Università degli studi di Udine

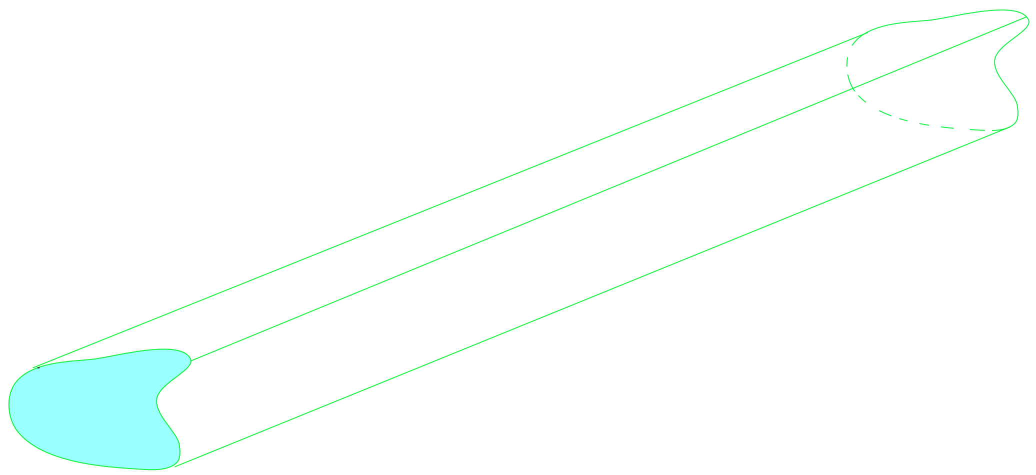
Derivazione per Γ-convergenza di modelli variazionali di travi e piastre

