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**Passing to the limit in thin cylinders in linear elasticity**

In this lecture, I will first report on joint work with Ali Sili (see [1]). We consider a linearly elastic thin cylinder $\Omega^\varepsilon = \varepsilon\omega \times (0, L)$ of basis $\varepsilon\omega$ (where $\omega \subset \mathbb{R}^2$ is given) and of given length $L$. The cylinder is fixed on its lower and upper bases, or only on one of them, and is subject to body forces and to other forces (including surface forces). As usual for such problems, we reconduct ourselves to the fixed geometry $\Omega = \omega \times (0, L)$ by a change of horizontal variables (an homothety), which leads us to the singular perturbation problem

$$(E^\varepsilon) \qquad \begin{cases} \displaystyle\int_\Omega A e^\varepsilon(u^\varepsilon) e^\varepsilon(\varphi) = \int_\Omega f\varphi + \int_\Omega h e^\varepsilon(\varphi), \\ u^\varepsilon \in (H^1_b(\Omega))^3, \quad \forall \varphi \in (H^1_b(\Omega))^3, \end{cases}$$

where $A$ is a very strong elliptic fourth-order tensor with $L^\infty(\Omega)$ coefficients, where $f$ belongs to $(L^2(\Omega))^3$ and $h$ to $(L^2(\Omega))^{3\times 3}$, where $H^1_b(\Omega)$ denotes the space

$$H^1_b(\Omega) = \{\varphi \in H^1(\Omega) \; : \; \varphi(x', 0) = \varphi(x', L) = 0, \; x' \in \omega\}$$

when the cylinder is fixed on its lower and upper bases, or the space

$$H^1_b(\Omega) = \{\varphi \in H^1(\Omega) \; : \; \varphi(x', L) = 0, \; x' \in \omega\}$$

when the cylinder is fixed only on its upper base, and where $e^\varepsilon(\varphi)$ denotes the tensor

$$e^\varepsilon(\varphi) = \begin{pmatrix} \frac{1}{\varepsilon^2} e_{\alpha\beta}(\varphi) & \frac{1}{\varepsilon} e_{\alpha 3}(\varphi) \\ \\ \frac{1}{\varepsilon} e_{\alpha 3}(\varphi) & e_{33}(\varphi) \end{pmatrix}, \quad \alpha, \beta = 1, 2.$$

In [1] we pass to the limit in $(E^\varepsilon)$ when $\varepsilon$ tends to zero and prove that the limit problem is given by

$(E)$

$$\begin{cases} \displaystyle\int_\Omega A \begin{pmatrix} e_{\alpha\beta}(w) & e_{\alpha 3}(v) \\ e_{\alpha 3}(v) & e_{33}(u) \end{pmatrix} \begin{pmatrix} e_{\alpha\beta}(\overline{w}) & e_{\alpha 3}(\overline{v}) \\ e_{\alpha 3}(\overline{v}) & e_{33}(\overline{u}) \end{pmatrix} = \int_\Omega f\overline{u} + \int_\Omega h \begin{pmatrix} e_{\alpha\beta}(\overline{w}) & e_{\alpha 3}(\overline{v}) \\ e_{\alpha 3}(\overline{v}) & e_{33}(\overline{u}) \end{pmatrix}, \\ (u, v, w) \in BN_b(\Omega) \times R_b(\Omega) \times RD_2^\perp(\Omega), \quad \forall (\overline{u}, \overline{v}, \overline{w}) \in BN_b(\Omega) \times R_b(\Omega) \times RD_2^\perp(\Omega), \end{cases}$$

where $BN_b(\Omega)$, $R_b(\Omega)$, and $RD_2^\perp(\Omega)$ respectively denote the spaces of the Bernouilli-Navier displacements, of the $2-d$ infinitesimal rotations, and of the space orthogonal to the $2-d$ rigid displacements. We moreover prove that $u^\varepsilon$ behaves like $u + \varepsilon v + \varepsilon^2 w$, in the sense that

$\|e^\varepsilon(u^\varepsilon) - e^\varepsilon(u + \varepsilon v + \varepsilon^2 w)\|_{(L^2(\Omega))^{3\times 3}}$ tends to zero when $v$ and $w$ are sufficiently smooth (i.e. belong to $(H^1(\Omega))^3$). This is a corrector result, which, making the inverse change of horizontal variables, allows us to describe the solution of the original problem posed in $\Omega^\varepsilon = \varepsilon\omega \times (0, L)$.

In a joint work [4] with Régis Monneau and Ali Sili, we further give an error estimate for this difference $\|e^\varepsilon(u^\varepsilon) - e^\varepsilon(u + \varepsilon v + \varepsilon^2 w)\|_{(L^2(\Omega))^{3\times 3}}$. The proof of this error estimate uses a new "partial Korn inequality", where "partial" refers to the fact that the difference between any $z$ and its projection on the Bernouilli-Navier displacements is controlled in the dual space of the space $H^1((0, L); L^2(\omega))$ by the sum in $\alpha, \beta = 1, 2$ of $\|e^\varepsilon_{\alpha\beta}(z)\|_{L^2(\Omega)}$ and $\|e^\varepsilon_{\alpha 3}(z)\|_{L^2(\Omega)}$.

The limit problem $(E)$ is a system of three systems of partial differential equations with three unknowns $u$, $v$, and $w$, in which we can eliminate $v$ and $w$ in order to obtain a reduced system of equations with the sole unknown $u$. When the fourth-order tensor $A$ is both anisotropic and heterogeneous, and when the cylinder is fixed on both its lower and upper bases, this reduced system is not in general a standard system of partial differential equations, since nonlocal terms can appear (see [2]).

Finally I will briefly describe a joint work [3] with Antonio Gaudiello, Régis Monneau, Jacqueline Mossino, and Ali Sili, in which we consider in linear elasticity the junction problem between a horizontal plate with fixed cross section and small thickness $h^\varepsilon$, and a vertical beam with fixed height and small cross section of radius $r^\varepsilon$, when the lateral boundary of the plate and the upper basis of the beam are fixed. When $h^\varepsilon$ and $r^\varepsilon$ tend to zero simultaneously with $(h^\varepsilon)^3 = q\,(h^\varepsilon)^2$ for some $0 \leq q \leq +\infty$, we identify the limit problem and prove that it involves six junction conditions.


[1] F. Murat & A. Sili,
Comportement asymptotique des solutions du système de l'élasticité linéarisée anisotrope hétérogène dans des cylindres minces.
C. R. Acad. Sci. Paris, Série I, 328, (1999), pp. 179-184.

[2] F. Murat & A. Sili,
Effets non locaux dans le passage $3d-1d$ en élasticité linéarisée anisotrope hétérogène.
C. R. Acad. Sci. Paris, Série I, 330, (2000), pp. 745-750.

[3] A. Gaudiello, R. Monneau, J. Mossino, F. Murat & A. Sili,
On the junction of elastic plates and beams.
C. R. Acad. Sci. Paris, Série I, 335, (2002), pp. 717-722.

4] R. Monneau, F. Murat & A. Sili,
Error estimates for the $3d-1d$ dimension reduction in anisotropic heterogeneous linearized elasticity.
To appear.