A network-decentralised strategy for shortest-path-flow routing

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***Abstract*— To control the flow in a dynamical network where the nodes are associated with buffer variables and the arcs with controlled flows, we consider a network-decentralised strategy such that each arc controller makes its decision exclusively based on local information about the levels of the buffers that it connects. We seek a flow control law that asymptotically minimises a cost specified in terms of a weighted L_1-norm. This approach has the advantage of providing a solution that is generally sparse, because it uses a limited number of controlled flows. In particular, in the presence of a resource demand applied on a single node, the asymptotic flow is concentrated along the shortest path.**

I. Introduction and Motivation

We consider a class of systems of the form

$$\begin{cases} \dot{x}(t) = Bu(t) - d \\ u(t) \in \mathbb{U} := \left\{ u \in \mathbb{R}^m : u^- \leq u \leq u^+ \right\}, \ \forall t\,, \end{cases} \tag{1}$$

where the inequalities hold component-wise, $x \in \mathbb{R}^n$ is the vector of buffer levels, $u \in \mathbb{R}^m$ is the vector of controlled flows, $B \in \mathbb{R}^{n\times m}$ is an assigned matrix that captures the topology of the interconnections among buffers and $d \in \mathbb{R}^n$ is an external, unknown constant demand. We seek a stabilising control strategy that is:

- *Network decentralised*: each control component u_k exploits local information only, i.e., depends on the state components x_i corresponding to nonzero entries B_{ik} of the k-th column of B, and it is independent of d.
- *Asymptotically optimal in the weighted L_1-norm*: the flows converge to a vector $\bar{u}$ such that

$$\bar{u} \in \left\{ \begin{array}{ll} \underset{u\in\mathbb{U}}{\operatorname{argmin}} & \sum_{k=1}^{m} \gamma_k \, |u_k| \\ \text{s.t.} & Bu = d. \end{array} \right. \tag{2}$$

Example 1: The source node 1 in Fig. 1 receives an external supply flow, while an outgoing demand is applied to the sink node 8. The arc controllers are network-decentralised; e.g., u_{10} only knows the buffer levels at nodes 5 and 6, hence $u_{10} = \Phi_{10}(x_5, x_6, \gamma_{10})$, where γ_{10} is a weight associated with arc 10. We wish to direct the whole steady-state flow (compatibly with other constraints on the network) through

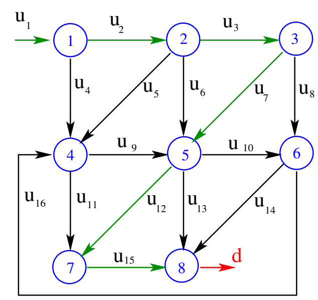

Fig. 1. The network in Example 1. Green arcs identify the shortest (or, more in general, minimum-cost) path from the source node to the sink node.

the shortest source-to-sink path, i.e., the path through nodes 1-2-3-5-7-8, in a decentralised way. Each arc outside this path may be initially activated to match the demand in the transient, but shall spontaneously drive its flow to 0, based on the *knowledge of the buffer levels at its extreme nodes only*, while each arc in the optimal path shall drive its flow to a steady-state value equal to the *unknown* demand d. □

Pioneering work on network-decentralised control [11], [12] has been later reconsidered in [2], [4], [5]. Important recent contributions about flow control include [8], [13], [14], [15]. As shown in [3], a saturated network-decentralised control $u = \text{sat}[-B^\top x]$ is asymptotically optimal in the L_2-norm. Extensions to more general classes of smooth and *strictly convex* functionals have been proposed in [7].

Minimising the L_2-norm "evenly distributes" the network flow. Contrarily, here we aim at concentrating the flow along a single path, preferably the shortest one, which can be identified by solving a linear programming problem, whose solution is typically sparse when the optimal point is unique (a generically satisfied condition, which holds with probability 1 for randomly generated instances).

Remark 1: Sparse solutions are fundamental when the network is subject to strong demands concentrated on a single node: if the flow reaches this node through the shortest path, only a subset of the arcs are permanently activated. □

The contributions of this paper are summarized next.

- Due to the lack of differentiability and strict convexity, we cannot directly apply the solution in [7]. First, we need to regularise the cost $\gamma|u|$ as $\gamma|u| + \delta u^2/2$, where $\delta > 0$ is small enough. We provide an upper bound for δ to ensure that the asymptotic optimal (sparse) solution coincides with that of the original linear problem.
- By assuming integer data and uniqueness of the solution, we show that δ can be determined based exclusively on upper bounds on the network size.
- We show that the proposed control law can satisfy flow demands concentrated at a sink node by directing the

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flow along the shortest source-to-sink path (when allowed by the arc flow constraints) and it is also resilient to modifications of the network topology (due to, e.g., failures or introduction of new network components).
- Since high values of the demand may lead to transient shortage, we propose also a different control strategy that considers the overall storage level in the network.

II. Problem Statement

Consider a directed network, represented as a graph $\mathcal{G} := (\mathcal{N}, \mathcal{A})$ where $\mathcal{N}$ is the set of nodes of cardinality n and $\mathcal{A}$ is set of arcs of cardinality m. We write $k = (i, j)$ when arc $k \in \mathcal{A}$ has initial node $i \in \mathcal{N}$ and final node $j \in \mathcal{N}$. We assume that the graph is strongly connected and also externally connected (i.e., there exists at least one arc whose initial node is the *external environment*; see, e.g., the arc associated with u_1 in Fig. 1). The nodes are associated with buffers, while the arcs with controlled flows, and an unknown uncontrolled flow demand d is applied to some nodes. We associate with each arc k a controller that regulates the intensity of the arc flow u_k by adopting a network-decentralised strategy [3], [5], [6], i.e., exclusively based on the buffer levels x_i, x_j at the extreme nodes i and j of arc k. We wish to asymptotically convey the overall flow (when possible, in the presence of flow constraints) through the path associated with minimum length, or time, or cost in general, and to match as much as possible the demand in the transient state. The control problem is the following.