Dottorato di Ricerca in Matematica e Fisica

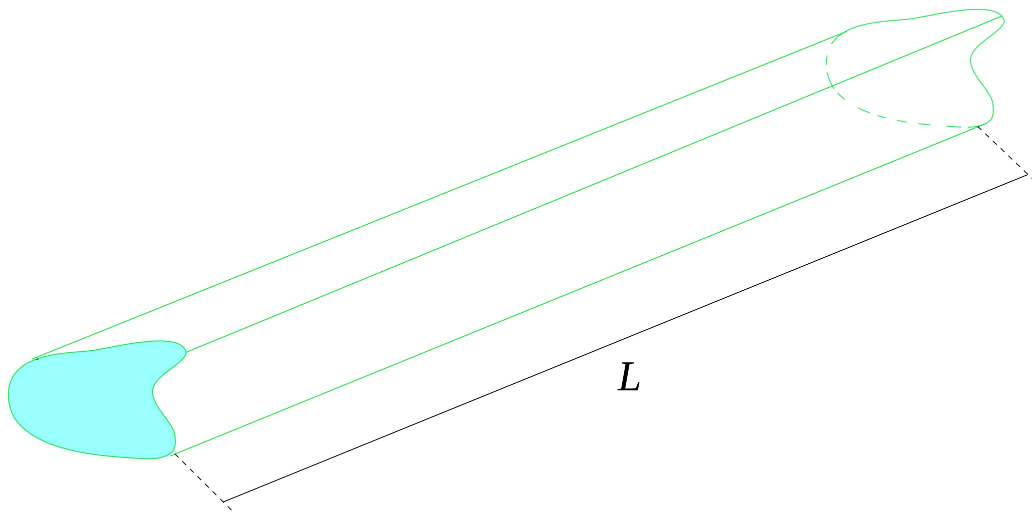
Alessandro Londero

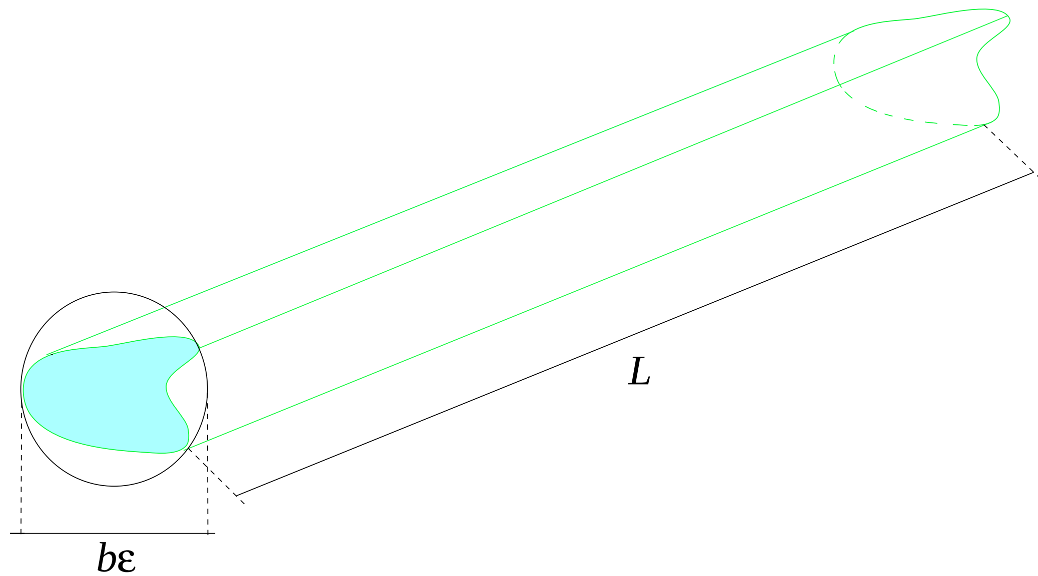
Introduzione

Introduzione

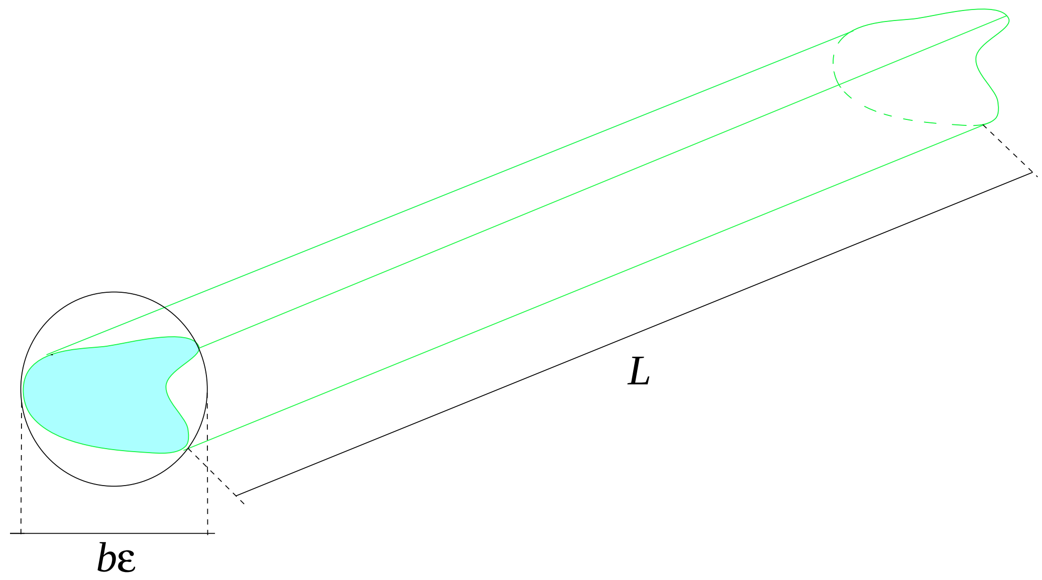


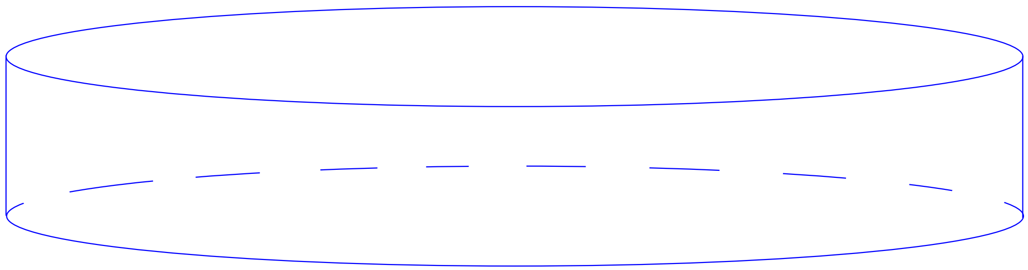
Introduzione



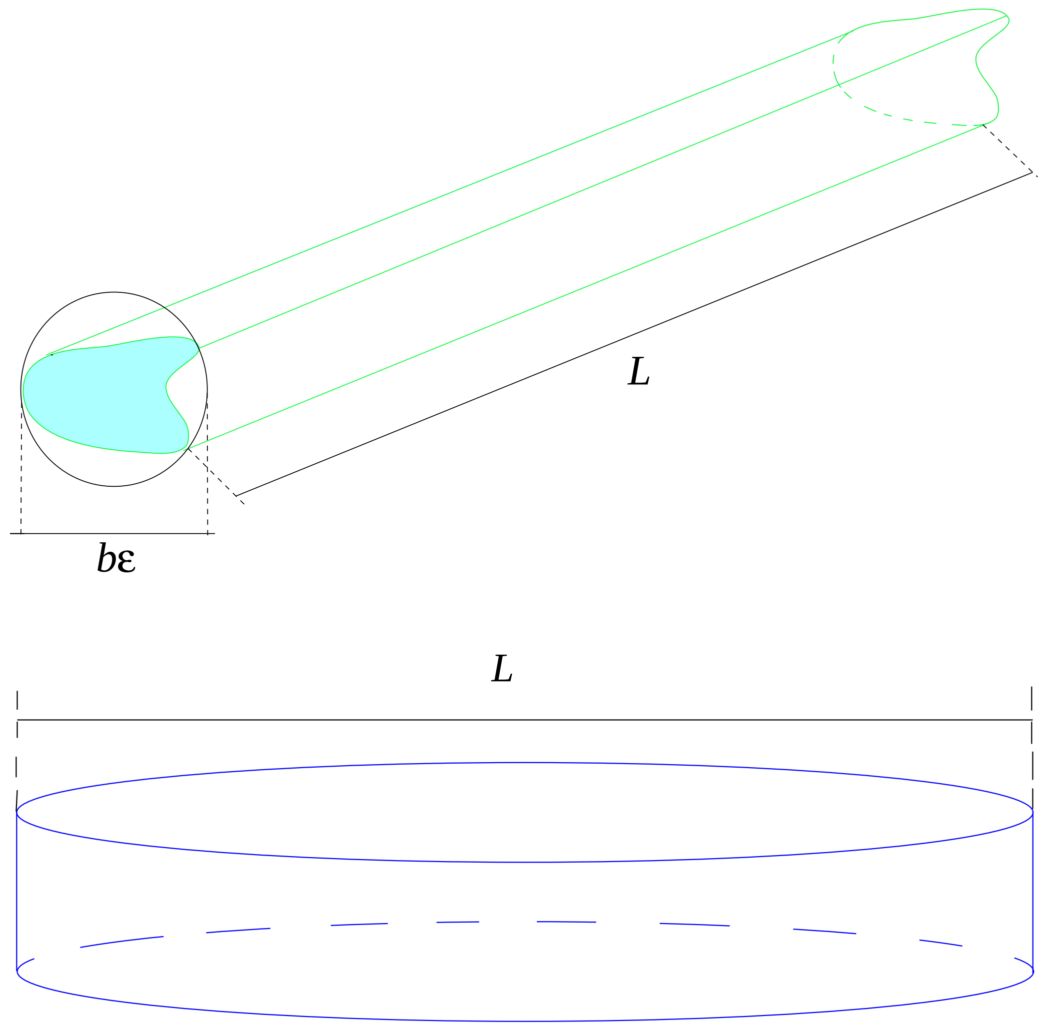

L
bε

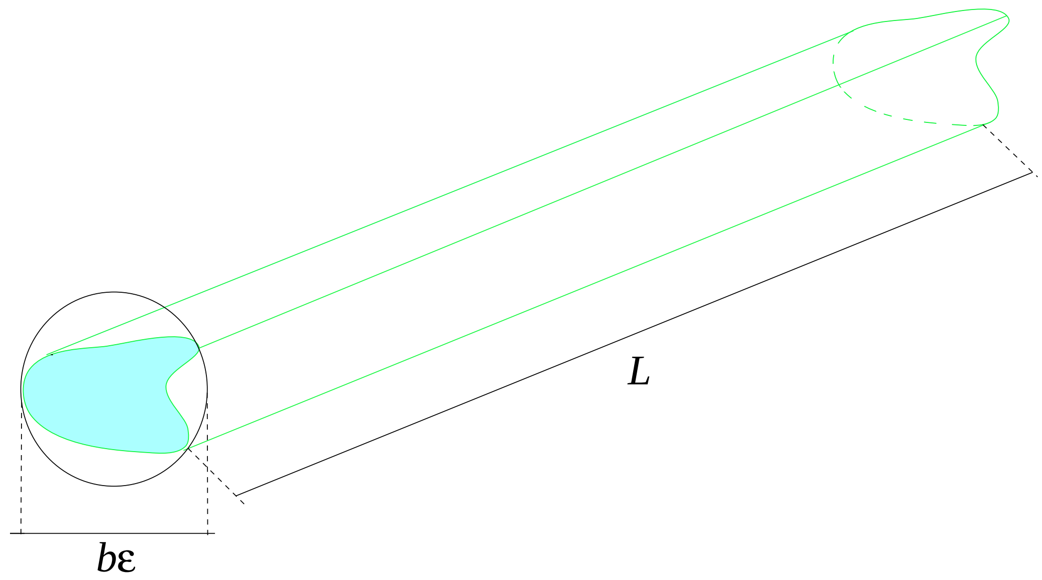
Introduzione

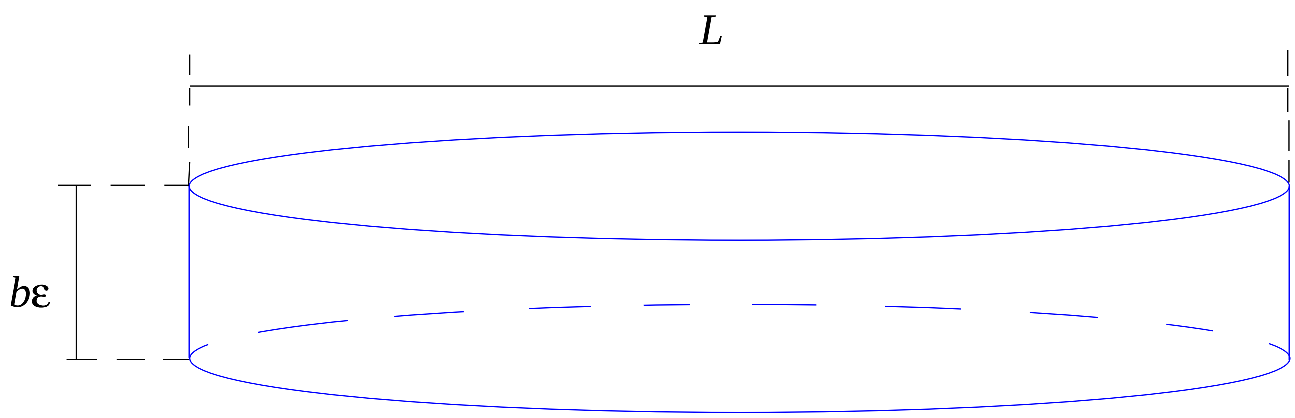




Introduzione

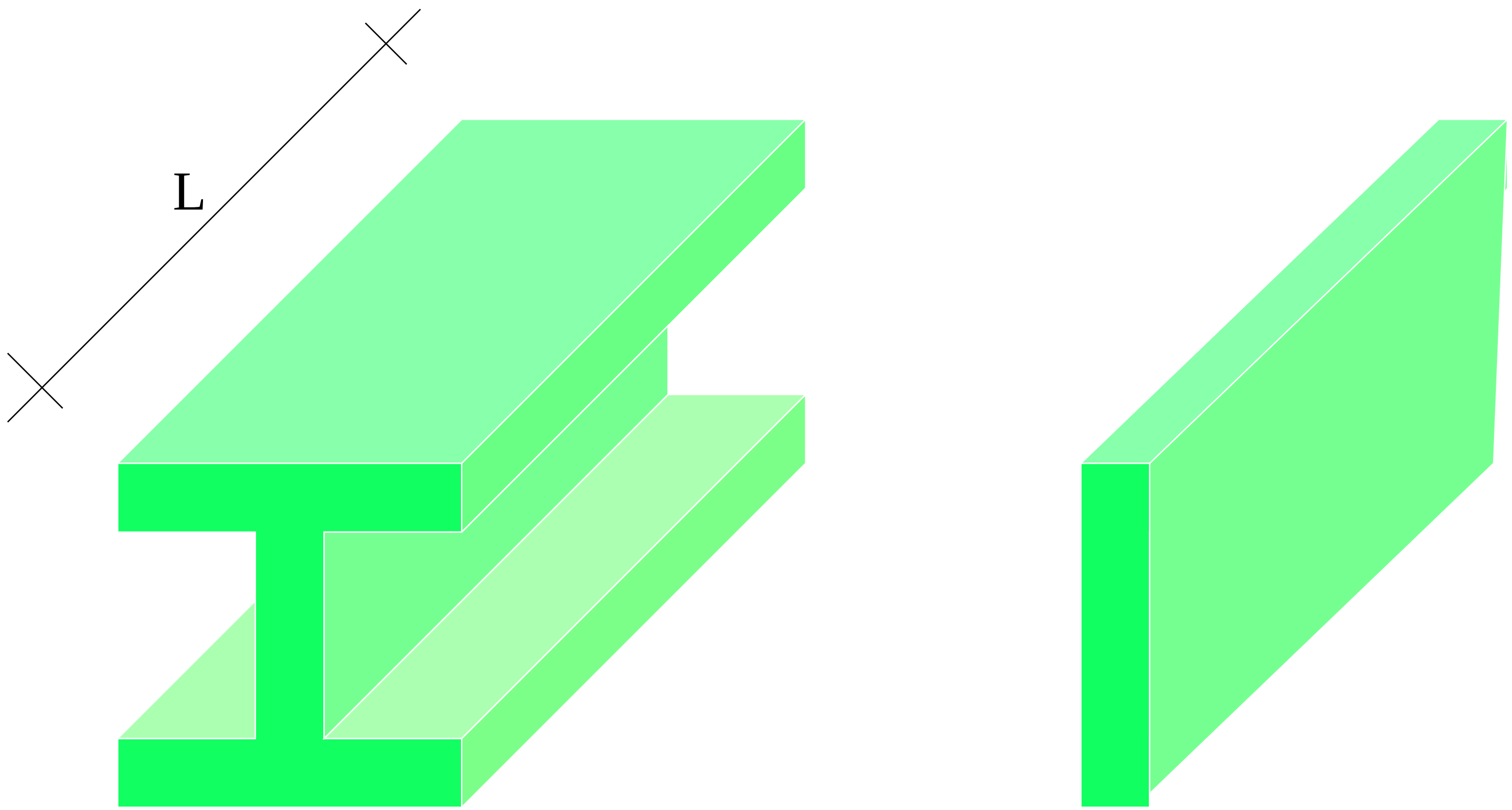


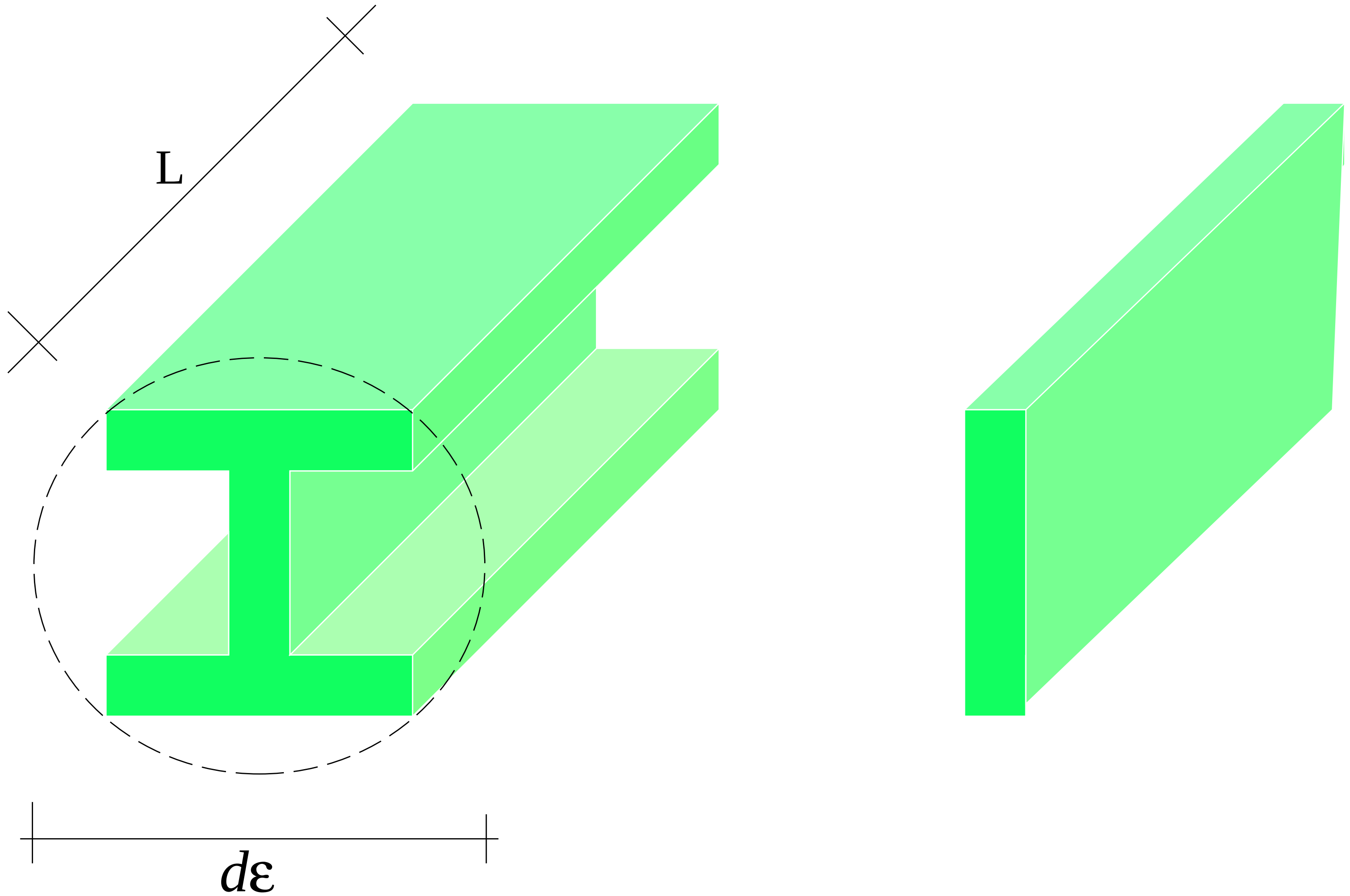

L
bε


L
bε

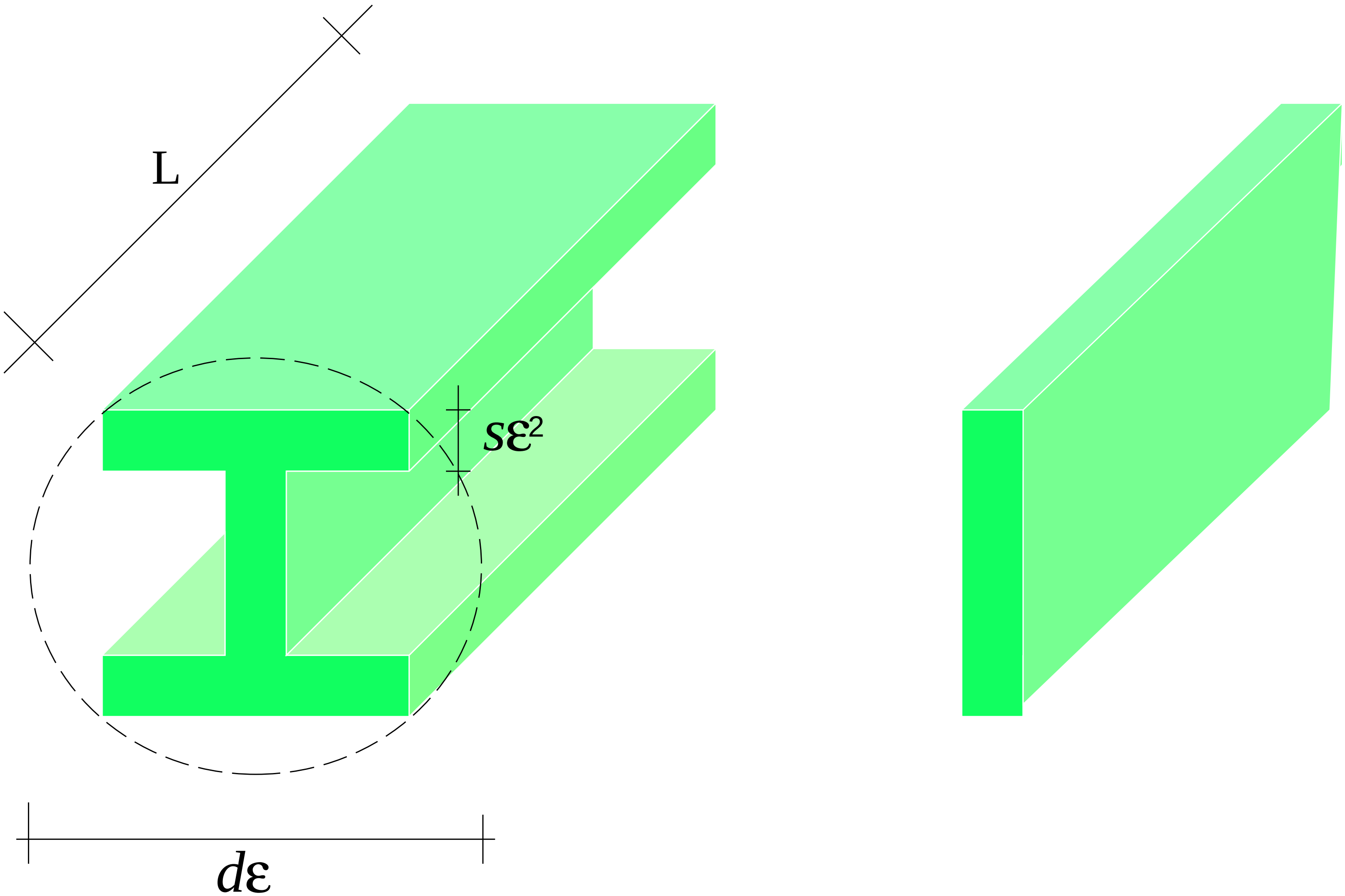
Introduzione

Introduzione



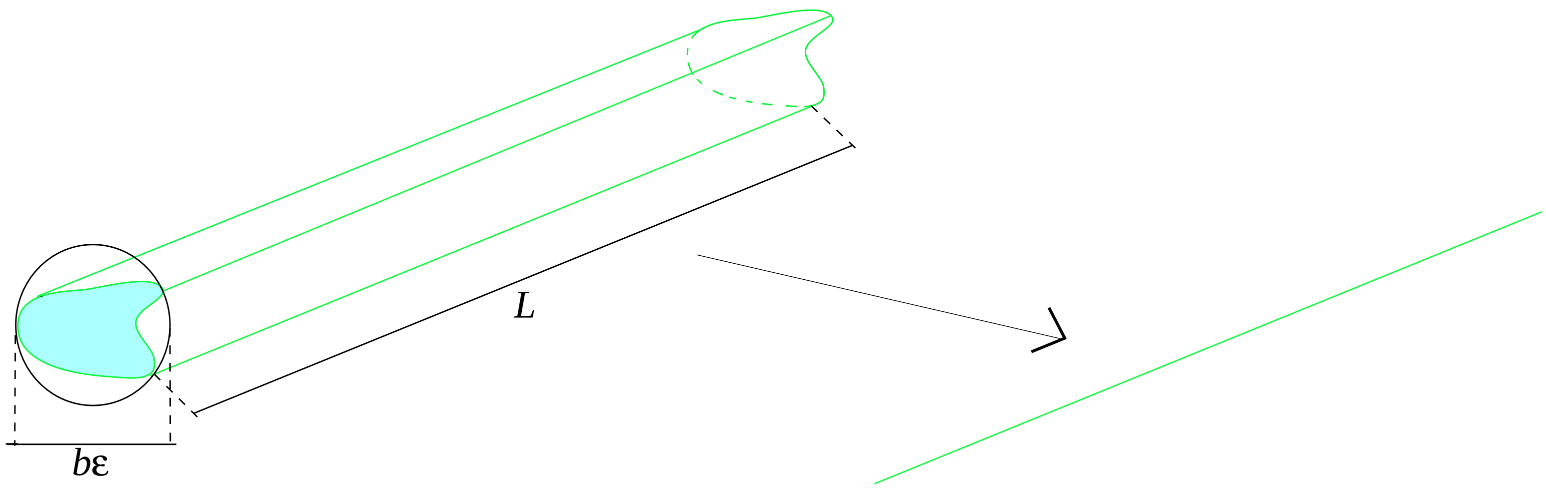


