Topology-Independent Robust Stability of Homogeneous Dynamic Networks ⋆

Franco Blanchini * **Daniele Casagrande** ** **Giulia Giordano** *** **Umberto Viaro** **

* *Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Università degli Studi di Udine, Via delle Scienze 206, 33100 Udine, Italy* blanchini@uniud.it.
** *Dipartimento Politecnico di Ingegneria e Architettura, Università degli Studi di Udine, Via delle Scienze 206, 33100 Udine, Italy* {daniele.casagrande,viaro}@uniud.it.
*** *Department of Automatic Control LTH and LCCC Linnaeus Center, Lund University, Box 118, SE 221 00 Lund, Sweden* giulia.giordano@control.lth.se.

Abstract: The paper presents conditions for the stability of a dynamical network described by a directed graph, whose nodes represent dynamical systems characterised by the same transfer function $F(s)$ and whose edges account for the interactions between pairs of nodes. In turn, these interactions depend via a transference $G(s)$ on the outputs of the subsystems associated with the connected nodes. The stability conditions are topology-independent, in that they hold for all possible connections of the nodes, and robust, in that they allow for uncertainties in the determination of the transferences. Two types of interactions are considered: bidirectional and unidirectional. In the first case, if nodes i and j are connected, both node i affects node j and node j affects node i, while in the second case only one of the two occurrences is admitted. The robust stability conditions are expressed as constraints for the Nyquist diagram of $H = FG$.

Keywords: Dynamical networks, Directed graphs, Feedback, Robust stability, Nyquist diagram.

1. INTRODUCTION

Recently, considerable attention has been paid to the analysis of dynamical networks consisting of a number of equal subsystems interacting dynamically with one another according to a similar mechanism. Indeed, models of this kind prove quite effective in describing a number of biological, economic, mathematical, chemical, and artificial distributed problems. Fairly ample and updated bibliographies on this subject are provided by Golovin et al. (2008), Cao et al. (2013), Nicolaides et al. (2015), Green and Sharpe (2015), Le Novère (2015), Giordano (2016). The application of dynamical networks is now spreading rapidly in diverse areas (cf., e.g., Jadbabaie et al. 2003; Smith and Hadaegh 2007; Del Vecchio et al. 2008; Wang et al. 2014) following a rather long latency, after the seminal papers by Turing (1952) on the chemical basis of morphogenesis (pattern formation) and Wolpert (1969) on positional information and the spacial pattern of cellular differentiation. Interesting early attempts at studying rigorously the dynamics of pattern formation are due to Gierer and Meinhardt (1972) and, for lateral-inhibition type homogeneous neural fields with general connections, to Amari (1977). Nor should we forget the fundamental book by Nicolis and Prigogine (1977) on self-organising systems. The subject has been lately reconsidered in more mathematical terms in many papers. Suffice it to recall the contribution of Golovin et al. (2008), Arcak (2011, 2013), Lin et al (2016), Lestas and Vinnicombe (2006), Pates and Vinnicombe (2012). The last two works, in particular, provide Nyquist–like conditions (the so–called "stability certificates") that guarantee the stability of the entire network by satisfying *local* rules involving each agent and the dynamics of its neighbours, so that these conditions *scale* with the network size. A number of interesting issues of biochemical networks relevant to the control engineering perspective are illustrated by Wolkenhauer et al. (2004). The present contribution has been stimulated by the last–mentioned papers as well as by Blanchini et al. (2015, 2016); Hori et al (2015); Miyazako et al. (2014), which apply concepts and tools of systems and control theory, such as feedback, decentralisation, stabilisation, root loci, linearisation and harmonic balance, to the stabilising control of decentralised systems and to the analysis of coordinated spacial pattern formation of biomolecular networks.

This paper focuses on homogeneous dynamical networks represented by directed graphs, whose nodes correspond to equal linear dynamic systems that are influenced by *flow variables* associated with the arcs connecting every node with its adjacent nodes. In turn, these flows depend dynamically, yet linearly, on the variables characterising the nodes, i.e., their outputs. The aim of the paper is to derive conditions that ensure the *robust* stability of the overall network *independently of its size and connectivity.* To this purpose, both node and arc transferences are

⋆ G.G. acknowledges support from the Swedish Research Council through the LCCC Linnaeus Center and the eLLIIT Excellence Center at Lund University.

considered to be uncertain, the only available information being the maximum number of arcs that may leave or enter a node.