Problem 1: Find a network-decentralised control law

$$u_k = \Phi_k(x_i, x_j, \gamma_k), \text{ with } k = (i, j), \; k \in \mathcal{A}, i, j \in \mathcal{N},$$

such that $\lim_{t\to\infty} u(t) = \bar{u}$, where $\bar{u}$ is a solution to (2). □

The next assumption guarantees the existence of a feasible solution to Problem 1 [7].

Assumption 1: Matrix B has full row rank, and there exists $u^0 \in \text{int}(\mathbb{U})$ for which $Bu^0 = d$. □

We denote by $\mathcal{U}$ the polyhedron of feasible solutions to (2) and by $\overline{\mathcal{U}}$ the polyhedron of the optimal solutions. Assumption 1 implies that $\mathcal{U}$ is a full dimensional polyhedron and that $\overline{\mathcal{U}}$ is not empty: $\overline{\mathcal{U}}$ is a singleton if (2) has a unique optimal solution, otherwise $\overline{\mathcal{U}}$ may include infinite elements.

Remark 2: In the instances of problem (2) considered herein, B is the generalised incidence matrix [10] associated with graph $\mathcal{G}$, hence its entries belong to $\{-1, 0, 1\}$. However, the theory developed here applies to any matrix $B \in \mathbb{R}^{n\times m}$. In general, the control u_k can be a function only of the state components x_i such that $B_{ik} \neq 0$. □

We stress that Problem 1 includes, but is more general than, the shortest-path problem discussed in Example 1. Indeed, given the demand d, assume that the flow u is bounded from below to be non negative, i.e., $u_k^- = 0$ for all $k \in \mathcal{A}$, but not upper bounded, i.e., $u_k^+ = +\infty$ for all $k \in \mathcal{A}$. Then, we make the following claims.

Claim 1: If $\mathcal{G}$ has a single external supply arc, and hence the demand vector d has a single non-zero entry $d_i^+ > 0$, then $\overline{\mathcal{U}}$ includes the solution to the shortest-path problem. □

More in general, the following result holds true.

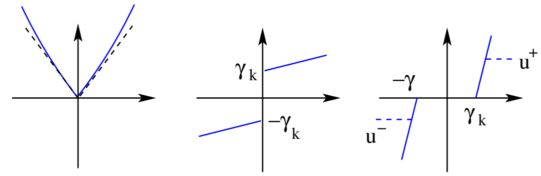

Fig. 2. Function f_k (left), its derivative g_k (center), and the inverse ϕ_k of the derivative (right) along with its saturation, dashed.

Claim 2: If $\mathcal{G}$ has several external supply arcs, the demand vector d may have more than one non-zero positive entry, and the costs γ_k correspond to distances or transit time, then the optimal solution to (2) corresponds to the solution that minimises the sum of traveled distances of the unit flow elements (packets, vehicles, unit volume of water...) or the overall permanence time in the network. □

Next, we determine an optimal control strategy.

III. A Regularised Cost Yields the True Optimum

The cost function in (2) is neither differentiable nor strictly convex. We regularise it as (Fig. 2, left):

$$J(u) = \sum_{k\in\mathcal{A}} f_k(u_k) := \sum_{k\in\mathcal{A}} \left(\gamma_k \; |u_k| + \frac{1}{2}\delta u_k^2\right),$$

for some $\delta > 0$. The function f_k is still not differentiable, but it is strongly convex. By considering $J(u)$ and under suitable assumptions, we show next that the proposed control law $u(t)$ converges to an optimal solution $\bar{u}$ to (2).

The generalised derivative of f_k (Fig. 2, center), $g_k(u_k) = \frac{df_k(u_k)}{du_k} = \gamma_k \text{sgn}(u_k) + \delta u_k$, has a discontinuity at zero and is invertible in $\mathbb{R} \setminus [-\gamma_k, \gamma_k]$. We extend g_k^{-1} to the whole $\mathbb{R}$ by defining the *dead-zone function* (Fig. 2, right):

$$\phi_k(\xi) := \begin{cases} \frac{\xi-\gamma_k}{\delta}, & \text{if} \quad \xi > \gamma_k, \\ 0, & \text{if} \quad |\xi| \leq \gamma_k, \\ \frac{\xi+\gamma_k}{\delta}, & \text{if} \quad \xi < -\gamma_k. \end{cases}$$

Moreover, we introduce the (component-wise) *saturation function* $\text{sat}_{[u^-,u^+]} : \mathbb{R}^m \to \mathbb{R}^m$ as

$$\text{sat}_{[u^-,u^+]}(y_k) := \begin{cases} u_k^-, & \text{if} \quad y_k < u_k^-, \\ y_k, & \text{if} \quad u_k^- \leq y_k \leq u_k^+, \\ u_k^+, & \text{if} \quad y_k > u_k^+. \end{cases}$$

We can then apply the network-decentralised strategy [7]:

$$u(t) = \text{sat}\left[\phi\left(-B^\top x(t)\right)\right], \tag{3}$$

where we alleviate the notation by omitting the subscripts u^- and u^+ denoting the saturation extrema.