L
$d\varepsilon$

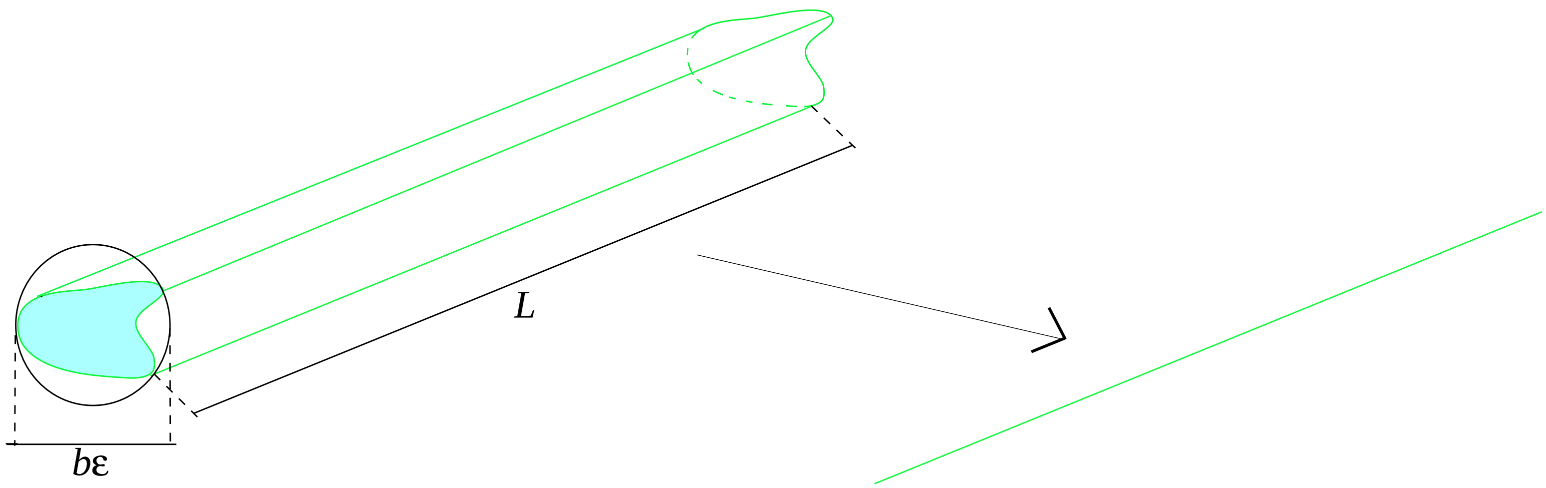


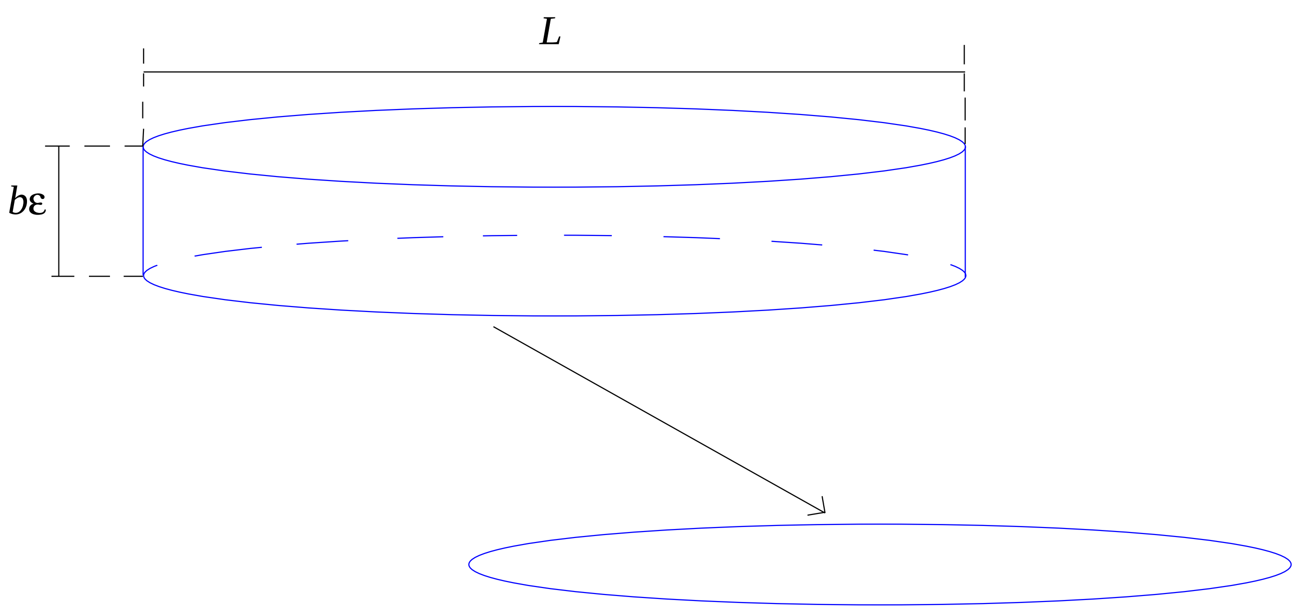
L
sε²
dε

Introduzione

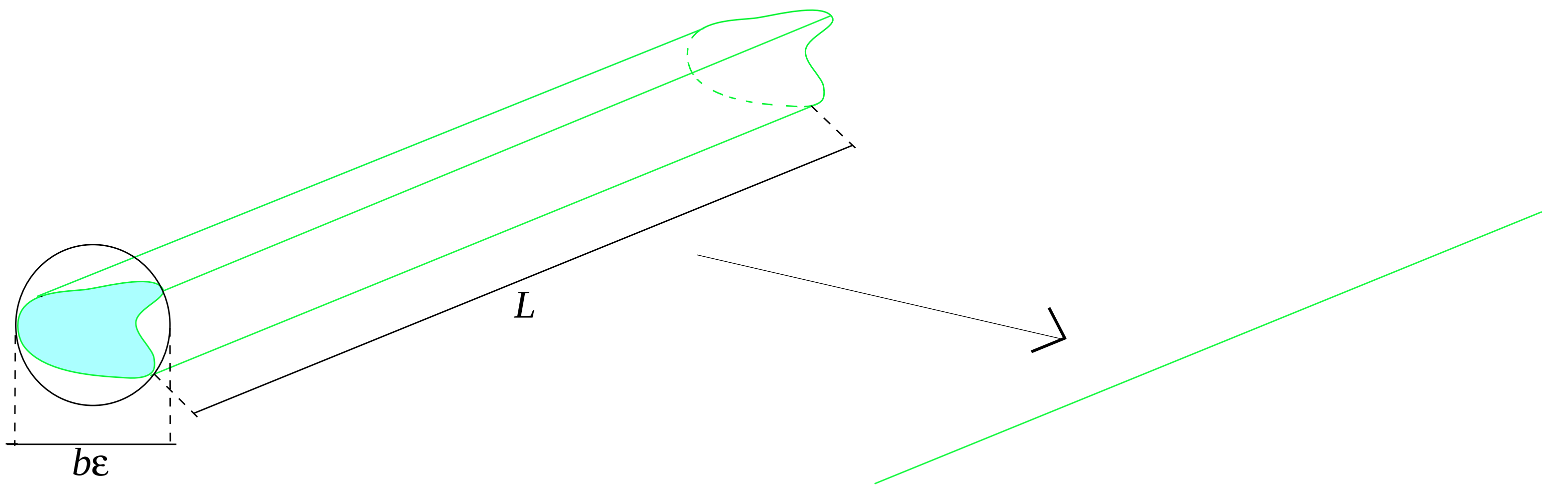


Introduzione

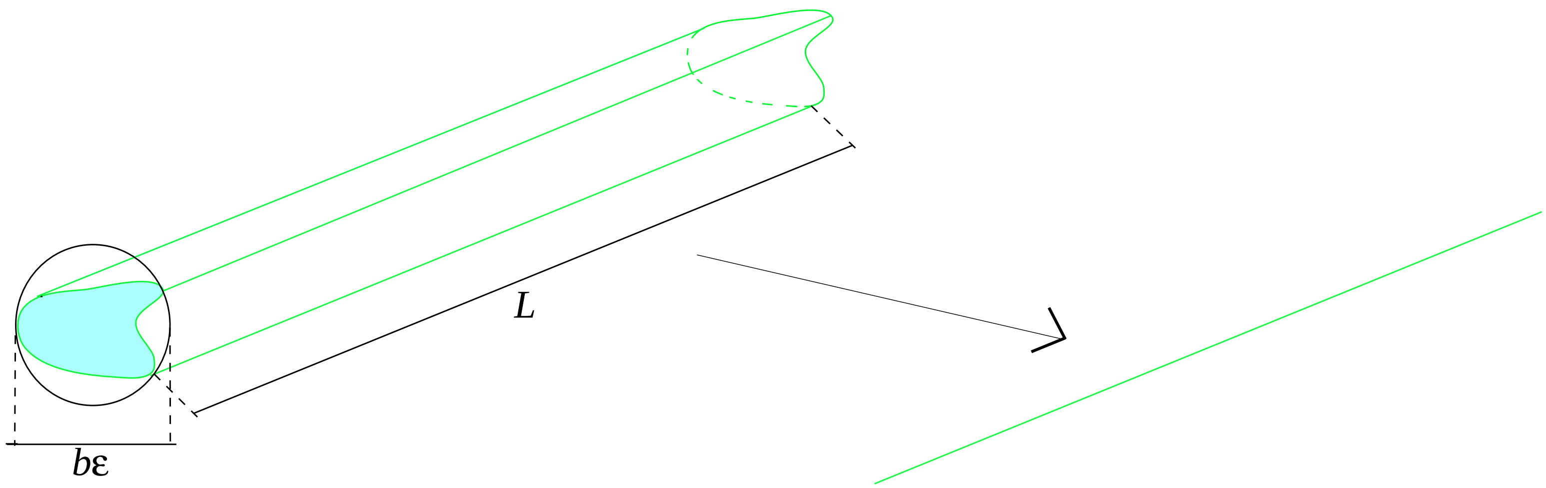




Introduzione



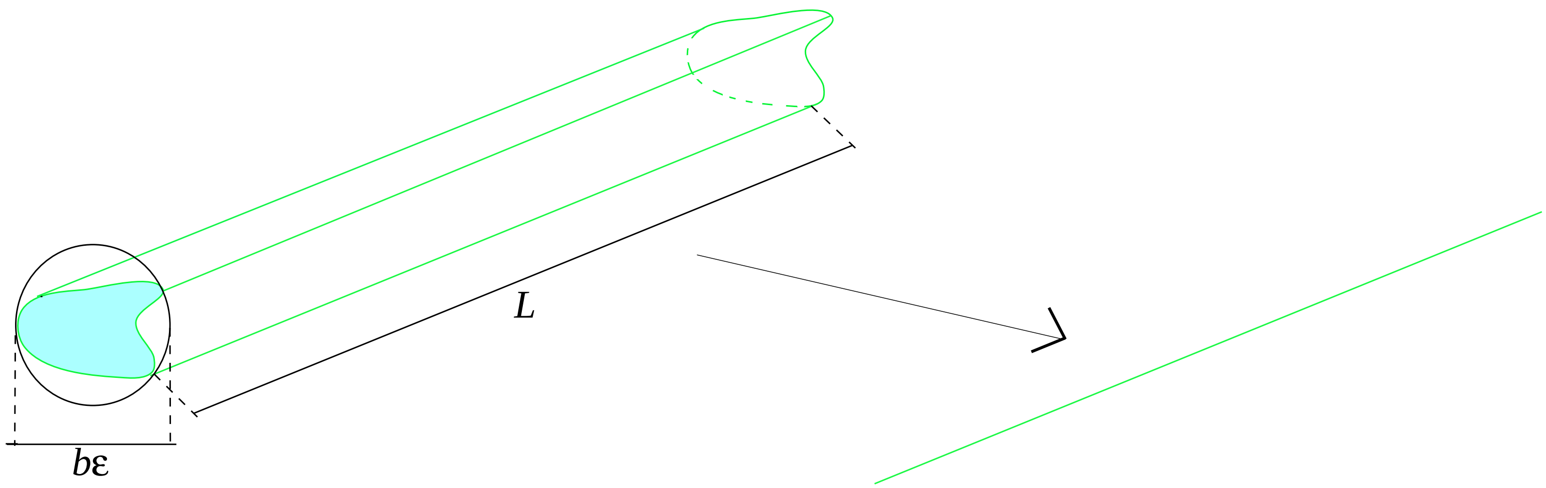
$$F_\varepsilon \xrightarrow{\Gamma} F$$



$$F_\varepsilon \xrightarrow{\Gamma} F$$

$$\min F_\varepsilon \longrightarrow \min F$$

Introduzione

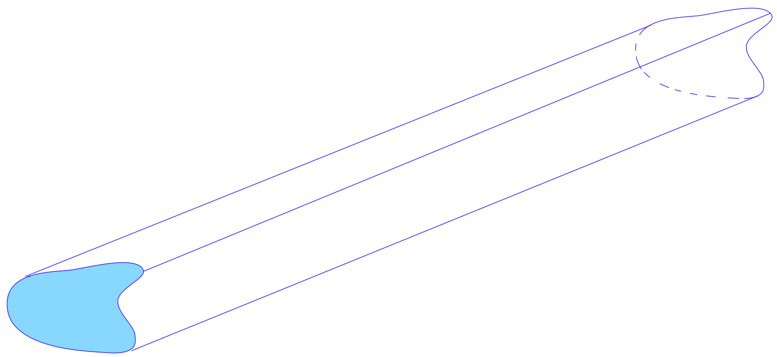


$$F_\varepsilon \xrightarrow{\Gamma} F$$

$$\min F_\varepsilon \longrightarrow \min F$$

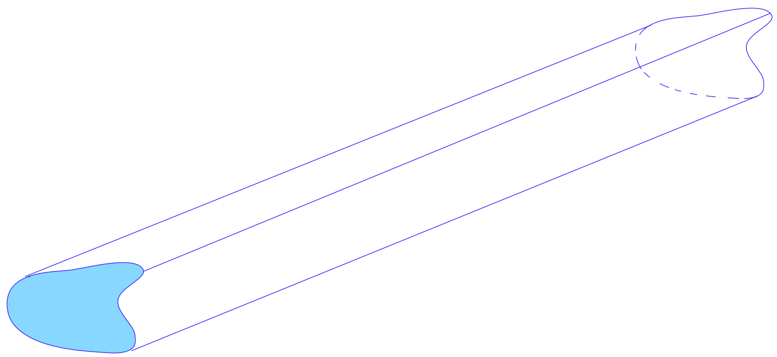