2. DEFINITIONS AND PROBLEM STATEMENT

A *directed graph* with n nodes and m arcs is the ordered pair $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ is the set of the *nodes* with cardinality $|\mathcal{N}| = n$ and $\mathcal{A} \subset \mathcal{N} \times \mathcal{N}$ is the set of the *arcs* with cardinality $|\mathcal{A}| = m$. Each arc is identified by a pair of nodes (i, j), with $i, j \in \mathcal{N}$ and $i \neq j$. It is assumed that two nodes are connected at most by one arc, so that $(i, j) \in \mathcal{A} \Rightarrow (j, i) \notin \mathcal{A}$. Two nodes $i, j \in \mathcal{N}$ are *adjacent* if they are connected by an arc, namely, if either $(i, j) \in \mathcal{A}$ or $(j, i) \in \mathcal{A}$. An arc $(i, j) \in \mathcal{A}$ is *incident* in both nodes i and j. For each node $i \in \mathcal{N}$, $\mathcal{L}_i \subseteq \mathcal{A}$ denotes the set of all the arcs that are incident in i, i.e.,

$$\mathcal{L}_i = \{(h, k) \in \mathcal{A} : \text{either } h = i \text{ or } k = i\}. \tag{1}$$

The *degree* of node i is the cardinality of $\mathcal{L}_i$. The *maximum connectivity degree* of the graph, denoted by $\mathcal{M}$, is the maximum of the degrees of its nodes. It is assumed that the graph is *connected*: given any pair of nodes $i, j \in \mathcal{N}$, there always exists a path, formed by a sequence of adjacent nodes, connecting node i to node j.

The dynamic behaviour of the graph is characterised by *scalar* variables associated with its nodes and arcs. Precisely, $y_i(t)$, $i = 1, \ldots, n$, denote the variable characterising node i and $u_h(t)$, $h = 1, \ldots, m$, the variable characterising arc h, where for notational simplicity a single index is used to identify arcs (e.g., according to a lexicographic order). Typically, node variables represent stored quantities (stocks), and arc variables represent flows.

It is assumed that the variable characterising each node i is related to those characterising its incident arcs via a balance-like equation that in the domain of Laplace transforms can be written as:

$$Y_i(s) = [F(s) + \Delta_F(s)] \sum_{h \in \mathcal{L}_i} \delta_{ih}\, U_h(s), \tag{2}$$

where:
- Y_i and U_h denote the Laplace transforms of y_i and u_h, respectively,
- F is the (scalar) *nominal* transfer function from each incident arc variable to the node variable,
- Δ_F represents the deviation of the actual transfer function from the nominal one F (common to all nodes),
- δ_{ih} accounts for the arc orientation, precisely:

$$\delta_{ih} = \begin{cases} 1 & \text{if arc } h \text{ is entering node } i\,, \\ -1 & \text{if arc } h \text{ is leaving node } i\,. \end{cases}$$

Equation (2) means that the *storage variable* y_i associated with node i depends dynamically on the *flow variables* u_h associated with the arcs entering and leaving node i.

In turn, the arc variables are made to depend dynamically, yet linearly, on the variables characterising the nodes connected by the arc. This dependence may be either *bidirectional* or *unidirectional*.

In the bidirectional case the variable associated with arc $h = (i, j)$, directed from node i to node j, depends on both the departure and the arrival node variables according to

$$U_h(s) = \mu\,[G(s) + \Delta_G(s)]\; [Y_i(s) - Y_j(s)], \tag{3}$$

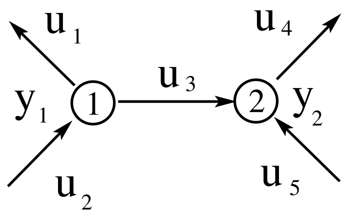

Fig. 1. Example of subgraph involving two nodes.

