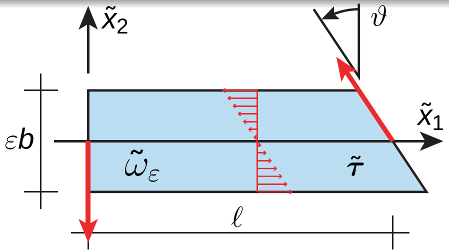
$$\tilde{\boldsymbol{\tau}} = \boldsymbol{\chi}_\sharp^\varepsilon\,\boldsymbol{\tau} = \left(-\frac{1}{\sqrt{g_\varepsilon}}2\,\mu\,\alpha\,x_2\,\boldsymbol{g}_1^\varepsilon\right)\circ\boldsymbol{\chi}_\varepsilon^{-1}$$

$$+\mu\,\alpha\left(\psi_m\,J^\varepsilon\,\boldsymbol{g}_2^\varepsilon\right)\circ\boldsymbol{\chi}_\varepsilon^{-1}\cdot\left(\mathcal{H}^1 L\,\boldsymbol{\chi}^\varepsilon\left(B_a^+\right) - \mathcal{H}^1 L\,\boldsymbol{\chi}^\varepsilon\left(B_a^-\right)\right)$$

Limit twisting moment

Shear traction contributions and torsional stiffness





$$\tau_1 = -2\mu\alpha\, x_2\, ; \qquad \tau_2 = \mu\,\alpha\,\psi_m\,\left(\mathcal{H}^1 L B_a^+ - \mathcal{H}^1 L B_a^-\right)$$

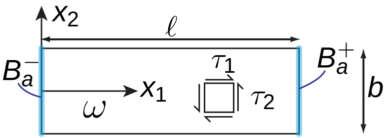
$$\frac{\tilde{M}^\varepsilon}{\varepsilon^3} = \frac{1}{\varepsilon^3}\int_{\tilde{\omega}_\varepsilon} \tilde{x}_1\,\tilde{\tau}_2^\varepsilon - \tilde{x}_2\,\tilde{\tau}_1^\varepsilon\, d\tilde{a} = \int_\omega x_1\,\tau_2^\varepsilon - x_2\,\tau_1^\varepsilon\, da \to M$$

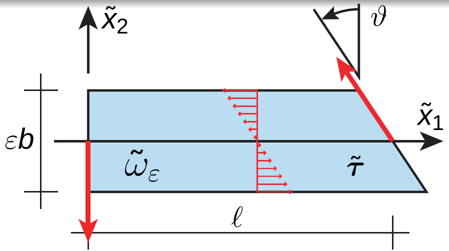
$$M = \int_\omega x_1\,\tau_2\, da - \int_\omega x_2\,\tau_1\, da \qquad \left(= 2\,\mu\,\alpha\int_\omega \psi_m\, da\right)$$

$$M = \mu\,\alpha\, a\, b^3\left(\frac{1}{6} + \frac{1}{6}\right) = \mu\,\alpha\,\frac{a\,b^3}{3}$$

Limit twisting moment

Shear traction contributions and torsional stiffness





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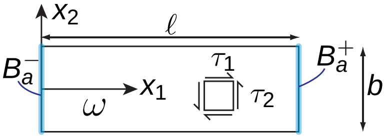

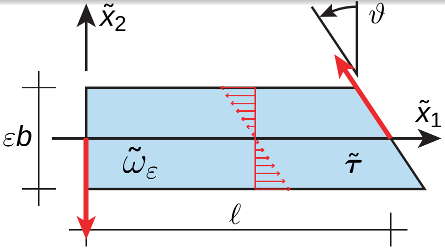

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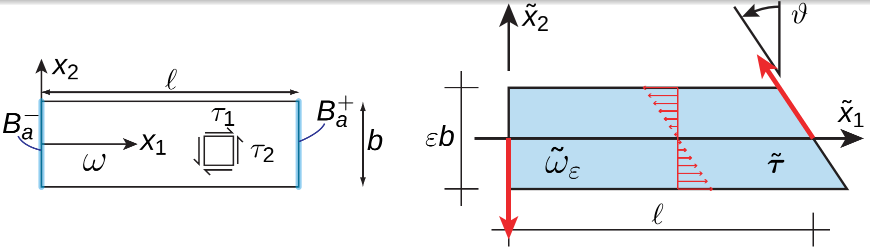