$$x_\varepsilon \in \operatorname{argmin} F_\varepsilon \longrightarrow x \in \operatorname{argmin} F$$

Travi a sezione compatta



diametro << lunghezza

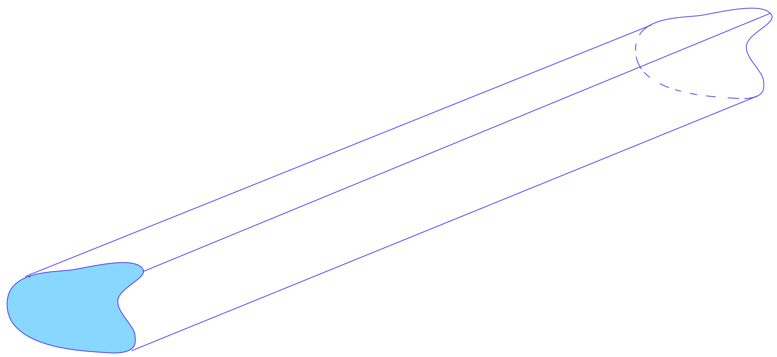
Travi a sezione compatta



diametro << lunghezza

Anzellotti, Baldo, Percivale (1994)

Travi a sezione compatta

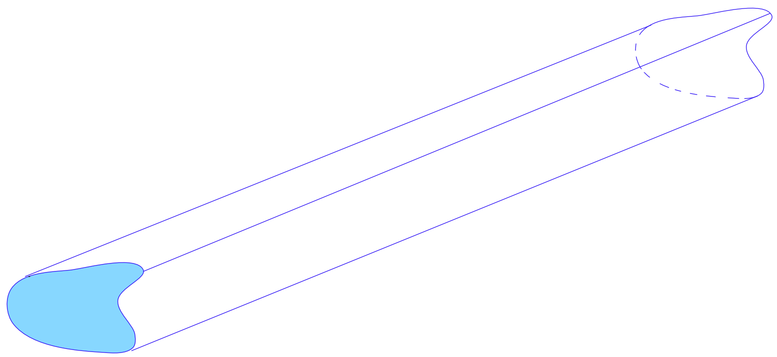


diametro << lunghezza

Anzellotti, Baldo, Percivale (1994)

- $\mathcal{F}_\varepsilon = \varepsilon^2 \mathcal{F}^{(1)} + \varepsilon^4 \mathcal{F}^{(2)} + O(\varepsilon^4)$

Travi a sezione compatta

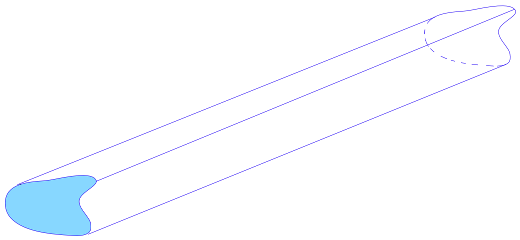


diametro << lunghezza

Anzellotti, Baldo, Percivale (1994)