Remark 3: For $\delta = 0$, the functions g_k are not invertible, and hence (3) could not be defined without regularisation. □

A control based on a dead-zone function has a strong practical motivation. First, it prevents a control from being active for small flow values, which leads to the sparsity of the solution. Also, when the buffer levels have similar value at steady state, small inaccuracies in the measurements would induce unnecessary circulations. For example, given Fig. 1, assume that the strategy in (3) is applied: consider arcs 9, 10 and 16, and let $\gamma_9 = \gamma_{10} = \gamma_{16} = 0$. By assuming a lack of accuracy that induces the sensor placed at arc 9 (respectively 10, 16) to overestimate the value of state x_4 (respectively x_5, x_6), a circulation would be triggered when $x_4 = x_5 = x_6$.

A. Convergence of the state

In this subsection, we show the convergence of the control (3) to the solution of the optimisation problem

$$\min_{u \in \mathbb{U}} J(u) \quad \text{s.t. } Bu = d. \tag{4}$$

We also show that the state is bounded. The next lemma characterises the optimal solution in terms of steady state.

Lemma 1 ([7]): The vector u^* is an optimal solution to (4) if and only if $u^* = \text{sat}[\phi(-B^\top x^*)]$, for some x^*, and

$$Bu^* = B\text{sat}[\phi(-B^\top x^*)] = d.$$

□

Proposition 1: The solution $x(t)$ of system (1) with the control (3) is bounded and converges to the non-empty set

$$\Xi := \left\{\xi \in \mathbb{R}^n : B\text{sat}\left[\phi(-B^\top \xi)\right] = d\right\},$$

while $u(t)$ converges to the solution u^* of (4). □

Proof: (Cf. [7]). First, Lemma 1 guarantees that Ξ is not empty. Then, consider $x^* \in \Xi$ and define $z(t) := x(t) - x^*$. Given (1) and (3), we have $\dot{z}(t) = B\text{sat}\left[\phi\left(-B^\top(x^* + z(t))\right)\right] - d = B\left(\text{sat}\left[\phi\left(-B^\top(x^* + z(t))\right)\right] - \text{sat}\left[\phi\left(-B^\top x^*\right)\right]\right) = -B\Delta(z(t))B^\top z(t)$, where $\Delta(z(t))$ is a diagonal matrix of nonnegative functions, which always exists because: a) both sat$(\cdot)$ and $\phi(\cdot)$ are non-decreasing, Lipschitz functions; b) their composition is non-decreasing and Lipschitz as well; c) for any non-decreasing Lipschitz function f, $f(p+q) - f(p) = ([f(p+q) - f(p)]/q)q = \delta(q)q$, $\delta(q) \geq 0$.

Now, consider $V(z) := \frac{1}{2}z^\top z$ as a candidate Lyapunov function. Its Lyapunov derivative is $\dot{V}(z(t)) = -z(t)^\top B\Delta(z(t))B^\top z(t) \leq 0$, which ensures that z is bounded and, due to LaSalle's principle, converges to the set where $\dot{V}(z) = -z^\top B\Delta(z)B^\top z = 0$. Since, for any symmetric positive (or negative) semidefinite matrix S, $z^\top Sz = 0$ if and only if $Sz = 0$, $z(t)$ converges to the set $\mathcal{Z} := \{z \in \mathbb{R}^n : B\Delta(z)B^\top z = 0\} = \{z \in \mathbb{R}^n : B\text{sat}\left[\phi\left(-B^\top(x^* + z)\right)\right] = d\}$, hence $x(t)$ converges to the set Ξ and $u(t)$ to u^*. ■

Remark 4: The limit solution achieved for $\delta \to 0$, i.e.,

$$u_k(-B_k^\top x) = \begin{cases} u_k^-, & \text{if} \quad -B_k^\top x < -\gamma_k, \\ 0, & \text{if} \quad -\gamma_k \leq -B_k^\top x \leq \gamma_k, \\ u_k^+, & \text{if} \quad -B_k^\top x > \gamma_k, \end{cases}$$

would lead to discontinuity and possibly chattering. Our goal, i.e., $u(t) \to \bar{u}$, would not be achieved with $\delta = 0$. □

B. Sub-optimality and sparse solution

Hereinafter, we denote by u^* the optimal solution to (4) when $\delta \neq 0$ and by $\bar{u}$ an optimal solution to (4) when $\delta = 0$, i.e., $\bar{u}$ is an optimal solution to (2). In addition, we always assume that $u^- = 0$. If this is not the case, we can double any arc k with $u_k^- < 0$ into two different arcs k' and k'', where k' has the same direction as k and flow $0 \leq u_{k'} \leq u_k^+$, k'' has the opposite direction of k and flow $0 \leq u_{k''} \leq -u_k^-$.

In general, $u^* \neq \bar{u}$. Here we determine sufficient conditions on the value of δ to guarantee $u^* = \bar{u}$.

Suppose that u^* is sub-optimal for (4) when $\delta = 0$. Then, we can provide an upper bound proportional to δ to the associated loss of optimality. We consider the *degradation index* $I_{\text{deg}} := \sum_{k \in \mathcal{A}} \gamma_k \left(|u_k^*| - |\bar{u}_k|\right) \geq 0$.

