Determining the anisotropic traction state in a membrane by boundary measurements

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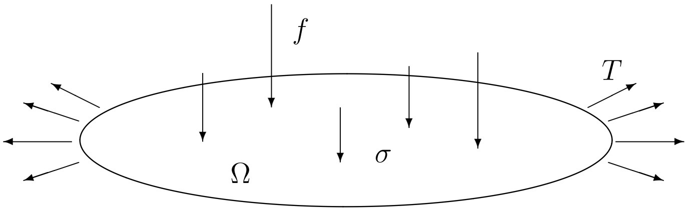
the classical equilibrium problem

Elastic or inextensible membrane $\Omega \subset \mathbf{R}^2$



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- Ω simply connected bounded open set
- $\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ stress tensor
- Assume $\sigma \in L^\infty(\Omega)$ be such that

$$K^{-1}|\xi|^2 \leq \sigma(x)\xi \cdot \xi \leq K|\xi|^2 , \qquad K \geq 1 ,$$

  and equilibrated with stretching forces T applied on $\partial\Omega$, so

$$\begin{cases} \operatorname{div} \sigma = 0 & \text{in } \Omega \\ \sigma\nu = T & \text{on } \partial\Omega \end{cases}$$

- f transverse load

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The transverse displacement u satisfies the boundary value problem (see [2] and [1])

$$\begin{cases} -\operatorname{div}(\sigma\nabla u) = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

Let $f = 0$ in the sequel.

Notice that σ divergencefree means

$$\operatorname{div}(\sigma\xi) = 0 \qquad \forall \xi \in \mathbf{R}^2$$

hence (crucial for a property in Γ-convergence [6])

$$\varphi(x) = \xi \cdot x \quad \text{on } \partial\Omega \quad \Rightarrow \quad u(x) = \xi \cdot x \quad \text{on } \Omega$$

equivalently, the operator is both variational and nonvariational

$$\sum_{ij=1}^{2} D_i(\sigma_{ij} D_j u) = \sum_{ij=1}^{2} \sigma_{ij} D_{ij}^2 u$$

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This model is derived in [2] from the Von Kármán plate

$$\begin{cases} \dfrac{Eh^3}{12(1-\nu)}\triangle^2 u - h\,\mathrm{div}(\sigma\nabla u) = f \\ \mathrm{div}\,\sigma = 0 \\ +b.c. \end{cases}$$

where the first term (longitudinal stress caused by bending) is neglected, it is considered small with respect to the external stretching forces applied on $\partial\Omega$.

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For every Dirichlet datum φ we measure the corresponding load ("Neumann output") on the boundary

$$\Lambda_\sigma \varphi = \sigma \nabla u \cdot \nu \qquad \text{on } \partial\Omega$$

The bounded linear operator $\Lambda_\sigma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is named Dirichlet-to-Neumann (D-N) map.
Weak definition of Λ_σ

$$< \Lambda_\sigma \varphi, v_{|\partial\Omega} > = \int_\Omega \sigma \nabla u \cdot \nabla v \qquad \forall v \in H^1(\Omega)$$

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Problem

Given Λ_σ, find $\sigma \in \Sigma_K$ where
$\Sigma_K = \{\sigma \in L^\infty(\Omega, Sym_{2\times 2}) \mid K^{-1} \leq \sigma \leq K, \ \mathrm{div}\, \sigma = 0\}$

Relevant aspects: uniqueness and, possibly, stability (i.e. continuous dependence).

In general:

- proposed by Calderón in 1980 [8] for a scalar (conductivity isotropic tensor) $\sigma : \Omega \rightarrow (0, +\infty)$, has been extensively studied for $k^{-1} \leq \sigma \leq k$, $k > 1$, see [3], [11] etc.
- for uniqueness, regularity assumptions on σ and $\partial\Omega$ were needed, now relaxed in [5] for the 2D case: $\sigma \in L^\infty$ and $\partial\Omega$ simple curve.
- Material instability may lead to perturbed anisotropic tensors, see [10], in the natural framework of G-convergence. For this reason, assuming additional bounds on the derivatives of σ play an essential rôle for stability, see [3].

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nonuniqueness if σ not isotropic, argument by L. Tartar in [9] for EIT:

Let $\Phi : \bar{\Omega} \to \bar{\Omega}$ be a C^∞ diffeomorphism such that $\Phi_{|\partial\Omega} = identity$

Apply the change of variables $v = u \circ \Phi^{-1}$ to the energy

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In [4] the converse: $\Lambda_{\tilde{\sigma}} = \Lambda_\sigma \Rightarrow \sigma, \tilde{\sigma}$ related by some Φ

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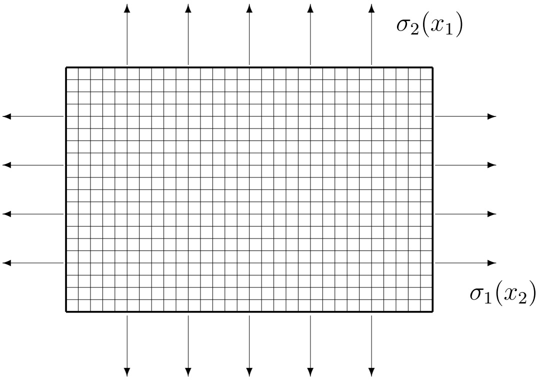
an explicit example