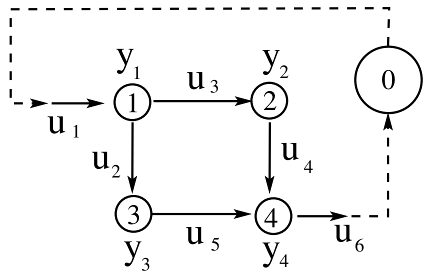

Fig. 2. Network connected with the external environment (denoted by node 0).

where G plays the role of a (scalar) nominal transference (common to all arcs), Δ_G is the deviation from the nominal transference G, and μ is a (possibly unknown) gain parameter.

In the unidirectional case, instead, an arc variable depends only on the variable associated with the departure node, i.e., for arc $h = (i, j)$:

$$U_h(s) = \mu[G(s) + \Delta_G(s)]\; Y_i(s). \tag{4}$$

Example 1. For the subgraph depicted in Figure 1, equations (2) particularise to

$$Y_1(s) = \mu[F(s) + \Delta_F(s)][U_2(s) - U_1(s) - U_3(s)],$$
$$Y_2(s) = \mu[F(s) + \Delta_F(s)][U_3(s) - U_4(s) + U_5(s)].$$

Let $B \in \{-1, 0, 1\}^{n \times m}$ denote the (generalised) incidence matrix of the directed graph, whose rows and columns correspond to the n nodes and to the m arcs, respectively. Each column has at most two non-zero entries. In each row, the entries corresponding to departure arcs are equal to -1, whereas the entries corresponding to arrival arcs are equal to 1.

To allow for possible interactions between some (or all) of the n nodes and the external environment, a further node, representing the environment and denoted by 0, may be added to the graph. This node is associated with variables that are *not* affected by the other graph variables and may account, e.g., for boundary conditions.

Example 2. The network in Figure 2 corresponds to the incidence matrix

$$B = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}. \tag{5}$$

No row is assigned to the external node 0, meaning that the (unique) variable associated with it is $y_0 \equiv 0$. Therefore, in the bidirectional case the flow *from* node 0 and the one *to* node 0 are given by: $U_1(s) = -\mu[G(s) + \Delta_G(s)]Y_1(s)$ and $U_6(s) = \mu[G(s) + \Delta_G(s)]Y_4(s)$, respectively.

Let $\tilde{B}$ denote the matrix whose entries are defined by:

$$\tilde{B}_{ij} = \min\{0, B_{ij}\}.$$

Example 3. For the graph considered in Example 2,

$$\tilde{B} = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Remark 1. Since the topology of a graph is specified by its incidence matrix, in the following, with some abuse of terminology, matrix B will also be referred to simply as the *topology*.

Given the vectors of node and arc transfer functions [1] :

$$Y(s) = [\ Y_1(s) \ \ Y_2(s) \ \ \dots \ \ Y_n(s)\ ]^\top ,$$
$$U(s) = [\ U_1(s) \ \ U_2(s) \ \ \dots \ \ U_m(s)\ ]^\top ,$$

the system of equations for the arc-to-node transferences can be expressed in compact form as

$$Y(s) = [F(s) + \Delta_F(s)]BU(s)\,, \qquad (6)$$

and the system of equations for the node-to-arc transferences as

$$U(s) = -\mu[G(s) + \Delta_G(s)]B^\top Y(s) \qquad (7)$$

in the bidirectional case, and

$$U(s) = -\mu[G(s) + \Delta_G(s)]\tilde{B}^\top Y(s) \qquad (8)$$

in the unidirectional case.

By combining (6) and (7), the characteristic equation in the bidirectional case turns out to be

$$\det[I_n + \mu H(s)L] = 0\,, \qquad (9)$$

where I_n is the n-dimensional identity matrix,

$$H(s) = [F(s) + \Delta_F(s)][G(s) + \Delta_G(s)]\,, \qquad (10)$$

and $L = BB^\top$ is the Laplacian matrix of the graph.

In the case of networks characterised by unidirectional arcs, instead, the characteristic equation is obtained by combining (6) and (8) and turns out to be

$$\det[I_n + \mu H(s)A] = 0\,, \qquad (11)$$

where $A = B\tilde{B}^\top$.