Proposition 2: Let $\omega = \max_{k \in \mathcal{A}} \max\{|u_k^-|, |u_k^+|\}$. Then, $I_{\text{deg}} \leq \frac{m\omega^2}{2}\delta$. □

Proof: Since $J(u^*) \leq J(\bar{u})$ in view of the optimality of u^*, we have: $I_{\text{deg}} = \sum_{k \in \mathcal{A}} \gamma_k |u_k^*| - \sum_{k \in \mathcal{A}} \gamma_k |\bar{u}_k| \leq \underbrace{\sum_{k \in \mathcal{A}} \left(\gamma_k |u_k^*| + \frac{1}{2}\delta(u_k^*)^2\right)}_{J(u^*)} - \sum_{k \in \mathcal{A}} \gamma_k |\bar{u}_k| \leq \underbrace{\sum_{k \in \mathcal{A}} \left(\gamma_k |\bar{u}_k| + \frac{1}{2}\delta(\bar{u}_k)^2\right)}_{J(\bar{u})} - \sum_{k \in \mathcal{A}} \gamma_k |\bar{u}_k| \leq \delta \sum_{k \in \mathcal{A}} \frac{1}{2}\omega^2$. ■

$\overline{\mathcal{U}}$ is the polyhedron of the optimal solutions to (4) also when $\delta = 0$. The proposition above can be used to prove that the optimal solution u^* of (4) satisfies $\lim_{\delta \to 0} u^*(\delta) \in \overline{\mathcal{U}}$. From now on, $\bar{u}$ denotes the optimal solution in $\overline{\mathcal{U}}$ such that $\bar{u} = \lim_{\delta \to 0} u^*(\delta)$.

A stronger result than the above limit may hold: if δ belongs to a sufficiently small right neighborhood of 0, we have $u^* = \bar{u}$. To this aim, we need additional assumptions.

Assumption 2: (a) d has a single nonzero component, $d_1 = d^+$; (b) $\overline{\mathcal{U}}$ is a singleton, i.e., $\bar{u}$ is the unique optimal solution to (4) when $\delta = 0$; (c) the network capacities and costs are such that the non-zero components of $\bar{u}$ describe a single shortest path from the source node to the sink node: $0 = u^- \leq d^+ \leq u^+$. □

Under Assumption 2, we can rewrite (4) with $\delta = 0$ as

$$\min_{u \in \mathbb{R}^m} \gamma^\top u \quad \text{s.t. } Bu = d, \ 0 \leq u \leq d^+\bar{1}, \tag{5}$$

where $\gamma^\top = [\gamma_1 \ \gamma_2 \ \ldots \ \gamma_m]$ and $\bar{1}^\top = [1 \ 1 \ \ldots \ 1]$. In addition, the components of the optimal solution $\bar{u}$ correspond either to arcs with zero flow, or to arcs with flow equal to d^+. As the arcs with $\bar{u}_k = d^+$ identify the shortest path, their number cannot be greater than the number n of nodes. Now, define $\Pi \in \mathbb{R}^{m \times m}$ as the diagonal matrix with components $\Pi_{kk} = 1$ if $\bar{u}_k = 0$ and $\Pi_{kk} = -1$ if $\bar{u}_k = d^+$. The set of all feasible solutions to (5) can be parameterised as $\bar{u} + v$ with $0 \leq \bar{u} + v \leq d^+\bar{1}$, where $v \in \ker(B)$. For $v = 0$ we have the optimal. The admissible infinitesimal variations of the solution to preserve feasibility are in the cone $\mathcal{C} = \{v \in \ker(B) \mid \Pi v \geq 0\}$ (only positive variations if $\bar{u}_i = 0$, and only negative variations if $\bar{u}_i = d^+$). Introduce the generator matrix C for the cone $\mathcal{C}$:

$$\mathcal{C} = \{v \in \ker(B) \mid v = Cw, \ w \geq 0\}. \tag{6}$$

Matrix C can be determined with an approach similar to the one used to compute the reduced costs in (5) as we will describe in Section III-C. Since we assume that (5) has a unique optimal solution $\bar{u}$, $\gamma^\top Cw > 0$ must hold for any $w \geq 0$. This condition in turn implies $\gamma^\top C > 0$.

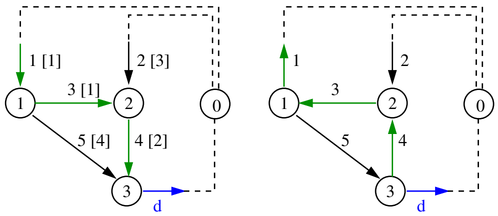

Fig. 3. Super-graph $\bar{\mathcal{G}}$ and an optimal flow $\bar{u}$ along the shortest path (left) and residual graph $\mathcal{R}(\bar{u})$ associated with the optimal flow $\bar{u}$ (right). Costs are in square brackets.

We can now provide a sufficient condition on the value of δ such that $u^* = \bar{u}$ in terms of the entries of C. We observe that a generic cost variation $J(u)$ in (4) is $\Delta J = J(u) - J(\bar{u}) = \gamma^\top u + (\delta/2)\|u\|_2^2 - \gamma^\top \bar{u} - (\delta/2)\|\bar{u}\|_2^2 = \gamma^\top(\bar{u}+v) + (\delta/2)\|\bar{u}+v\|_2^2 - \gamma^\top\bar{u} - (\delta/2)\|\bar{u}\|_2^2 = \gamma^\top v + \delta\bar{u}^\top v + (\delta/2)\|v\|_2^2 = [\gamma + \delta\bar{u} + (\delta/2)v]^\top Cw,\ w \geq 0$. Therefore $J(\bar{u})$ is the optimal cost for (4) if:

$$(\gamma + \delta\bar{u})^\top C > 0. \tag{7}$$

As $\bar{u}$ has at most n non-zero components, all equal to d^+, a sufficient condition for (7) to hold is $\gamma^\top C + \delta n d^+ \underline{C}\bar{1} > 0$, where $\underline{C}$ is the element of matrix C with minimum value.