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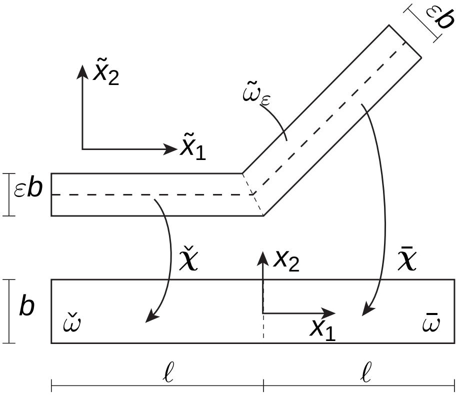
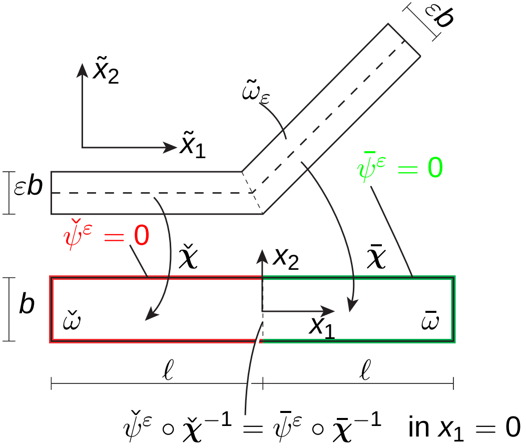
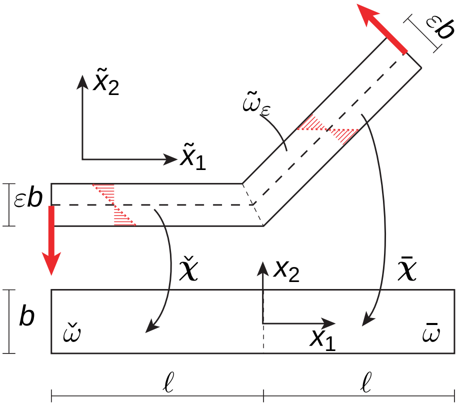


$$\tilde{\mathcal{F}}^\varepsilon\left(\tilde{\psi}^\varepsilon\right) = \int_{\tilde{\omega}_\varepsilon} \left|\nabla\tilde{\psi}^\varepsilon\right|^2 - 4\tilde{\psi}^\varepsilon \, d\tilde{a}$$

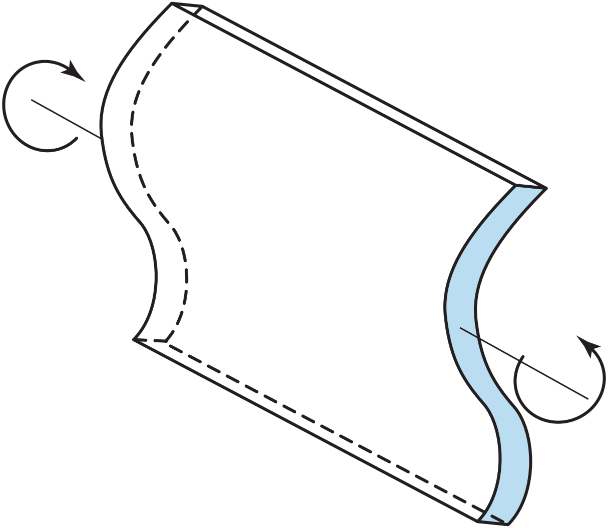


$$\mathcal{F}^{\varepsilon}\left(\check{\psi}^{\varepsilon}, \bar{\psi}^{\varepsilon}\right) :=$$