Remark 2. Matrix $L = BB^\top$ has non-positive off-diagonal entries and positive diagonal entries. It is symmetric and positive semi-definite. It is positive definite if and only if the system is externally connected (see e.g. Merris 1994).

Remark 3. Matrix $A = B\tilde{B}^\top$ has non-positive off-diagonal entries and positive diagonal entries. It is also column diagonally dominant since, for all j, $\sum_{i\neq j} |A_{ij}| \leq A_{jj}$. Therefore, $-A$ is a *compartmental matrix* (see e.g. De Leenheer and Aeyels 2001).

Example 4. The L and A matrices for the graph considered in Example 2 are

$$L = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}.$$

[1] For notational simplicity the variable or variables associated with the external node 0, if any, are assumed to be identically zero and thus neglected.

In general, however, A is not triangular.

The following assumptions are adopted in the remainder of this article.

Assumption 1. The nominal transferences F and G are both proper and asymptotically stable, and at least one of them is strictly proper.

Assumption 2. The terms Δ_F and Δ_G are bounded as

$$\sup_{\omega\in\mathbb{R}^+} \left| \frac{\Delta_F(j\omega)}{F(j\omega)} \right| < K_F\,, \quad \sup_{\omega\in\mathbb{R}^+} \left| \frac{\Delta_G(j\omega)}{G(j\omega)} \right| < K_G\,,$$

with $K \triangleq K_F + K_G + K_F K_G < 1$.

As will be shown in Section 3, the value of K provides a measure of the uncertainty induced on (10) by the uncertainties on F and G.

With a slight abuse of terminology, in the following the system with characteristic equation (9) or (11) will be referred to simply as system (9) or system (11).

Definition 1. Let the value of μ be fixed and let $\mathcal{B}$ be a family of incidence matrices associated with the same set of n nodes. System (9) (or system (11)) is *robustly topology-invariant stable in* $\mathcal{B}$ ($\mathcal{B}$-RTIS) if, for all Δ_F and Δ_G satisfying Assumption 2, asymptotic stability is ensured for all incidence matrices $B \in \mathcal{B}$. System (9) (or system (11)) is *robustly topology-invariant stable* (RTIS) if, for all Δ_F and Δ_G satisfying Assumption 2, asymptotic stability is ensured for all possible incidence matrices.

Definition 2. Let $\mathcal{B}$ be a family of incidence matrices. System (9) (or system (11)) is *μ-robustly topology-invariant stable in* $\mathcal{B}$ ($\mathcal{B}$-μ-RTIS) if, for all Δ_F and Δ_G satisfying Assumption 2, asymptotic stability is ensured for all incidence matrices $B \in \mathcal{B}$ and for all $\mu > 0$. System (9) (or system (11)) is *μ-robustly topology-invariant stable* (μ-RTIS) if, for all Δ_F and Δ_G satisfying Assumption 2, asymptotic stability is ensured for all possible incidence matrices and for all $\mu > 0$.

Since both systems (9) and (11) depend on the topology B, even if in different ways, it is natural to wonder whether and how the network topology affects system stability. In the following two sections, it is shown that, under mild hypotheses, stability does not depend on B in both the bidirectional and the unidirectional case.

Remark 4. The assumption of homogeneous uncertainties may be reasonable in some cases, for instance in dealing with swarms of robots (Jadbabaie et al. 2003), cellular dynamics (Turing 1952), consensus (Olfati-Saber and Murray 2004) or distributed estimation (Giordano et al. 2016), where the component subsystems are uncertain but identical. Still, it remains a restriction; a more general setup, where uncertainties are heterogeneous, is considered by Blanchini et al. (2017).

3. TOPOLOGY-INVARIANT STABILITY FOR BIDIRECTIONAL NETWORKS

Let $T \in \mathbb{R}^{n\times n}$ be an invertible matrix such that $T^{-1}LT = \Gamma$ is a diagonal matrix whose diagonal entries are the real nonnegative eigenvalues of $L = BB^\top$. The solutions of (9) clearly coincide with those of

$$\det[I_n + \mu H(s)\Gamma] = 0\,,$$

which is equivalent to the set of n equations

$$1 + \mu H(s)\gamma_k = 0, \quad k = 1, \ldots, n\,, \tag{12}$$

where γ_k is the k-th diagonal entry of Γ (see Hori et al 2015). The asymptotic stability of the overall system requires that these solutions have negative real part.