Explicit identification for a rectangular network, a portion of fabrics (see [7] for an optimal traction design problem)



In this case the traction tensor field takes the form

$$\sigma = \begin{pmatrix} \sigma_{11}(x_2) & 0 \\ 0 & \sigma_{22}(x_1) \end{pmatrix} = (\sigma_1(x_2), \sigma_2(x_1)), \quad x_1, x_2 \in [0,1]$$

Given $\Lambda_\sigma(\varphi) = \sigma \nabla u \cdot \nu = \psi \;\; \forall \varphi$ on $\partial\Omega$, take φ linear

$$\varphi(x) = \xi \cdot x \;\Rightarrow\; u(x) = \xi \cdot x$$

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choose $\xi = (1,0)$ on $x_1 = 1$, $\nu = (1,0)$ and $\psi_\xi(x) = \psi(x_2)$, then

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similar argument for the other edges.

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Theorem

Λ_σ uniquely determines σ among all tensors in Σ_K

We work with $\sigma_{ij}, \partial\Omega \in C^\infty$, but our result carries over to the case when $\sigma \in L^\infty(\Omega)$ and $\partial\Omega$ arbitrary closed curve.

- pushforward of σ by $\Phi : \bar{\Omega} \to \bar{D}$

$$T_\Phi \sigma(y) = \frac{\nabla\Phi\sigma\nabla^T\Phi}{\det \nabla\Phi}(\Phi^{-1}(y)) , \qquad y \in D$$

- $T_\Phi\sigma$ is still symmetric and elliptic with a (possibly new) constant $K \geq 1$
- T_σ preserves the bilinear Dirichlet form associated to σ, i.e.

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Theorem

Λ_σ uniquely determines σ among all tensors in Σ_K

We work with $\sigma_{ij}, \partial\Omega \in C^\infty$, but our result carries over to the case when $\sigma \in L^\infty(\Omega)$ and $\partial\Omega$ arbitrary closed curve.

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Lemma

If $\operatorname{div}\sigma = 0$ *in* Ω *then* $\operatorname{div}(T_\Phi \sigma \nabla^T \Phi^{-1}) = 0$ *in* D

Proof.

Put $u = x_r$ in the previous bilinear form

$$\int_\Omega \sigma_{rj} D_j v = \int_D \frac{\Phi_{i,h}\sigma_{hk}\Phi_{j,k}}{\det \nabla\Phi} \circ \Phi^{-1} D_i \Phi_r^{-1}(y) D_j v(\Phi^{-1}(y))$$

If

$$\int_\Omega \sigma_{rj} D_j v = 0 \quad \forall v \in H_0^1(\Omega)$$

the same holds true for the right hand side

$$\int_D T_\Phi \sigma(\Phi^{-1}(y))_{ij} D_i \Phi_r^{-1}(y) D_j v = 0$$

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If $\operatorname{div} T_\Phi \sigma = 0$ *in* D *for some* Φ, *then* $\operatorname{div}(\sigma \nabla^T \Phi) = 0$ *in* Ω

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From [11] and [12] Λ_σ determines uniquely the class

$$\{T_\Phi \sigma \mid \Phi : \bar{\Omega} \to \bar{\Omega} \text{ is a diffeomorphism such that } \Phi_{\partial\Omega} = I\}$$

which contains at most one divergencefree element.

In fact, if $\operatorname{div} T_\Phi \sigma = 0$ for some diffeomorphism Φ which fixes the boundary, by the II lemma

$$\begin{cases} \operatorname{div}(\sigma \nabla \Phi^T) = 0 & \text{in } \Omega \\ \Phi = I & \text{on } \partial\Omega \end{cases}$$

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Theorem

The mapping $\sigma \to \Lambda_\sigma$ from Σ_K to $\mathscr{L}(H^{1/2}(\Omega), H^{-1/2}(\Omega))$ has a continuous inverse when Σ_K is endowed with G-convergence topology

Proof.

Σ_K closed, hence G-compact.
Let $\Lambda_{\sigma_h}, \Lambda_\sigma$ be D-N maps such that $\|\Lambda_{\sigma_h} - \Lambda_\sigma\| \to 0$
By compactness (see [6]) $\exists \sigma_{r_h} \stackrel{G}{\to} \sigma' \in \Sigma_K$ and we prove $\sigma' = \sigma$
$\forall \varphi \in H^{1/2}(\partial\Omega)$ Dirichlet datum, let u_h, u' solutions with σ_h, σ'

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This result explains previous *instability* results ([10] and others).
For isotropic conductivity problems, stability fails because when σ is known to be isotropic any tensor can be G-approximated by isotropic tensors. Hence stability for isotropic $\Rightarrow$ uniqueness in its G-closure, in contradiction with Tartar's example.

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G-convergence on Σ_K equivalent to $L^\infty(\Omega)$-weak* convergence.
Hence, previous theorem $\Rightarrow \forall \psi \in L^1(\Omega)$

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If $\psi = 1/|\Omega|$ we get the average, which depends Lipschitz on on $D - N$ map.

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$\forall \sigma, \sigma' \in \Sigma$ we have

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Proof.

$u = x_i$, $v = x_j$

$$\int_\Omega (\sigma - \sigma')_{ij} = < (\Lambda_\sigma - \Lambda_{\sigma'}) x_i, x_j >$$

□

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