$$\int_{\check{\omega}} \left(\left| \check{\psi}^{\varepsilon}{}_{,\alpha} \boldsymbol{g}_{\varepsilon}^{\alpha} \right|^2 - 4\check{\psi}^{\varepsilon} \right) \sqrt{g_{\varepsilon}} d\check{a} + \int_{\bar{\omega}} \left(\left| \bar{\psi}^{\varepsilon}{}_{,\alpha} \boldsymbol{g}_{\varepsilon}^{\alpha} \right|^2 - 4\bar{\psi}^{\varepsilon} \right) \sqrt{g_{\varepsilon}} d\bar{a}$$



$$\frac{\mathcal{F}^\varepsilon\left(\check{\psi}^\varepsilon,\bar{\psi}^\varepsilon\right)}{\varepsilon^3} \xrightarrow{\Gamma} \mathcal{F}^0\left(\check{\psi},\bar{\psi}\right) = \int_{\check{\omega}} \left(\check{\psi}_{,2}\right)^2 - 4\check{\psi}\, d\check{a} + \int_{\bar{\omega}} \left(\bar{\psi}_{,2}\right)^2 - 4\bar{\psi}\, d\bar{a}$$



- Lipschitz mean line
- union of C^2 arcs

Lemma (Coercivity)

$$\exists\, c > 0 \text{ s.t. } \int_\omega |\psi^\varepsilon{}_{,\alpha}\, \boldsymbol{g}^\alpha_\varepsilon|^2\, da \;\geq\; c \int_\omega \left(\psi^\varepsilon{}_{,1}{}^2 + \left(\frac{\psi^\varepsilon{}_{,2}}{\varepsilon}\right)^2 \right) da$$

Back

Proof.

$$|\psi^\varepsilon{}_{,\alpha}\, \boldsymbol{g}^\alpha_\varepsilon|^2 = \left| \frac{1}{1-\varepsilon x_2/\rho} \left(\psi^\varepsilon{}_{,1}\, , \frac{\psi^\varepsilon{}_{,1} x_1}{\rho} \right) + \left(0, \frac{\psi^\varepsilon{}_{,2}}{\varepsilon} \right) \right|^2$$

$$= \frac{1}{(1-\varepsilon x_2/\rho)^2} \left[(\psi^\varepsilon{}_{,1})^2 + \frac{(\psi^\varepsilon{}_{,1})^2 x_1^2}{\rho^2} + \frac{(\psi^\varepsilon{}_{,2})^2}{\varepsilon^2} \left(1 - \varepsilon \frac{x_2}{\rho}\right)^2 + 2 \frac{\psi^\varepsilon{}_{,1} x_1}{\rho} \frac{\psi^\varepsilon{}_{,2}}{\varepsilon} \left(1 - \varepsilon \frac{x_2}{\rho}\right) \right]$$

$$\overset{\text{Young}}{\geq} \frac{1}{(1-\varepsilon x_2/\rho)^2} \left[\psi^\varepsilon{}_{,1}{}^2 \left(1 - \frac{x_1^2}{\rho^2} \left(\frac{1}{\eta} - 1 \right) \right) + \left(\frac{\psi^\varepsilon{}_{,2}}{\varepsilon} \right)^2 \left(1 - \varepsilon \frac{x_2}{\rho} \right)^2 (1 - \eta) \right]$$

$$(1-\eta) > 0 \;\text{ and }\; \left(1 - \frac{x_1^2}{\rho^2} \left(\frac{1}{\eta} - 1 \right) \right) > 0 \Leftrightarrow \frac{\operatorname{tg}^2\theta}{1+\operatorname{tg}^2\theta} < \eta < 1$$

□

Boundedness

Lemma (Boundedness)

$$W \equiv L^2\left((0,\ell); H_0^1\left(-\tfrac{b}{2}, \tfrac{b}{2}\right)\right), \quad \{\psi^\varepsilon\} \subset H_0^1(\omega) \ \ s.t. \ \ \sup_\varepsilon \frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} < +\infty$$

$$\Rightarrow \sup_\varepsilon \left\| \frac{\psi^\varepsilon{}_{,1}}{\varepsilon} \right\|_{L^2(\omega)} < +\infty \quad , \quad \sup_\varepsilon \left\| \frac{\psi^\varepsilon}{\varepsilon^2} \right\|_W < +\infty$$

Back

Proof.