Theorem 1. Under Assumption 1, system (9) is μ-RTIS if and only if the Nyquist plot of $W \triangleq FG$ lies outside the closed sector

$$\mathcal{S}_0 = \{\Re(W) < 0\,,\, |\sin(\arg(W))| \le K\} \tag{13}$$

(shaded region in Figure 3).

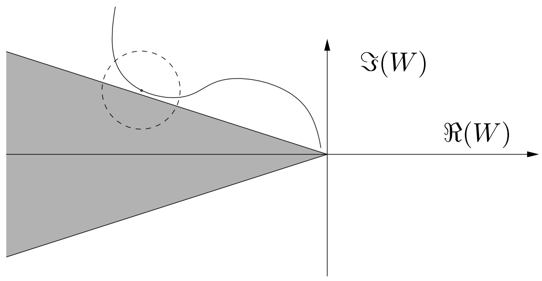

Fig. 3. Nyquist plot of W not entering the "forbidden" sector $\mathcal{S}_0$ (shaded). The disk inside the dashed circle represents the uncertainty at a particular frequency.

Proof. Sufficiency. If the Nyquist plot of W lies outside $\mathcal{S}_0$, then, for all $\omega \ge 0$,

$$K < |\sin(\arg(W(j\omega)))|. \tag{14}$$

Recalling that $\Im(W(j\omega)) = |W(j\omega)|\sin(\arg(W(j\omega)))$, inequality (14) implies

$$|W(j\omega)|K < |\Im(W(j\omega))|\,. \tag{15}$$

Since

$$H = W + W\left(\frac{\Delta_G}{G} + \frac{\Delta_F}{F} + \frac{\Delta_F\,\Delta_G}{FG}\right), \tag{16}$$

function H differs from the "nominal" function W by

$$\Delta_H \triangleq W\left[\frac{\Delta_G}{G} + \frac{\Delta_F}{F} + \frac{\Delta_F\Delta_G}{FG}\right]. \tag{17}$$

For all $\omega \ge 0$, the term $\Delta_H(j\omega)$ is contained in a circle of radius

$$R(j\omega) \triangleq |W(j\omega)|\,K \tag{18}$$

(see the circle inside the dashed contour in Figure 3). It follows that, if (15) holds, the Nyquist plot of H cannot intersect the negative real axis. Therefore, for any choice of μ and for any eigenvalue γ_k of L, the Nyquist plot of $\mu\gamma_k H$ cannot encircle the point -1 and, according to the Nyquist criterion, asymptotic stability is ensured.

Necessity. Suppose that a point of the Nyquist plot of W lies in the sector $\mathcal{S}_0$, i.e., that, for some $\omega^* \ge 0$, $K \ge |\sin(\arg(W(j\omega^*)))|$. Then, the circle of radius $R(j\omega^*)$ centred at $W(j\omega^*)$ intersects the negative real semi–axis (see Figure 4). As a consequence, there exist $\overline{\Delta_F}$ and $\overline{\Delta_G}$ such that

$$H(j\omega^*) = [F(j\omega^*) + \overline{\Delta_F}(j\omega^*)][G(j\omega^*) + \overline{\Delta_G}(j\omega^*)] = -p < 0\,. \tag{19}$$

Considering that at least one eigenvalue of the Laplacian matrix $L = BB^\top$ of any non–trivial graph is strictly

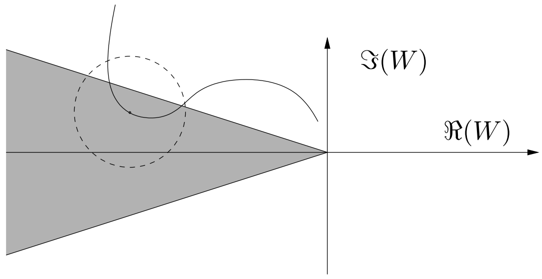

Fig. 4. Nyquist plot of W entering the "forbidden" sector $\mathcal{S}_0$ and uncertainty disk for $\omega = \omega^*$.