The above condition implies $u^* = \bar{u}$ for any $\delta \geq 0$ if $\underline{C} \geq 0$. This situation occurs, for example, when the graph $\mathcal{G}$ is a tree. Differently, if $\underline{C} < 0$, the above condition guarantees that $u^* = \bar{u}$ at least when

$$\delta < \frac{\min_r\{[\gamma^\top C]_r\}}{-nd^+\underline{C}}, \tag{8}$$

where $[\gamma^\top C]_r$ is r-th component of vector $\gamma^\top C$.

When the demand vector d has more than one non-zero component, the previous arguments should be repeated by considering all the non-zero demands d_k acting on the nodes. The intrinsic limit of bound (8) is that knowledge of the network structure is needed and a global re-computation must take place at any change or variation (due, e.g., to an arc failure). We deal with this problem next.

C. A decentralised choice of δ

In this subsection, we understand that Assumptions 1 and 2 hold together with the following assumption.

Assumption 3: Vectors u^+ and γ have integer entries. □

We consider the *super-graph* $\bar{\mathcal{G}}$ as the graph derived from $\mathcal{G}$ by adding an external node, hereinafter referred as node 0, which represents the external environment. In the super-graph $\bar{\mathcal{G}}$, any arc entering or leaving the original graph $\mathcal{G}$ is not "floating", but is connected with node 0. See, e.g., Fig. 3. In view of Assumption 2, we know that the optimal solution $\bar{u}$ to (5) is unique and that we can consider the network capacity constraints to be $0 \leq u \leq d^+\bar{1}$.

Then, we define the residual graph $\mathcal{R}(\bar{u})$ associated with the flow $\bar{u}$ [1]: $\mathcal{R}(\bar{u})$ includes the same nodes and arcs as $\bar{\mathcal{G}}$. In addition, each arc of $\mathcal{R}(\bar{u})$ has the same capacity as the corresponding arc of $\bar{\mathcal{G}}$, and they have:

- opposite direction if $\bar{u}_k = d^+$,
- the same direction if $\bar{u}_k = 0$.

It follows that a feasible flow $\bar{u}+v$ in $\bar{\mathcal{G}}$ is associated with any feasible flow v in the residual graph $\mathcal{R}(\bar{u})$.

As an example, given the super-graph $\bar{\mathcal{G}}$ depicted on the left in Fig. 3 with a flow $\bar{u}$ along the shortest path, which is the one connecting nodes 1-2-3, the corresponding residual graph $\mathcal{R}(\bar{u})$ is the one on the right in Fig. 3.

Define a circulation in the residual graph as a flow along an oriented cycle of $\mathcal{R}(\bar{u})$. Let the vector c represent the associated cycle as follows. For each arc k in $\mathcal{R}(\bar{u})$

$$c_k := \begin{cases} 1 & \text{if } k \text{ is in the cycle with same orientation in } \mathcal{R}(\bar{u}) \text{ and } \bar{\mathcal{G}}, \\ -1 & \text{if } k \text{ is in the cycle with opposite orientation in } \mathcal{R}(\bar{u}) \text{ and } \bar{\mathcal{G}}, \\ 0 & \text{if } k \text{ does not belong to the cycle.} \end{cases}$$

For example, the vectors associated with the two circulations 1-0-2-1 and 1-3-2-1 in the residual graph $\mathcal{R}(\bar{u})$ of Fig. 3 are $[-1\ 1\ -1\ 0\ 0]^\top$ and $[0\ 0\ -1\ -1\ 1]^\top$, respectively.

Theorem 1 ([1]): A flow $\bar{u}$ is optimal if and only if the residual graph has no circulation with negative cost: $\gamma^\top c \geq 0$ for any circulation. Moreover, any other non-optimal flow is $u = \bar{u} + v$ and has cost $\gamma^\top u = \gamma^\top \bar{u} + \gamma^\top v = \gamma^\top \bar{u} + \sum_k \gamma_k c_k w_k,\ \ w_k \geq 0$, i.e., vector v is a positive combination of the circulations. □

As a consequence [1], the solution is strict if and only if the circulation cost is strictly positive: $\gamma^\top c > 0$ for any circulation in the residual graph. It turns out that the cone in (6) is generated by circulations. Then if the entries of γ are integers, and $\bar{u}$ is strictly optimal, the vector $\gamma^\top C$ has positive integer components that are all lower bounded by 1, and condition (8) is implied by the stronger condition:

$$\delta < \frac{1}{nd^+}. \tag{9}$$

Therefore we provide a bound for δ that does not depend on the network structure.