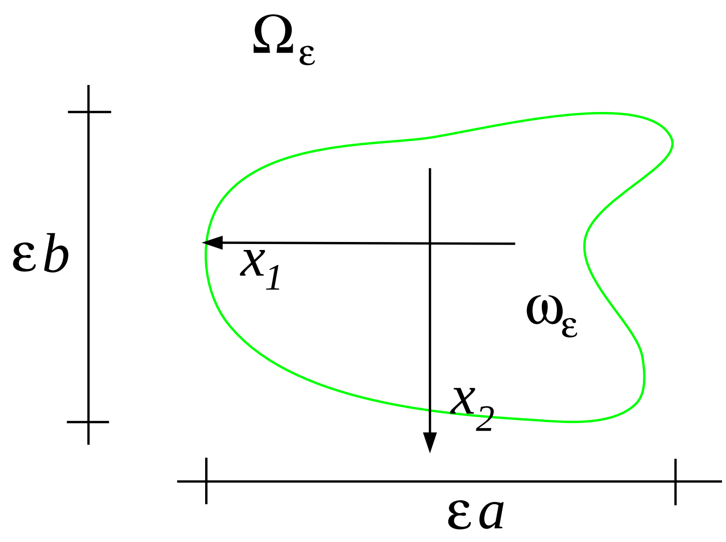
- $\mathcal{F}_\varepsilon = \varepsilon^2 \mathcal{F}^{(1)} + \varepsilon^4 \mathcal{F}^{(2)} + O(\varepsilon^4)$

Percivale (1999)



diametro << lunghezza

Studiamo il caso in cui la sezione sia



Travi a sezione compatta

Energia della trave

$$\mathcal{F}_\varepsilon(u) := \frac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}Eu \cdot Eu\, dx - \int_{\Omega_\varepsilon} b^\varepsilon \cdot u\, dx$$

dove

$$\mathbb{C}Eu = 2\mu Eu + \lambda(\mathbf{tr}Eu)I$$

Energia della trave

$$\mathcal{F}_\varepsilon(u) := \frac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}Eu \cdot Eu\,dx - \int_{\Omega_\varepsilon} b^\varepsilon \cdot u\,dx$$

dove

$$\mathbb{C}Eu = 2\mu Eu + \lambda(\mathbf{tr}Eu)I$$

$$H^1_\#(\Omega_\varepsilon;\mathbb{R}^3) := \{u \in H^1(\Omega_\varepsilon;\mathbb{R}^3) : u = 0 \textbf{ su } S_\varepsilon(0)\}$$

Travi a sezione compatta

Energia della trave

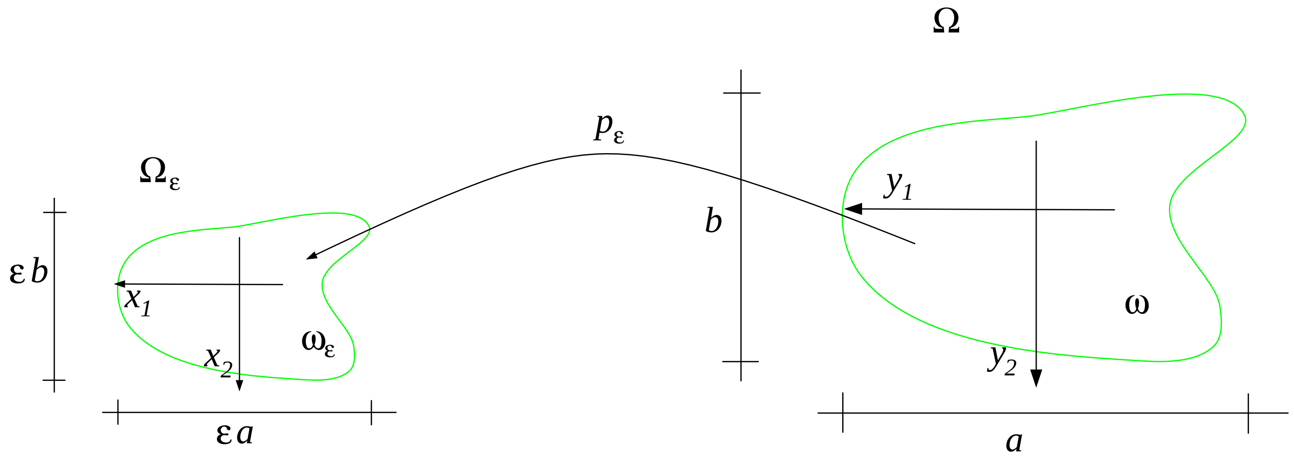
$$\mathcal{F}_\varepsilon(u) := \frac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}Eu \cdot Eu\,dx - \int_{\Omega_\varepsilon} b^\varepsilon \cdot u\,dx$$

dove

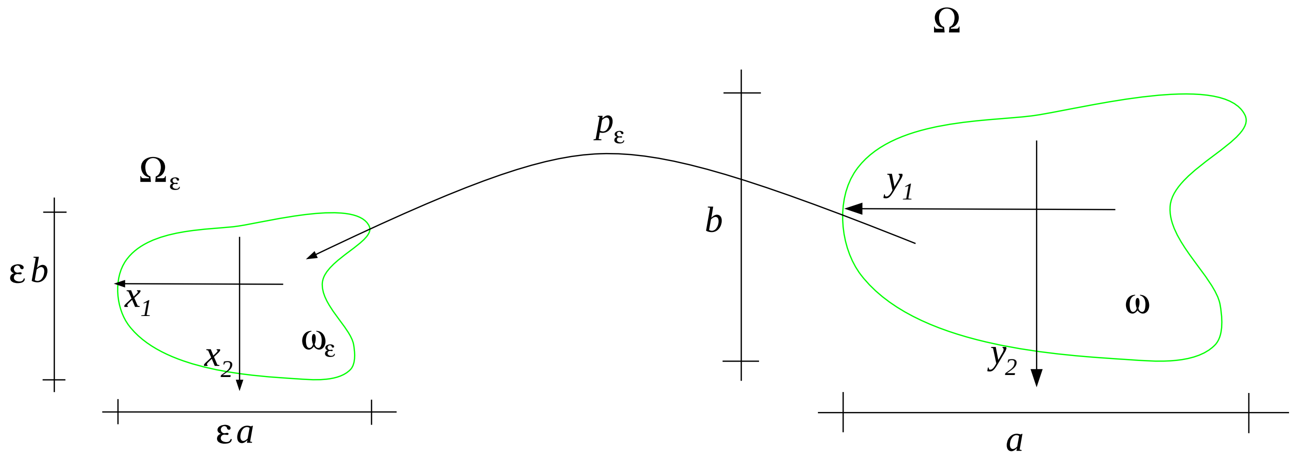
$$\mathbb{C}Eu = 2\mu Eu + \lambda(\mathbf{tr}Eu)I$$

$$H^1_\#(\Omega_\varepsilon;\mathbb{R}^3) := \{u \in H^1(\Omega_\varepsilon;\mathbb{R}^3) : u = 0 \textbf{ su } S_\varepsilon(0)\}$$

Travi a sezione compatta

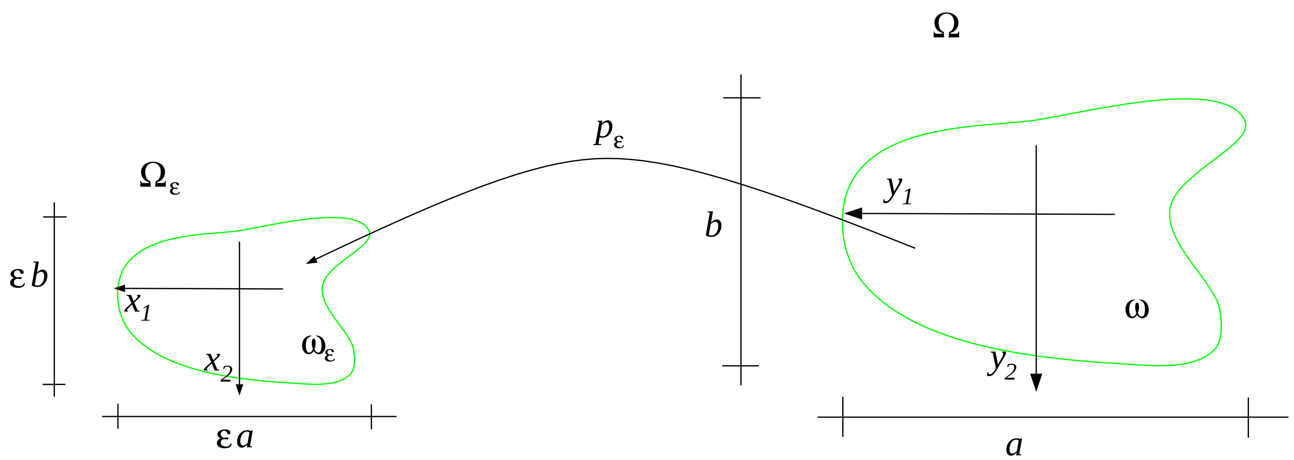


Travi a sezione compatta



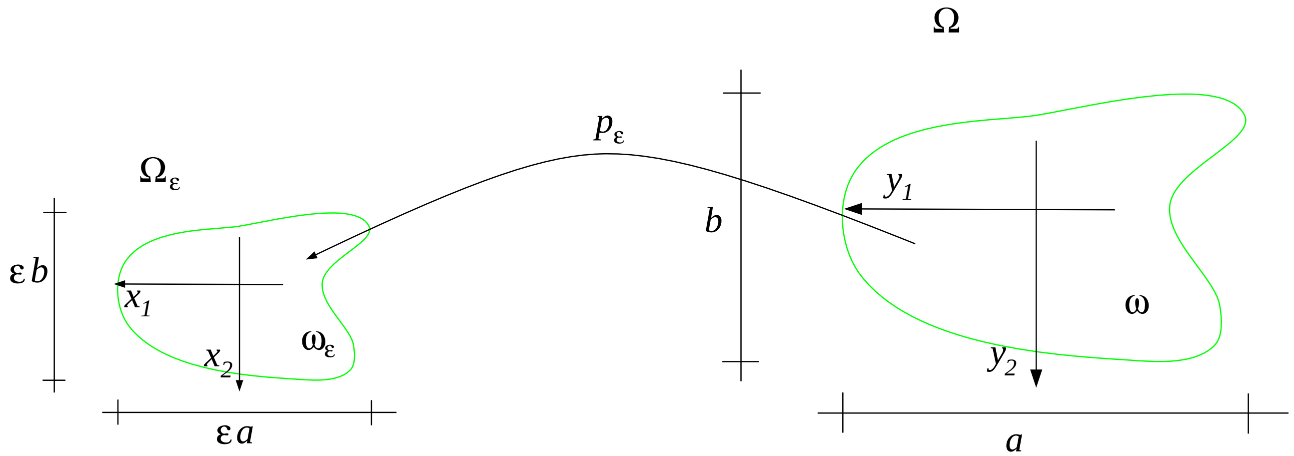
$$u : \Omega_\varepsilon \longrightarrow \mathbb{R}^3 \qquad v : \Omega \longrightarrow \mathbb{R}^3$$