$$+\infty > \frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} = \frac{1}{\varepsilon^3} \int_\omega \left(|\psi^\varepsilon{}_{,\alpha} \boldsymbol{g}_\varepsilon^\alpha|^2 - 4\psi^\varepsilon \right) g_\varepsilon \, da$$

$$+\infty > \frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} \overset{\text{coercivity}}{\geq} \int_\omega \left(c\left(\frac{\psi^\varepsilon{}_{,1}}{\varepsilon}\right)^2 + c\left(\frac{\psi^\varepsilon{}_{,2}}{\varepsilon^2}\right)^2 - 4\psi^\varepsilon \right) \left(1 - \frac{\varepsilon x_2}{\rho}\right) da$$

$$+\infty > \frac{2}{c}\frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} \geq \int_\omega \left(\frac{\psi^\varepsilon{}_{,1}}{\varepsilon}\right)^2 + \frac{1}{2}\left(\frac{\psi^\varepsilon{}_{,2}}{\varepsilon^2}\right)^2 + \frac{1}{2b^2}\left(\frac{\psi^\varepsilon}{\varepsilon^2}\right)^2 - \delta\left(\frac{\psi^\varepsilon}{\varepsilon^2}\right)^2 - \frac{1}{\delta}\left(\frac{4}{c}\right)^2 da$$

$$+\infty > \left(8\frac{b}{c}\right)^2 + \frac{2}{c}\frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} \geq \int_\omega \left(\frac{\psi^\varepsilon{}_{,1}}{\varepsilon}\right)^2 + \frac{1}{2}\left(\frac{\psi^\varepsilon{}_{,2}}{\varepsilon^2}\right)^2 + \frac{1}{4b^2}\left(\frac{\psi^\varepsilon}{\varepsilon^2}\right)^2 da \qquad \square$$

Used inequalities

$$\|\psi^\varepsilon\|_{L^2(\omega)} \leq b \, \|\psi^\varepsilon{}_{,2}\|_{L^2(\omega)} \ \ \forall \psi^\varepsilon \in H_0^1(\omega) \quad ; \quad a\,b \leq \delta\, a^2 + \frac{1}{\delta}\, b^2$$

Compactness

Lemma (Compactness)

$W \equiv L^2\left((0,\ell)\,;H^1_0\left(-\frac{b}{2},\frac{b}{2}\right)\right)$, $\forall\,\{\psi^\varepsilon\}\subset H^1_0(\omega)$ *s.t.* $\sup_\varepsilon \frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} < +\infty$,

$\exists\,\psi\in W$ *and a subsequence of* $\{\psi^\varepsilon\}$*, not re-labeled, s.t.*

$\frac{\psi^\varepsilon{}_{,1}}{\varepsilon} \xrightarrow{L^2(\omega)} 0$, $\frac{\psi^\varepsilon}{\varepsilon^2} \xrightarrow{W} \psi$

Back

Proof.

By boundedness lemma:

- $\exists\psi\in W$ s.t. $\frac{\psi^\varepsilon}{\varepsilon^2} \xrightarrow{W} \psi$
- $\exists\xi\in L^2(\omega)$ s.t. $\frac{\psi^\varepsilon{}_{,1}}{\varepsilon} \xrightarrow{L^2(\omega)} \xi$

  Hence: $0 \leftarrow -\varepsilon\int_\omega \frac{\psi^\varepsilon}{\varepsilon^2}\,\eta_{,1} = \int_\omega \frac{\psi^\varepsilon{}_{,1}}{\varepsilon}\,\eta \rightarrow \int_\omega \xi\,\eta \quad \forall\eta\in\mathcal{C}^\infty_0(\omega)$

  By density $\frac{\psi^\varepsilon{}_{,1}}{\varepsilon} \xrightarrow{L^2(\omega)} 0$

□

$W \equiv L^2\left((0,\ell)\,;H_0^1\left(-\frac{b}{2},\frac{b}{2}\right)\right)$
$\mathcal{F}^0 : W \to \mathbb{R}\,; \quad \mathcal{F}^0(\psi) = \int_\omega (\psi_{,2})^2 - 4\psi\, da$