Proposition 3: Under Assumptions 1-3, let $\bar{u}$ be the optimal solution to (4) with $\delta = 0$. Then, $\bar{u}$ is optimal for the same problem when $\delta \neq 0$, if condition (9) holds. □

Since Assumption 2 is quite strong, let us now drop it and compute again the value of ΔJ for $u \in \mathcal{U} \setminus \overline{\mathcal{U}}$, that is, for u that can be expressed as $u = \bar{u} + \hat{v}$, where $\hat{v}$ is any feasible flow in $\mathcal{R}(\bar{u})$ such that $\gamma^\top \bar{u} \neq 0$. Let $\hat{C}$ be the generating matrix of the flows $\hat{v}$. Then, for $u \in \mathcal{U} \setminus \overline{\mathcal{U}}$, we have $\Delta J = J(u) - J(\bar{u}) = \gamma^\top u + (\delta/2)\|u\|_2^2 - \gamma^\top \bar{u} - (\delta/2)\|\bar{u}\|^2 = (\gamma^\top u - \gamma^\top \bar{u}) + (\delta/2)(\|u\|_2^2 - \|\bar{u}\|_2^2) = \gamma^\top \hat{v} + \delta\bar{u}^\top\hat{v} + (\delta/2)\|\hat{v}\|_2^2 = \gamma^\top\hat{v} + \delta\bar{u}^\top\hat{v} = [\gamma + \delta\bar{u}]^\top \hat{C}\hat{w}$, $\hat{w} \geq 0$. Assumption 3 guarantees that $\gamma^\top \hat{C} \geq 1$. Also, the components of the vector $\bar{u}^\top\hat{C}$ can be lower bounded by $-md^+$. Indeed, all entries of $\hat{C}$ are in the set $\{-1, 0, 1\}$ and the bounds u^+ on the arc capacities may force all the entries of $\bar{u}$ to be different from 0.

Hence, we can conclude that $\Delta J > 0$, for $u \in \mathcal{U} \setminus \overline{\mathcal{U}}$, if

$$[\gamma + \delta\bar{u}]^\top \hat{C} \geq (1 - md^+\delta)\bar{1}^\top \ \Rightarrow\ \delta < \frac{1}{md^+}. \tag{10}$$

The optimal solution to (4) cannot belong to $\mathcal{U} \setminus \overline{\mathcal{U}}$ when condition (10) holds, as formalised next.

Proposition 4: Under Assumptions 1 and 3, let $\overline{\mathcal{U}}$ be the polyhedron of the optimal solutions to (4) with $\delta = 0$. Then,

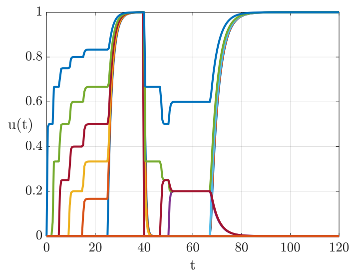

Fig. 4. The time evolution of the arc flows u_i in the graph in Fig. 1 according to our dynamic algorithm spontaneously leads to the exact minimum cost solution. At time $T = 40$, arc 7 undergoes a failure.

if condition (10) holds, $\overline{\mathcal{U}}$ includes the optimal solution to the same problem when $\delta \neq 0$. □

D. Robustness, multiple demands and shortest path

The proposed control provides sparse flow solutions. If

a) d has a single nonzero component $d_1 = d^+$,
b) B is a generalised incidence matrix,
c) $u^- = 0$ (positivity constraint),

then the optimal solution asymptotically converges to the flow corresponding to the shortest path from the source node to the sink node *as long as this flow does not saturate the capacity of any arc*. If we consider larger values of the demand such that some link saturates, then the flow is automatically re-distributed to other arcs. The proposed control scheme is then able to dynamically determine the shortest path in a decentralised fashion, without explicitly resorting to standard shortest path algorithms, such as Dijkstra's.

The proposed flow algorithm has a different nature with respect to primal shortest path algorithms. No information about the distance to the sink node is available, not even asymptotically, at the nodes, which ignore the identity of the sink. Moreover, the algorithm is robust to failures. If at some point an arc fails, the flow is re-directed to a different path (the new optimal one), achieving a resilient network [9].

Example 2: Reconsider the problem in Example 1, along with Fig. 1. If the weights are $\gamma = [1\ 1\ 1\ 3\ 3\ 3\ 1\ 3\ 3\ 3\ 3\ 1\ 3\ 3\ 1\ 3]^\top$, and the demand is $d_8 = 2.5$, the exact solution to the optimisation problem is: $\bar{u} = [1\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0]^\top$, which corresponds the shortest path. We apply the proposed dynamic flow algorithm with $\delta = 0.1$: the corresponding time evolution of u is shown in Fig. 4, and the asymptotic value is identical to $\bar{u}$. Indeed, asymptotically only u_1, u_2, u_3, u_7, u_{12} and u_{15} are equal to 2.5, while all the other flows converge to zero. At time $T = 40$, arc 7 undergoes a failure. The flow is then redirected. The new path crosses the arcs 1, 2, 6, 12 and 15, corresponding to new (optimal) $\bar{u} = [1\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0]^\top$. □

IV. Transient Optimality

We now analyse the behaviour of the proposed decentralised strategy in the transient, before the steady-state is reached. We have seen that our algorithm, in the presence of concentrated demands, supplies the flow along the shortest path. Clearly, highly concentrated demands may create a network shortage (or excess) in terms of stored goods.