Travi a sezione compatta



$$u : \Omega_\varepsilon \longrightarrow \mathbb{R}^3 \qquad v : \Omega \longrightarrow \mathbb{R}^3$$

$$v = u \circ p_\varepsilon = u(\varepsilon y_1, \varepsilon y_2, y_3)$$

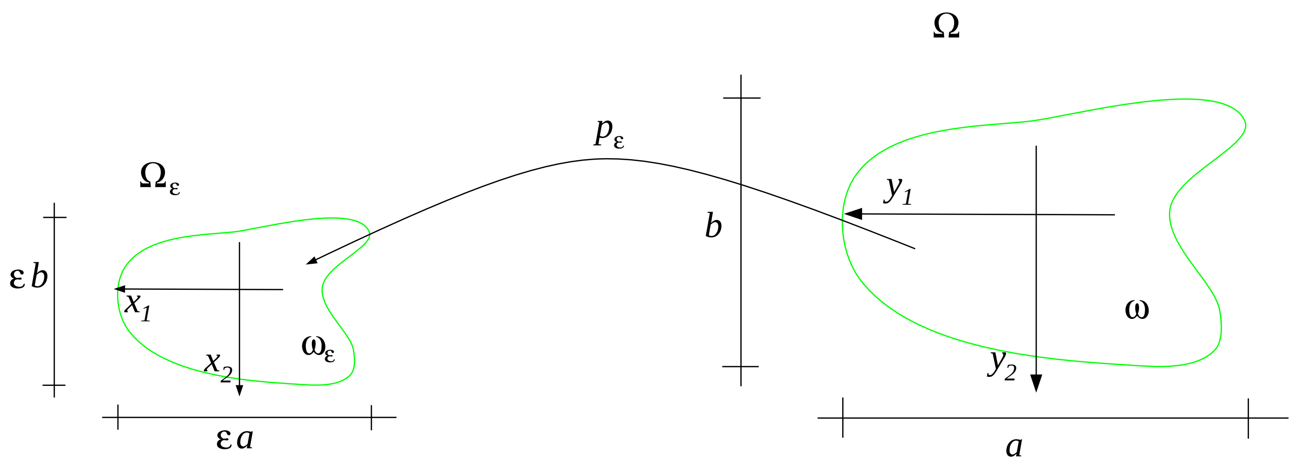


$$u : \Omega_\varepsilon \longrightarrow \mathbb{R}^3 \qquad v : \Omega \longrightarrow \mathbb{R}^3$$

$$v = u \circ p_\varepsilon = u(\varepsilon y_1, \varepsilon y_2, y_3)$$

$$H^\varepsilon v := \left(\frac{D_1 v}{\varepsilon}, \frac{D_2 v}{\varepsilon}, D_3 v\right)$$

$$E^\varepsilon v := \mathbf{sym}(H^\varepsilon v) \qquad W^\varepsilon v := \mathbf{skw}(H^\varepsilon v)$$



Energia riscalata

$$F_\varepsilon(v) = \frac{1}{2}\int_\Omega \mathbb{C}E^\varepsilon v \cdot E^\varepsilon v \, dy - \int_\Omega b^\varepsilon \circ p_\varepsilon \cdot v \, dy$$

Diseguaglianza di Korn $(u(y_1, y_2, 0) = 0)$

$$\int_\Omega \left(\left| \left(u_1, u_2, \frac{u_3}{\varepsilon} \right) \right|^2 + |H^\varepsilon u|^2 \right) dy \leq \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy$$

Diseguaglianza di Korn $(u(y_1, y_2, 0) = 0)$

$$\boxed{\int_\Omega \left(\left| \left(u_1, u_2, \frac{u_3}{\varepsilon} \right) \right|^2 + |H^\varepsilon u|^2 \right) dy \leq \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy \leq C}$$

Diseguaglianza di Korn $(u(y_1, y_2, 0) = 0)$

$$\boxed{\int_\Omega \left(\left| \left(u_1, u_2, \frac{u_3}{\varepsilon} \right) \right|^2 + |H^\varepsilon u|^2 \right) dy \leq \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy \leq C}$$

$$u_1^\varepsilon \overset{H^1}{\rightharpoonup} v_1$$

$$u_2^\varepsilon \overset{H^1}{\rightharpoonup} v_2$$

$$\frac{u_3^\varepsilon}{\varepsilon} \overset{H^1}{\rightharpoonup} v_3$$

Diseguaglianza di Korn $(u(y_1, y_2, 0) = 0)$

$$\boxed{\int_\Omega \left(\left| \left(u_1, u_2, \frac{u_3}{\varepsilon} \right) \right|^2 + |H^\varepsilon u|^2 \right) dy \leq \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy \leq C}$$

$$u_1^\varepsilon \overset{H^1}{\rightharpoonup} v_1$$

$$u_2^\varepsilon \overset{H^1}{\rightharpoonup} v_2$$

$$\frac{u_3^\varepsilon}{\varepsilon} \overset{H^1}{\rightharpoonup} v_3$$

$$v = (v_1, v_2, v_3) \in H_{BN}$$

Diseguaglianza di Korn $(u(y_1, y_2, 0) = 0)$

$$\boxed{\int_\Omega \left(\left| \left(u_1, u_2, \frac{u_3}{\varepsilon} \right) \right|^2 + |H^\varepsilon u|^2 \right) dy \le \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy \le C}$$

$$v_1 = \xi_1(y_3) \in H^2_\#$$

$$v_2 = \xi_2(y_3) \in H^2_\#$$

$$v_3 = \xi_3(y_3) - y_1 \xi_1'(y_3) - y_2 \xi_2'(y_3) \qquad \xi_3 \in H^1_\#$$

Diseguaglianza di Korn $(u(y_1, y_2, 0) = 0)$

$$\boxed{\int_\Omega \left(\left| \left(u_1, u_2, \frac{u_3}{\varepsilon} \right) \right|^2 + |H^\varepsilon u|^2 \right) dy \le \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy \le C}$$

$$W^\varepsilon u^\varepsilon \overset{L^2}{\rightharpoonup} \begin{pmatrix} 0 & ??? & D_3 v_1 \\ ??? & 0 & D_3 v_2 \\ -D_3 v_1 & -D_3 v_2 & 0 \end{pmatrix}$$

Diseguaglianza di Korn $(u(y_1, y_2, 0) = 0)$

$$\boxed{\int_\Omega \left(\left| \left(u_1, u_2, \frac{u_3}{\varepsilon} \right) \right|^2 + |H^\varepsilon u|^2 \right) dy \leq \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy \leq C}$$

$$W^\varepsilon u^\varepsilon \overset{L^2}{\rightharpoonup} \begin{pmatrix} 0 & -\vartheta & D_3 v_1 \\ \vartheta & 0 & D_3 v_2 \\ -D_3 v_1 & -D_3 v_2 & 0 \end{pmatrix}$$

Diseguaglianza di Korn $(u(y_1, y_2, 0) = 0)$

$$\boxed{\int_\Omega \left(\left| \left(u_1, u_2, \frac{u_3}{\varepsilon} \right) \right|^2 + |H^\varepsilon u|^2 \right) dy \leq \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy \leq C}$$

$$W^\varepsilon u^\varepsilon \overset{L^2}{\rightharpoonup} \begin{pmatrix} 0 & -\vartheta & D_3 v_1 \\ \vartheta & 0 & D_3 v_2 \\ -D_3 v_1 & -D_3 v_2 & 0 \end{pmatrix}$$

$$\vartheta \in H^1_\#(\Omega)$$

ϑ **non dipende da** y_1 **e** y_2

Diseguaglianza di Korn $(u(y_1, y_2, 0) = 0)$

$$\boxed{\int_\Omega \left(\left| \left(u_1, u_2, \frac{u_3}{\varepsilon} \right) \right|^2 + |H^\varepsilon u|^2 \right) dy \le \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy \le C}$$

$$E_{33} = D_3 v_3$$

$$-D_2 E_{13} + D_1 E_{23} = D_3 \vartheta$$

Travi a sezione compatta

$$(1/\varepsilon^2)F_\varepsilon$$

$$\downarrow\Gamma$$

$$F(v,\vartheta) = \int_\Omega \left(\frac{E}{2}|D_3 v_3|^2 + \frac{\mu}{2}|D\psi|^2|D_3\vartheta|^2\right) dy +$$

$$- \int_\Omega b \cdot v \, dy - \int_0^\ell m\,\vartheta\, dy_3$$

$$(1/\varepsilon^2)F_\varepsilon$$

$$\downarrow\Gamma$$

$$F(v,\vartheta) = \int_\Omega \left(\frac{E}{2}|D_3v_3|^2 + \frac{\mu}{2}|D\psi|^2|D_3\vartheta|^2\right) dy +$$

$$- \int_\Omega b \cdot v \, dy - \int_0^\ell m \, \vartheta \, dy_3$$

Funzione di stress di Prandtl

$$\begin{cases} \triangle\psi = -2 \\ \psi \in H_0^1(\omega) \end{cases}$$

Travi a sezione compatta

- **Liminf inequality:** $\forall(v,\vartheta)\ \forall u^\varepsilon \subset H^1_\#$ **tale che**

$$(u_1^\varepsilon, u_2^\varepsilon, \frac{u_3^\varepsilon}{\varepsilon}) \overset{H^1}{\rightharpoonup} v$$

$$(W^\varepsilon u^\varepsilon)_{12} \overset{L^2}{\rightharpoonup} -\vartheta$$

$$\Rightarrow \liminf \frac{1}{\varepsilon^2} F_\varepsilon(u^\varepsilon) \geq F(v,\vartheta)$$

- **Recovery sequence:** $\forall(v,\vartheta)\ \exists\ u^\varepsilon \subset H^1_\#$ **tale che**

$$(u_1^\varepsilon, u_2^\varepsilon, \frac{u_3^\varepsilon}{\varepsilon}) \overset{H^1}{\rightharpoonup} v$$

$$(W^\varepsilon u^\varepsilon)_{12} \overset{L^2}{\rightharpoonup} -\vartheta$$

$$\lim \frac{1}{\varepsilon^2} F_\varepsilon(u^\varepsilon) = F(v,\vartheta)$$

Recovery sequence:

$$
\begin{aligned}
u_1^\varepsilon &= \xi_1 - \varepsilon y_2 \vartheta + \varepsilon^2 \frac{\nu}{2}\big(- y_2^2 \xi_1'' + y_1^2 \xi_1'' + 2 y_1 y_2 \xi_2'' \big) - \varepsilon^2 \nu y_1 \xi_3' \\
u_2^\varepsilon &= \xi_2 + \varepsilon y_1 \vartheta + \varepsilon^2 \frac{\nu}{2}\big(- y_1^2 \xi_2'' + y_2^2 \xi_2'' + 2 y_1 y_2 \xi_1'' \big) - \varepsilon^2 \nu y_2 \xi_3' \\
u_3^\varepsilon &= \varepsilon(\xi_3 - y_1 \xi_1' - y_2 \xi_2') + \varepsilon^2 \varphi D_3 \vartheta
\end{aligned}
$$

Recovery sequence:

$$u_1^\varepsilon = \xi_1 - \varepsilon y_2 \vartheta + \varepsilon^2 \frac{\nu}{2}\big(- y_2^2 \xi_1'' + y_1^2 \xi_1'' + 2 y_1 y_2 \xi_2'' \big) - \varepsilon^2 \nu y_1 \xi_3'$$

$$u_2^\varepsilon = \xi_2 + \varepsilon y_1 \vartheta + \varepsilon^2 \frac{\nu}{2}\big(- y_1^2 \xi_2'' + y_2^2 \xi_2'' + 2 y_1 y_2 \xi_1'' \big) - \varepsilon^2 \nu y_2 \xi_3'$$

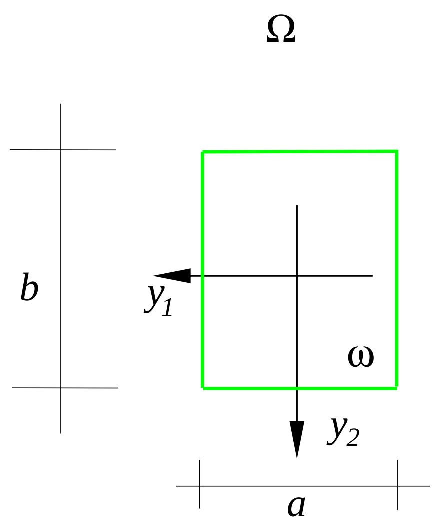
$$u_3^\varepsilon = \varepsilon(\xi_3 - y_1 \xi_1' - y_2 \xi_2') + \varepsilon^2 \varphi D_3 \vartheta$$