positive (see Remark 2), denote by $\bar{\gamma}$ such eigenvalue. Since for $\mu = 1/(\bar{\gamma}\,p)$ equation (12) admits the solution $s = j\omega^*$, at least one root of the characteristic equation has a nonnegative real part. ∎

Corollary 1. Let γ_M be the maximum eigenvalue of L and let

$$\rho_M \triangleq \left[\frac{K}{\gamma_M(1-K^2)}\right].$$

Under Assumption 1, system (9) is RTIS for $\mu = 1$ if the Nyquist plot of W lies outside the region $\mathcal{R} = \mathcal{S}_M \cup \mathcal{C}_M$, (shaded region in Figure 5) where

$$\mathcal{S}_M \triangleq \left\{\Re(W) < -\frac{1}{\gamma_M}\,,\, |\sin(\arg(W))| \le K\right\}$$

and $\mathcal{C}_M$ is the (closed) disk defined by

$$\Im(W)^2 + \left[\Re(W) + \frac{1}{\gamma_M(1-K^2)}\right]^2 \le \rho_M^2\,.$$

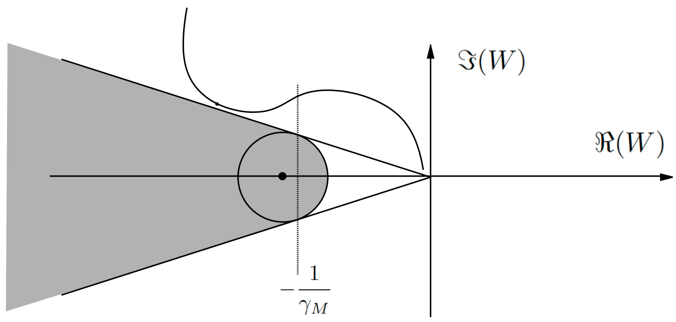

Fig. 5. Nyquist plot of H and "forbidden" region (shaded) for Corollary 1.

Proof. [sketch] It is enough to prove that, under the considered hypotheses, the Nyquist plot of W cannot cross the negative real semi–axis in the infinite interval $(-\infty, -1/\gamma_M]$, since the thesis then follows immediately from Nyquist–plot arguments as in the sufficiency proof of Theorem 1. To this purpose, consider the point $P \triangleq (-1/\gamma_M, 0)$. If ω^* is such that $\Re(W(j\omega^*)) \le -1/\gamma_M$, and $W(j\omega^*)$ lies outside sector $\mathcal{S}_M$, then the Nyquist plot of H does not intersect the negative real semi–axis (on the left of P) (see the proof of Theorem 1). For $\Re(W(j\omega^*)) > -1/\gamma_M$, instead, in order to avoid intersections with the negative real semi-axis on the left of P, the magnitude of the uncertainty, see (17), must be less than the distance between P and $W(j\omega^*)$ (see Figure 5), which is guaranteed, see (18), if

$$|W(j\omega^*)|\,K < \sqrt{\left[\Re(W(j\omega^*)) + \frac{1}{\gamma_M}\right]^2 + \Im(W(j\omega^*))^2}\,. \tag{20}$$

Since both sides are positive, by using the simpler notation $x \triangleq \Re(W(j\omega^*))$ and $y \triangleq \Im(W(j\omega^*))$, (20) is equivalent to

$$K^2(x^2+y^2) < \left[x + \frac{1}{\gamma_M}\right]^2 + y^2 \,. \qquad (21)$$

Equation (21) can be rewritten as

$$\left[x + \frac{1}{\gamma_M(1-K^2)}\right]^2 + y^2 > \left[\frac{K}{\gamma_M(1-K^2)}\right]^2 , \qquad (22)$$

which corresponds to the region outside the disk $\mathcal{C}_M$. Therefore, on the boundary of disk (22) we have

$$y = \pm \frac{\sqrt{K^2 - [1+\gamma_M(1-K^2)x]^2}}{\gamma_M(1-K^2)} , \qquad (23)$$

whose value at $x = -1/\gamma_M$ coincides with the ordinate of the boundary of sector $\mathcal{S}_K$ at the same point. Moreover, the derivative of (23) at $x = -1/\gamma_M$ is

$$\left.\frac{dy}{dx}\right|_{x=-\frac{1}{\gamma_M}} = \mp \frac{K}{\sqrt{1-K^2}}$$

which coincides with the derivative of the boundary of sector $\mathcal{S}_K$, so that the boundary of the forbidden region is smooth there, as shown in Figure 5. ∎

Corollary 1 is useful to establish the following interesting link between the stability of the network and the maximum connectivity degree of its corresponding graph.