To consider optimality in the transient, we introduce an auxiliary scalar variable $\xi(t) := \eta^\top x(t) - \rho$, where $\eta \in \mathbb{R}^n$ is a weighting vector and ρ is a constant, with dynamics

$$\dot{\xi}(t) = \eta^\top \dot{x}(t) = \eta^\top (Bu(t) - d). \tag{11}$$

The variable $\eta^\top x(t)$ is the weighted sum of buffer levels, whose desired value is ρ, so that $\xi = 0$ is the target level for ξ. For instance, if $\eta = [1\ 1\ \dots\ 1]^\top$, then $\xi = \eta^\top x - \rho = 0$ means that the total amount stored in the buffers is ρ. Without restriction (since it can always be achieved by translating the reference) we assume $\rho = 0$. If a deviation occurs, due to an accidental event (flood), then suddenly $\xi \neq 0$, and the controls should restore the zero condition as soon as possible. This can be done by the control

$$u_{bb}(t) = \text{ext}\left[-B^\top \eta \xi(t)\right], \tag{12}$$

where the *extremum* function ext$[\cdot]$ is defined as

$$\text{ext}[v_i] := \begin{cases} u_i^+, & \text{if} \quad v_i \geq 0, \\ u_i^-, & \text{if} \quad v_i < 0. \end{cases}$$

Theorem 2: The control (12) ensures that the target $\xi = 0$ is monotonically reached in minimum time, and minimises $\int_0^\infty |\xi(t)|dt$, as well as $\int_0^\infty |\xi(t)|^2 dt$. □

Proof: Consider the candidate Lyapunov function $V(\xi) := \xi^2$. Its time derivative is $\dot{V} = 2\xi\dot{\xi} = 2\xi\eta^\top\dot{x} = 2\xi\eta^\top(Bu - d)$. Consider the case $\xi(0) > 0$ (an analogous reasoning holds in the case $\xi(0) < 0$). The control (12) minimises the Lyapunov derivative: $\min_{u \in \mathbb{U}} 2\xi\eta^\top(Bu - d) = 2\xi\eta^\top(Bu_{bb} - d) < 0$, where the negative sign depends on Assumption 1, since $\eta^\top(B\bar{u} - d) = 0$ and $\bar{u} \in \text{int}(\mathbb{U})$.

Also, u_{bb} is constant as long as $\xi(t) > 0$ and does not depend on $x(t)$. Hence, $\xi(t)$ converges to 0 with constant negative slope $\eta^\top(Bu_{bb} - d)$, $\xi_{opt}(t) = \xi(0) + \eta^\top(Bu_{bb} - d)\,t$, and in finite time $T(\xi(0)) = -\frac{\xi(0)}{\eta^\top(Bu_{bb} - d)} > 0$. Any other choice of u leads to a trajectory such that $\xi(t) \geq \xi_{opt}(t)$, hence optimality follows. ■

The dynamics in (11) can be added to the original system (1) and the above theory applies to the extended system

$$\begin{cases} \dot{\xi}(t) = \eta^\top \dot{x}(t) = \eta^\top (Bu(t) - d) \\ \dot{x}(t) = Bu(t) - d \end{cases} \tag{13}$$

The resulting control is

$$u(t) = \text{sat}\left[\phi\left(-B^\top x(t) - B^\top \eta \xi(t)\right)\right]. \tag{14}$$

By scaling $\eta = \alpha\eta_0$ for some $\alpha > 0$ large enough, the dominant portion of the control is $u(t) \approx \text{sat}\left[\phi\left(-B^\top \alpha\eta_0 \xi(t)\right)\right]$, which, for $\alpha \to \infty$, converges point-wise (or in L_1-norm) to the extremum function: sat$[\phi_k(-\alpha v)] \to$ ext$[v]$. Therefore, the control compromises the buffer and the links performances. The implementation of the technique requires that the information about the variable $\xi(t)$ is communicated to all local controllers. To analyse convergence we perform a change of variables and we consider the new state vector

$$\begin{bmatrix} \omega(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \xi(t) - \eta^\top x(t) \\ (I + \eta\eta^\top)x(t) \end{bmatrix}.$$

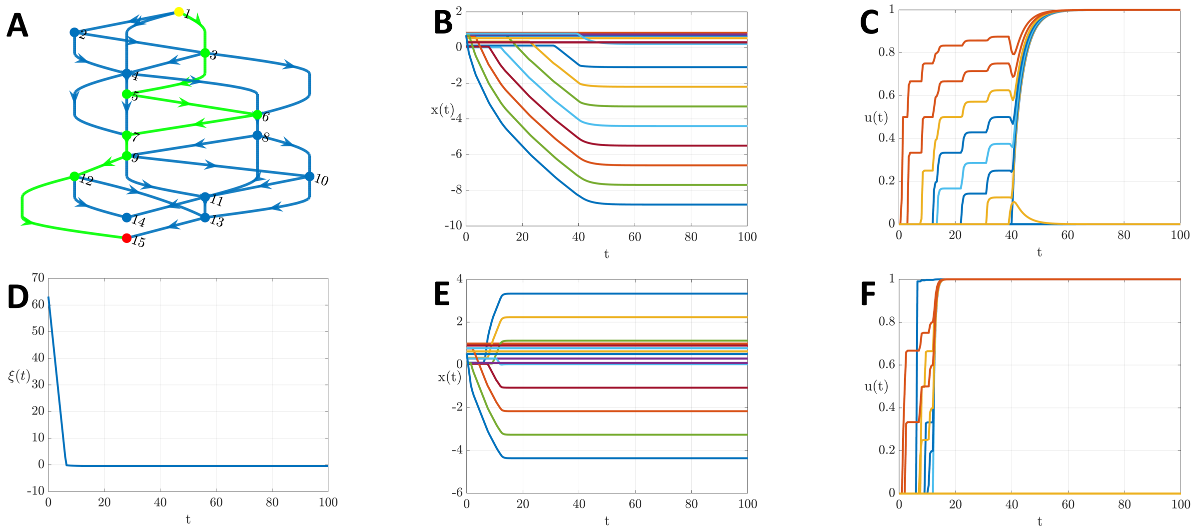

Fig. 5. A: Graph with 15 nodes and 30 arcs. Green arcs identify the shortest (minimum-cost) path from the source node (yellow) to the sink node (red). The arrows on the arcs represent the direction of the flow. Evolution of state (B) and arc flows (C) for system (1) with control strategy (3). Evolution of auxiliary state variable (D), state (E) and arc flows (F) for system (13) with control strategy (14) (η scaled with $\alpha = 10$).