Funzione di ingobbamento

$$\begin{cases} D_1 \psi = -D_2 \varphi - y_1 \\ D_2 \psi = D_1 \varphi - y_2 \end{cases}$$

$$F(v,\vartheta) = \int_\Omega \left(\frac{E}{2} |D_3 v_3|^2 + \frac{\mu}{2} |D\psi|^2 |D_3\vartheta|^2 \right) dy + \\ - \int_\Omega b \cdot v \, dy - \int_0^\ell m \, \vartheta \, dy_3$$

Supponiamo la sezione



Travi a sezione compatta

$$F(v,\vartheta)=\int_\Omega\left(\frac{E}{2}|D_3v_3|^2+\frac{\mu}{2}|D\psi|^2|D_3\vartheta|^2\right)dy+ \\ -\int_\Omega b\cdot v\,dy-\int_0^\ell m\,\vartheta\,dy_3$$

$$F(\xi,\vartheta)=\int_0^\ell\frac{E}{2}A\xi_3'^2+\frac{E}{2}J_2\xi_1''^2+\frac{E}{2}J_1\xi_2''^2+\frac{1}{2}\mu\tau(\omega)\vartheta'^2\,dy_3+ \\ -\int_0^\ell\langle b_i\rangle\xi_i-\langle y_\alpha b_3\rangle\xi_\alpha'+m\vartheta\,dy_3$$

Funzione di rigidità torsionale

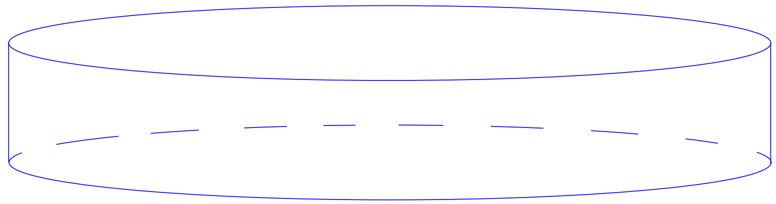
$$\mu\tau(\omega):=\mu\int_\omega|D\psi|^2dy_1\,dy_2$$

Equazioni di Eulero-Lagrange:

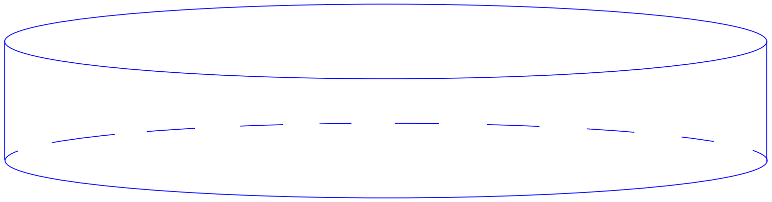
$$\begin{cases} EJ_2\xi_1'''' - \langle b_1 \rangle - \langle y_1 b_3 \rangle' = 0 \\ EJ_1\xi_2'''' - \langle b_2 \rangle - \langle y_2 b_3 \rangle' = 0 \\ EA\xi_3'' + \langle b_3 \rangle = 0 \\ \mu\tau(\omega)\vartheta'' + m = 0 \end{cases}$$

Piastre

Piastre



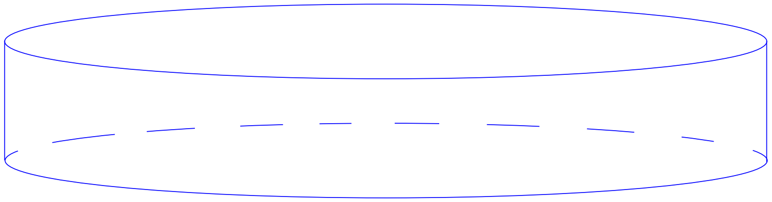
spessore $<<$ diametro



spessore << diametro

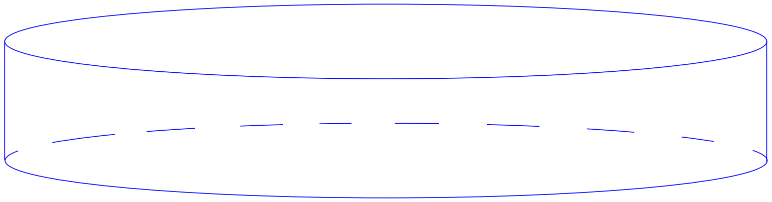
Bourquin, Ciarlet, Geymonat, Raoult (1992)

Anzellotti, Baldo, Percivale (1994)



Energia della piastra

$$\mathcal{F}_{\varepsilon}(u) := \frac{1}{2}\int_{\Omega_{\varepsilon}} \mathbb{C}Eu \cdot Eu\,dx - \int_{\Omega_{\varepsilon}} b^{\varepsilon} \cdot u\,dx$$

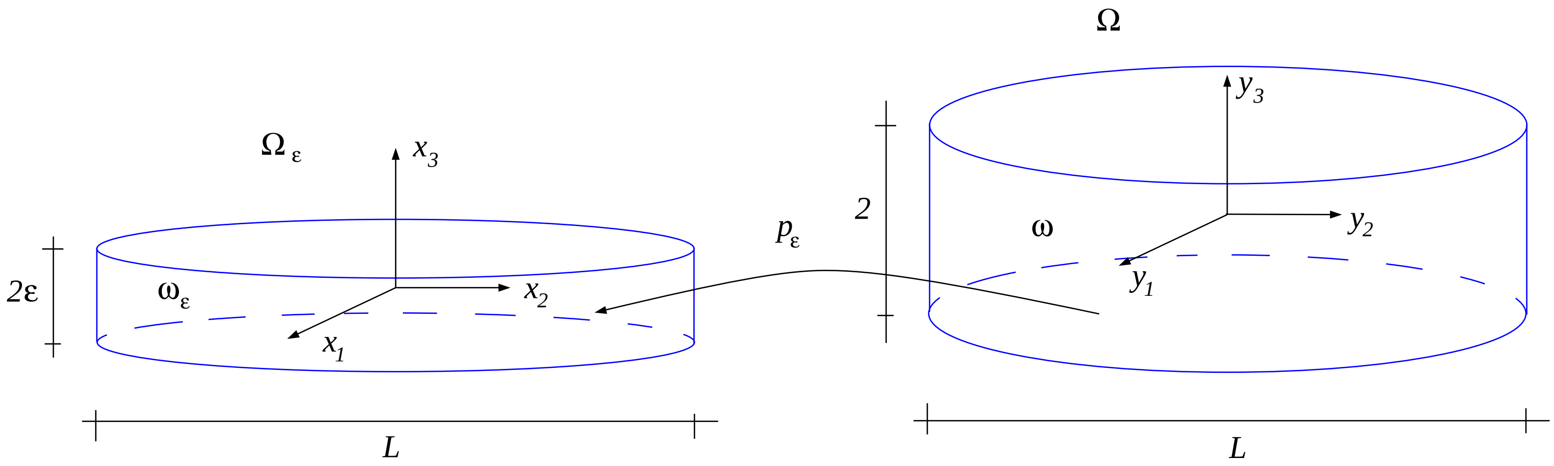


Energia della piastra

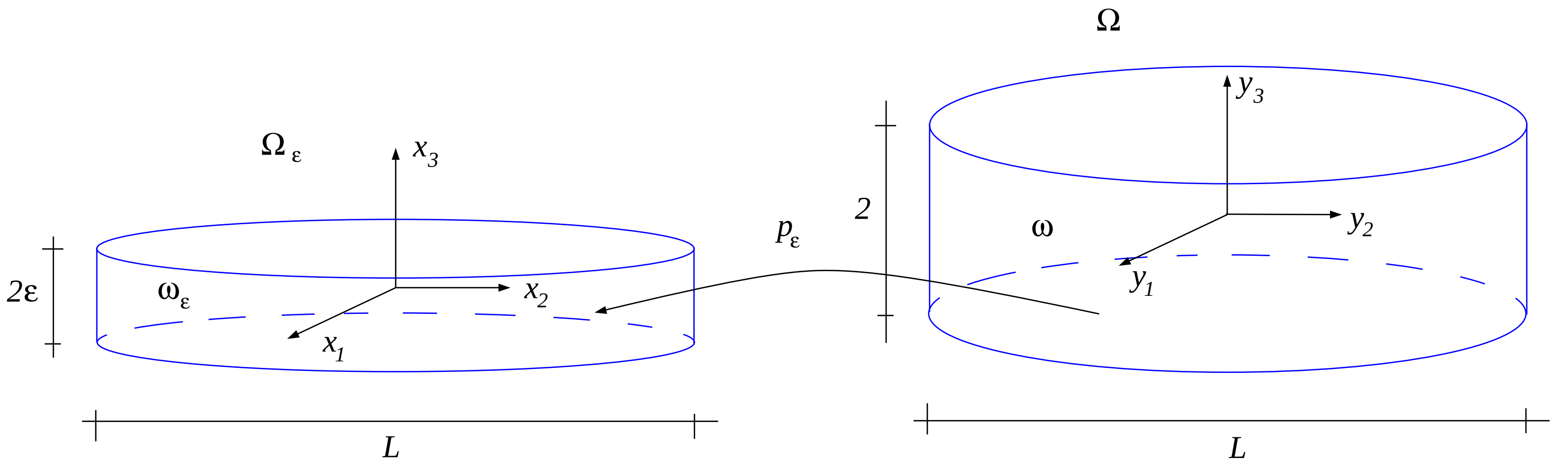
$$\mathcal{F}_\varepsilon(u) := \frac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}Eu \cdot Eu\,dx - \int_{\Omega_\varepsilon} b^\varepsilon \cdot u\,dx$$

$$H^1_\#(\Omega_\varepsilon;\mathbb{R}^3) := \{u \in H^1(\Omega_\varepsilon;\mathbb{R}^3) : u = 0 \textbf{ su } \partial\omega \times (-\varepsilon,\varepsilon)\}$$