Theorem 2. Let $\mathcal{B}_\mathcal{M}$ denote the set of all possible graphs with maximum connectivity degree $\mathcal{M}$ for a given set of nodes. System (9) with $\mu = 1$ is $\mathcal{B}_\mathcal{M}$-RTIS if the Nyquist plot of W lies outside the region

$$\left\{\Re(W) < -\frac{1}{2\mathcal{M}} \,,\, |\sin(\arg(W))| \le K\right\} \cup \mathcal{D} ,$$

where $\mathcal{D}$ is the disk defined by

$$\Im(W)^2 + \left[\Re(W) + \frac{1}{2\mathcal{M}(1-K^2)}\right]^2 \le \left[\frac{K}{2\mathcal{M}(1-K^2)}\right]^2 .$$

Proof. Recall that the ith diagonal entry L_{ii} of the Laplacian matrix $L = BB^\top$ is equal to the degree $\mathcal{M}_i$ of the ith node and

$$\sum_{\substack{j=1\\ j\neq i}}^{n} |L_{ij}| \le L_{ii} ,$$

where the equality holds if the ith node has no external connections. Moreover, all of the eigenvalues of L are (real) nonnegative and, by assumption, $L_{ii} \le \mathcal{M}$ for all i. In view of Gershgorin's Circle Theorem, the eigenvalues of L lie in the union of the n circles centred at $L_{ii} \le \mathcal{M}$ with radii $r_i = \sum_{j\neq i} |L_{ij}| \le \mathcal{M}$. Hence, the k-th eigenvalue satisfies the double inequality $0 \le \gamma_k \le 2\mathcal{M}$, which implies that

$$\left(-\frac{1}{\gamma_m}, -\frac{1}{\gamma_M}\right] \subset \left(-\infty, -\frac{1}{2\mathcal{M}}\right] ,$$

where γ_m and γ_M represent the minimum and the maximum eigenvalue, respectively. The application of Corollary 1 ends the proof. ∎

Remark 5. In view of Theorem 2, the greater the connectivity is, the more the system is prone to instability. For instance, if the network corresponds to a discretisation grid for a field model (describing, e.g., fluid dynamics) in which the nodes are placed at the vertices of square (2D) or cubic (3D) cells, then 2-dimensional grids (whose maximum connectivity degree is $\mathcal{M} = 4$) are less prone to instability than 3-dimensional grids (whose maximum connectivity degree is $\mathcal{M} = 6$).

4. TOPOLOGY-INDEPENDENT STABILITY FOR UNIDIRECTIONAL NETWORKS

Proving RTIS of unidirectional networks, whose characteristic equation is (11), is slightly more involved. In this case, a transformation matrix T can be found such that $T^{-1}AT$ is triangular (in general, $A = B\tilde{B}^\top$ is not diagonalisable), leading to the n scalar equations

$$1 + \mu H(s)\tilde{\gamma}_k = 0 \,, \quad k = 1, \ldots, n \,, \qquad (24)$$

where the $\tilde{\gamma}_k$'s are the eigenvalues of A, which are not necessarily real. Clearly, asymptotic stability entails that the solutions of (24) have negative real part for all $\tilde{\gamma}_k$'s. The following technical lemma, whose proof is omitted for brevity, will be useful to prove the main result.