The first component ω obeys $\dot{\omega}(t) = \dot{\xi}(t) - \eta^\top \dot{x}(t) = 0$, and therefore $\omega(t) = \bar{\omega} = 0$. So we need to analyse the behaviour of x. If we initialise $\xi(0)$ so that $\omega(0) = \bar{\omega} = 0$, namely as $\xi(0) = \eta^\top x(0)$, the variable $y(t)$ evolves according to $\dot{y}(t) = (I+\eta\eta^\top)\left[B\text{sat}\left[\phi\left(-B^\top(I+\eta\eta^\top)x\right)\right]-d\right]$. Introduce $\hat{B} = (I+\eta\eta^\top)B$ and $\hat{d} = (I+\eta\eta^\top)d$ to get a system exactly in the previous form $\dot{y}(t) = \hat{B}\text{sat}\left[\phi\left(-\hat{B}^\top y\right)\right] - \hat{d}$. Hence stability is ensured by adopting $V(y) = y^\top y/2$.

Remark 5: In the x space, the level curves of the Lyapunov function $y^\top y/2 = x^\top(I+\eta\eta^\top)^2 x$ are squashed in the direction η, hence are similar to a "flat ellipsoid" with minor axis directed as η: this explains the behaviour of x. $\square$

V. Numerical simulations

We consider the network in Fig. 5A and compare the behaviour of the original system (1) with the control (3), and the behaviour of the augmented system (13) with the control (14). As expected, the transient behaviour of the two systems is considerably different, both for the state $x(t)$ and the control input $u(t)$; as shown in Fig. 5D, the auxiliary variable $\xi(t) \to 0$. However, even though the values achieved as $x(t) \to \infty$ are different (see Fig. 5B and E), both control strategies correctly activate the arcs associated with the shortest path (see Fig. 5C and F). As evidenced in Fig. 5F, the effect of the correction term $-B^\top\eta\xi$ in the control (14) not only does not compromise the convergence of $u(t) \to \bar{u}$, but speeds up the transient with respect to control (3).

VI. Conclusion

We have proposed a network-decentralised flow control strategy capable of directing the flow along the shortest path. This strategy has the advantage that, in the presence of large demands concentrated in a single node, the supply follows the fastest route. This strategy is resilient since, in case of failure of some arcs, the flow is dynamically redirected along the new shortest path. We have also shown that a suitable modification of this control keeps the overall amount of resource stored in the network close to a target level.

References

[1] R K. Ahuja, T L. Magnanti and J.B. Orlin, "Network flow", Prentice Hall, Upper Saddle River, New Iersey, 1993.

[2] F. Blanchini, S. Miani, and W. Ukovich, "Control of production-distribution systems with unknown inputs and system failures", *IEEE Trans. Automatic Control*, 45(6):1072–1081, 2000.

[3] D. Bauso, F. Blanchini, L. Giarré and R. Pesenti, "The linear saturated control strategy for constrained flow control is asymptotically optimal," *Automatica*, 49(7):2206–2212, 2013.

[4] F. Blanchini, E. Franco and G. Giordano, "Structured-LMI conditions for stabilizing network-decentralized control," *Proc. IEEE Conf. Dec. Control*, pp. 6880–6885, 2013.

[5] F. Blanchini, E. Franco and G. Giordano, "Network-decentralized control strategies for stabilization," *IEEE Trans. Automatic Control*, 60(2):491–496, 2015.

[6] F. Blanchini, E. Franco, G. Giordano, V. Mardanlou, and P. L. Montessoro, "Compartmental flow control: decentralization, robustness and optimality" *Automatica*, 64(2):18–28, 2016

[7] F. Blanchini, D. Casagrande, F. Fabiani, G. Giordano and R. Pesenti, "Network-decentralised optimisation and control: an explicit saturated solution", *Automatica*, 103:379-389, 2019.

[8] M. Bürger and C. De Persis, "Dynamic coupling design for nonlinear output agreement and time-varying flow control", *Automatica*, 51:210–222, 2015.

[9] G. Como, "On resilient control of dynamical flow networks", *Annual Reviews in Control*, 43:80–90, 2017.

[10] G. Giordano, F. Blanchini, E. Franco, V. Mardanlou, P. L. Montessoro, "The smallest eigenvalue of the generalized Laplacian matrix, with application to network-decentralized estimation for homogeneous systems", *IEEE Trans. Network Science Eng.*, 3(4), pp. 312-324, 2016.

[11] A. Iftar, "A linear programming based decentralized routing controller for congested highways", *Automatica*, 35(2):279–292, 1999.

[12] A. Iftar and E. J. Davison, "Decentralized control strategies for dynamic routing", *Optim. Contr. Appl. Met.*, 23(6):329–355, 2002.

[13] T. Scholten, C. De Persis, and P. Tesi, "Optimal steady state regulation of distribution networks with input and flow constraints", *Proc. American Control Conference*, pp. 6953–6958, 2016.

[14] S. Trip, T. Scholten and C. De Persis, "Optimal regulation of flow networks with transient constraints", *Automatica* ,104:141–153, 2019.

[15] J. Wei and A. J. van der Schaft, "Load balancing of dynamical distribution networks with flow constraints and unknown in/outflows", *Systems & Control Letters*, 62(11):1001–1008, 2013.