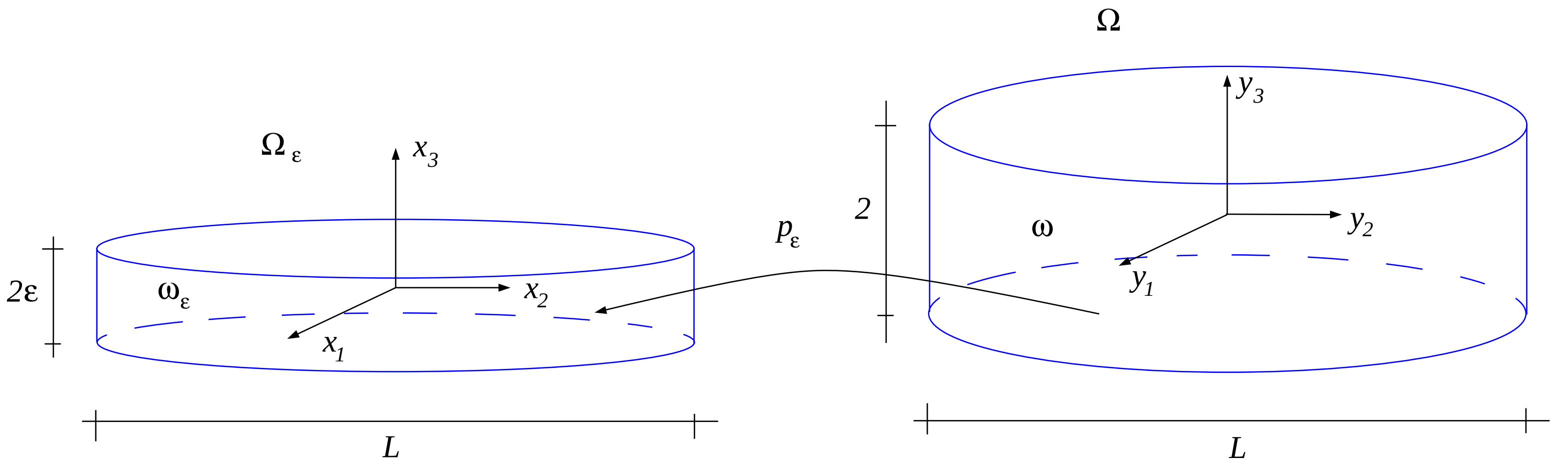
Piastre



Piastre



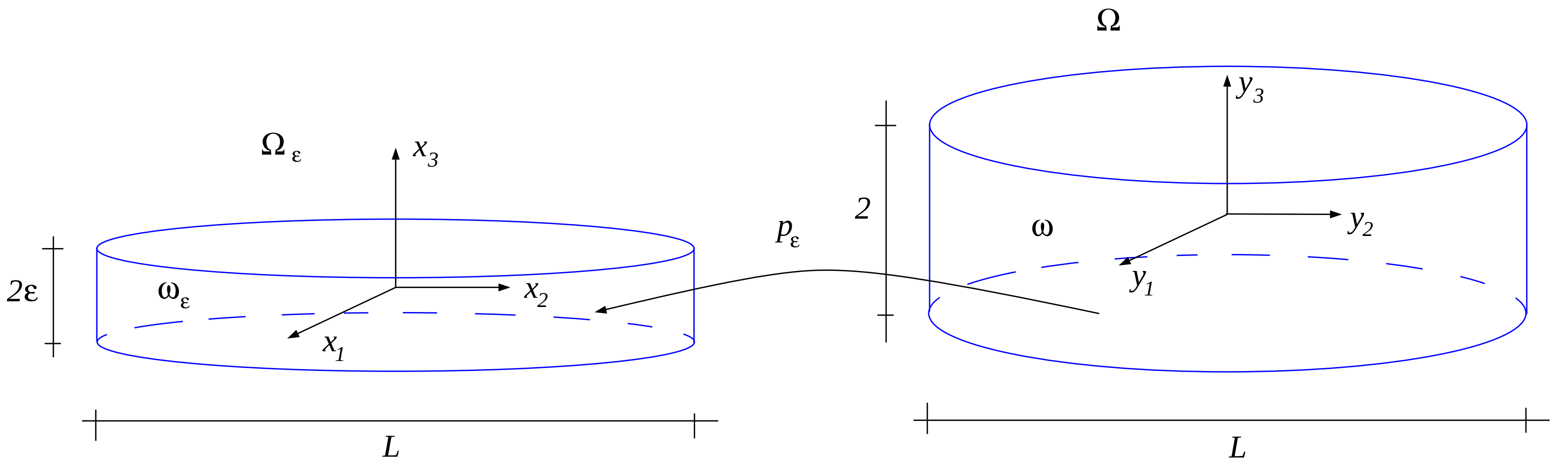
$$u : \Omega_\varepsilon \longrightarrow \mathbb{R}^3 \qquad v : \Omega \longrightarrow \mathbb{R}^3$$



$$u : \Omega_\varepsilon \longrightarrow \mathbb{R}^3 \qquad v : \Omega \longrightarrow \mathbb{R}^3$$

$$v = u \circ p_\varepsilon = u(y_1, y_2, \varepsilon y_3)$$

Piastre

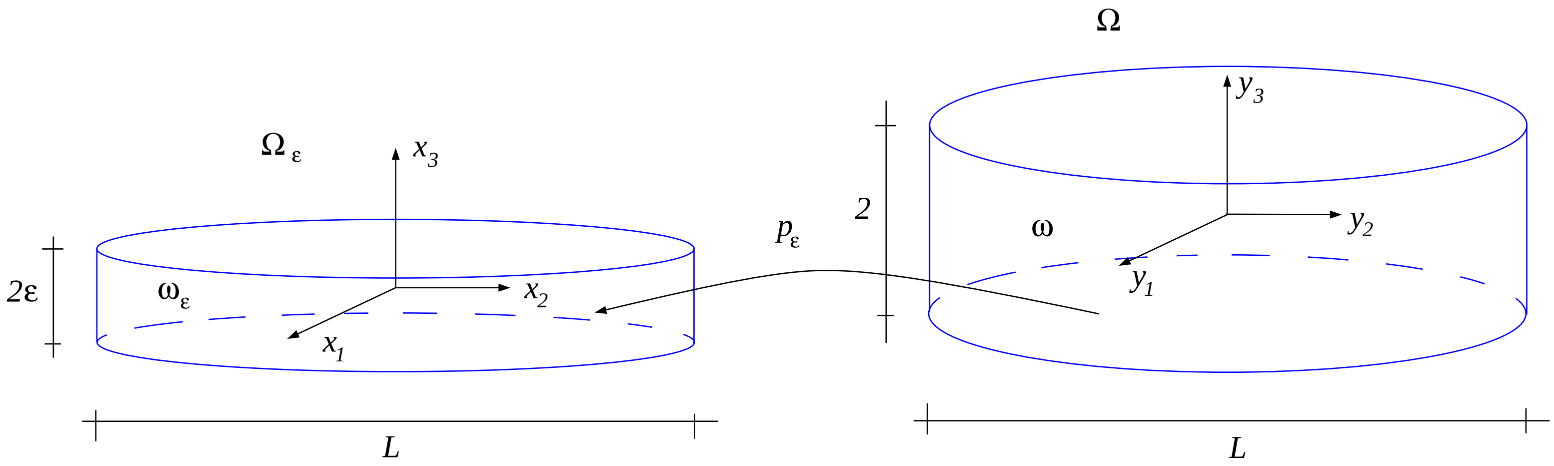


$$u : \Omega_\varepsilon \longrightarrow \mathbb{R}^3 \qquad v : \Omega \longrightarrow \mathbb{R}^3$$

$$v = u \circ p_\varepsilon = u(y_1, y_2, \varepsilon y_3)$$

$$H^\varepsilon v := (D_1 v, D_2 v, \frac{D_3 v}{\varepsilon})$$

$$E^\varepsilon v := \mathbf{sym}(H^\varepsilon v) \qquad W^\varepsilon v := \mathbf{skw}(H^\varepsilon v)$$



Energia riscalata

$$F_\varepsilon(v) = \frac{1}{2}\int_\Omega \mathbb{C}E^\varepsilon v \cdot E^\varepsilon v \, dy - \int_\Omega b^\varepsilon \circ p_\varepsilon \cdot v \, dy,$$

Diseguaglianza di Korn $(u \in H^1_\#)$

$$\int_\Omega \left(\left| \left(\frac{u_1}{\varepsilon}, \frac{u_2}{\varepsilon}, u_3\right) \right|^2 + |H^\varepsilon u|^2 \right) dy \leq \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy$$

Diseguaglianza di Korn $(u \in H^1_\#)$

$$\int_\Omega \left(\left| \left(\frac{u_1}{\varepsilon}, \frac{u_2}{\varepsilon}, u_3 \right) \right|^2 + |H^\varepsilon u|^2 \right) dy \leq \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy \leq C$$

Diseguaglianza di Korn $(u \in H^1_\#)$

$$\boxed{\int_\Omega \Big(\big|(\frac{u_1}{\varepsilon}, \frac{u_2}{\varepsilon}, u_3)\big|^2 + |H^\varepsilon u|^2 \Big) dy \leq \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy \leq C}$$

$$\frac{u_1^\varepsilon}{\varepsilon} \overset{H^1}{\rightharpoonup} v_1$$

$$\frac{u_2^\varepsilon}{\varepsilon} \overset{H^1}{\rightharpoonup} v_2$$

$$u_3^\varepsilon \overset{H^1}{\rightharpoonup} v_3$$

Diseguaglianza di Korn $(u \in H^1_\#)$

$$\int_\Omega \Big(\big| (\frac{u_1}{\varepsilon}, \frac{u_2}{\varepsilon}, u_3) \big|^2 + |H^\varepsilon u|^2 \Big) dy \leq \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy \leq C$$

$$v_3(y) = \xi_3(y_1, y_2) \in H^2_\#$$

$$v_1(y) = \xi_1(y_1, y_2) - y_3 \, \xi_{3,1}(y_1, y_2) \qquad \xi_1 \in H^1_\#$$

$$v_2(y) = \xi_2(y_1, y_2) - y_3 \, \xi_{3,2}(y_1, y_2) \qquad \xi_2 \in H^1_\#$$

Diseguaglianza di Korn $(u \in H^1_\#)$

$$\boxed{\int_\Omega \left(\left| \left(\frac{u_1}{\varepsilon}, \frac{u_2}{\varepsilon}, u_3\right) \right|^2 + |H^\varepsilon u|^2 \right) dy \le \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy \le C}$$

$$E_{11} = D_1 v_1$$

$$E_{22} = D_2 v_2$$

$$E_{12} = \frac{1}{2}(D_1 v_2 + D_2 v_1)$$

$$(1/\varepsilon^2)F_\varepsilon$$

$$\downarrow \Gamma$$

$$F(v) := \int_\Omega \mu(D_1 v_2 + D_2 v_1)^2 + \mu[(D_1 v_1)^2 + (D_2 v_2)^2)]\, dy +$$

$$+ \int_\Omega \frac{\mu\lambda}{2\mu+\lambda}(D_1 v_1 + D_2 v_2)^2\, dy - \int_\Omega b \cdot v\, dy$$

Recovery sequence:

$$u_1^\varepsilon = \varepsilon(\xi_1 - y_3\xi_{3,1})$$

$$u_2^\varepsilon = \varepsilon(\xi_2 - y_3\xi_{3,2})$$

$$u_3^\varepsilon = \xi_3 - \frac{\mu}{2\mu+\lambda}\varepsilon^2(y_3(\xi_{1,1}+\xi_{2,2}) - \frac{1}{2}y_3^2(\xi_{3,11}+\xi_{3,22}))$$

Equazioni di Eulero-Lagrange:

Equazioni di Eulero-Lagrange:

$$\begin{cases} \mu(\xi_{2,12} + \xi_{1,22}) + 2\mu\,\xi_{1,11} + \dfrac{2\mu\lambda}{2\mu + \lambda}(\xi_{1,11} + \xi_{2,21}) + \langle b_1 \rangle = 0 \\ \mu(\xi_{1,21} + \xi_{2,11}) + 2\mu\,\xi_{2,22} + \dfrac{2\mu\lambda}{2\mu + \lambda}(\xi_{2,22} + \xi_{1,12}) + \langle b_2 \rangle = 0 \end{cases}$$

Equazioni di Eulero-Lagrange:

$$\begin{cases} \mu(\xi_{2,12} + \xi_{1,22}) + 2\mu\, \xi_{1,11} + \dfrac{2\mu\lambda}{2\mu + \lambda}(\xi_{1,11} + \xi_{2,21}) + \langle b_1 \rangle = 0 \\ \mu(\xi_{1,21} + \xi_{2,11}) + 2\mu\, \xi_{2,22} + \dfrac{2\mu\lambda}{2\mu + \lambda}(\xi_{2,22} + \xi_{1,12}) + \langle b_2 \rangle = 0 \end{cases}$$

$$\frac{2\mu}{3}(\xi_{3,1221} + \xi_{3,1212} + \xi_{3,2121} + \xi_{3,2112}) + \frac{4\mu}{3}\xi_{3,\alpha\alpha\alpha\alpha} +$$

$$+ \frac{4\mu\lambda}{3(2\mu + \lambda)}\xi_{3,\alpha\alpha\beta\beta} - \langle b_3 \rangle - \langle y_3 b_\alpha \rangle_{,\alpha} = 0$$

Fine