Theorem (Γ– convergence: lim inf)

$$\frac{\psi^\varepsilon{}_{,1}}{\varepsilon} \rightharpoonup 0 \textit{ in } L^2(\omega) \quad \textit{and} \quad \frac{\psi^\varepsilon}{\varepsilon^2} \rightharpoonup \psi \textit{ in } W \;\Rightarrow \liminf_{\varepsilon\longrightarrow 0} \frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} \geq \mathcal{F}^0(\psi)$$

Proof.

$$\frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} = \int_\omega \left(\left| \frac{\psi^\varepsilon{}_{,\alpha}}{\varepsilon} \boldsymbol{g}_\varepsilon^\alpha \right|^2 - 4\frac{\psi^\varepsilon}{\varepsilon^2} \right) \frac{g_\varepsilon}{\varepsilon}\, da$$

$$\frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} = \int_\omega \left| \frac{\psi^\varepsilon{}_{,1}}{\varepsilon}\left(1,\frac{x_1}{\rho}\right) + \frac{\psi^\varepsilon{}_{,2}}{\varepsilon^2}(0,1)\left(1-\varepsilon\frac{x_2}{\rho}\right)\right|^2 - 4\frac{\psi^\varepsilon}{\varepsilon^2}\left(1-\varepsilon\frac{x_2}{\rho}\right)\, da$$

Due to the weak lower semi-continuity of $\mathcal{F}^\varepsilon$:

$$\liminf_{\varepsilon\longrightarrow 0} \frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} \geq \mathcal{F}^\varepsilon\left(w-\lim_{\varepsilon\longrightarrow 0}\psi^\varepsilon\right) = \int_\omega \psi_{,2}^{\,2} - 4\,\psi\, da = \mathcal{F}^0(\psi)$$

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$$W \equiv L^2\left((0,\ell)\,; H_0^1\left(-\frac{b}{2},\frac{b}{2}\right)\right)\ ;\ \mathcal{F}^0 : W \to \mathbb{R}\,;\ \mathcal{F}^0(\psi) = \int_\omega (\psi_{,2})^2 - 4\psi\, da$$

Theorem (Γ– convergence: recovery)

$\forall\ \psi \in W,\ \ \exists\ \psi^\varepsilon \in H_0^1(\omega)$ *s. t.*

$$\frac{\psi^\varepsilon{}_{,1}}{\varepsilon} \rightharpoonup 0 \textit{ in } L^2(\omega) \quad ; \quad \frac{\psi^\varepsilon}{\varepsilon^2} \rightharpoonup \psi \textit{ in } W \quad ; \quad \limsup_{\varepsilon\to 0} \frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} \le \mathcal{F}^0(\psi)$$

Proof.

$\forall \psi_k \in \mathcal{C}_0^\infty(\omega)$, let $\psi_k^\varepsilon = \varepsilon^2\,\psi_k$, then $\lim_{\varepsilon\to 0} \frac{\mathcal{F}^\varepsilon(\psi_k^\varepsilon)}{\varepsilon^3} = \mathcal{F}^0(\psi_k)$

$$\frac{\mathcal{F}^\varepsilon(\psi_k^\varepsilon)}{\varepsilon^3} = \int_\omega \left|\varepsilon\,\psi_{k,1}\left(1,\frac{x_1}{\rho}\right) + \psi_{k,2}\,(0,1)\left(1-\varepsilon\frac{x_2}{\rho}\right)\right|^2 - 4\psi_k\left(1-\varepsilon\frac{x_2}{\rho}\right) da$$

$\forall\psi \in W$, $\exists\psi_k \in \mathcal{C}_0^\infty(\omega)$ & $\psi_k^\varepsilon = \varepsilon^2\,\psi_k$ s.t. $\frac{\psi_k^\varepsilon}{\varepsilon^2} = \psi_k \xrightarrow{W} \psi$

$$\lim_k \lim_\varepsilon \frac{\mathcal{F}^\varepsilon(\psi_k^\varepsilon)}{\varepsilon^3} = \lim_k \mathcal{F}^0(\psi_k) = \mathcal{F}^0(\psi)$$

diagonalise . . .

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