Lemma 1. For any $z \in \mathbb{C}$ with $\Re(z) < 0$ and any arbitrarily small $\epsilon > 0$, there exist an incident matrix B and a real number $\mu > 0$ such that $\mu B\tilde{B}^\top$ admits at least one eigenvalue γ satisfying the inequality

$$\left|\frac{1}{\gamma} + z\right| < \epsilon \,. \qquad (25)$$

Theorem 3. Under Assumption 1, let $W(0) > 0$. Then, the system represented by equation (11) is μ-RTIS if and only if the Nyquist plot of W lies in the open sector (see Figure 6)

$$\mathcal{S}_u = \{\Re(W) > 0 \,,\, |\cos(\arg(W))| > K\} \,.$$

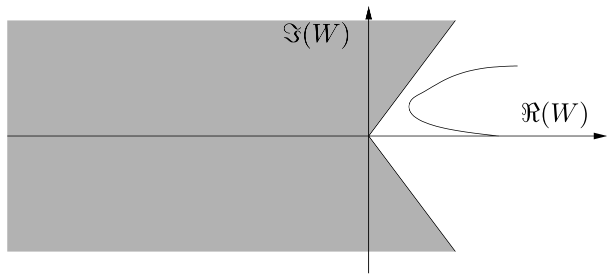

Fig. 6. Allowed region $\mathcal{S}_u$ (white minor sector) for Theorem 3.

Proof. Sufficiency. According to Gershgorin's Circle Theorem, the eigenvalues of A lie in the union of the n circles centred at the diagonal entries A_{ii} of A with radii $r_i = \sum_{i\neq j} |A_{ij}|$ so that, according to Remark 3, $r_i \le A_{ii}$. It follows that the eigenvalues of A lie inside the circle with centre $\max_i\{A_{ii}\}$ and radius $\max_i\{A_{ii}\}$. Therefore, given any eigenvalue $\tilde{\gamma}_k$ of A, the point $-1/\tilde{\gamma}_k$ lies in the interior of the LHP. Now, the condition $\mu H(j\omega) \neq -1/\tilde{\gamma}_k$, for all $\mu > 0$, may be rewritten as

$$W(j\omega) + \Delta_H(j\omega) \neq -\frac{1}{\mu\tilde{\gamma}_k} \,, \qquad \text{for all } \mu \,.$$

Since $|\Delta_H(j\omega)| \le |W(j\omega)|K$, the claim can be proved analogously to the sufficiency proof of Theorem 1.

Necessity. Assume, by contradiction, that, for some ω^*, $W(j\omega^*) \notin \mathcal{S}_u$. This means that the uncertainties Δ_F and Δ_G can be such that $\Re(H(j\omega^*)) < 0$ and, by the continuity of H, there is a neighbourhood $\mathcal{U}$ of $H(j\omega^*)$ such that $\Re(v) < 0$ for all $v \in \mathcal{U}$. If H has a finite number of zeros and poles, it is always possible to choose $\mathcal{U}$ such that the Nyquist plot of H divides $\mathcal{U}$ into two (and not more than two) regions $\mathcal{U}_1$ and $\mathcal{U}_2$. Observe, now, that the winding numbers, with respect to the Nyquist plot, of any two adjacent regions into which the plane is partitioned by the plot itself, differ exactly by 1 (Munkres 2000). Hence, one region among $\mathcal{U}_1$ and $\mathcal{U}_2$ must be encircled by the Nyquist plot. Suppose it is $\mathcal{U}_1$; take z and ϵ such that the open ball with centre z and radius ϵ, denoted by $\mathcal{S}_z(\epsilon)$, belongs to $\mathcal{U}_1$. In view of Lemma 1, there exist μ and B such that, at least for one eigenvalue γ of $\mu B\tilde{B}^\top$, $-1/\gamma \in \mathcal{S}_z(\epsilon)$. The overall Nyquist plot of H encircles the point $-1/\gamma$ (hence, the Nyquist plot of γH encircles the point -1) in the clockwise direction. Since H is assumed to be stable, this implies that (24) admits RHP solutions. ■

5. CONCLUDING REMARKS

The stability of homogeneous dynamical networks, where nodes and arcs are associated with uncertain transfer functions $F + \Delta_F$ and $G + \Delta_G$, respectively, has been investigated for F, G stable and Δ_F, Δ_G suitably bounded. A necessary and sufficient stability condition, robust against variations of the gain μ, is that the Nyquist plot of the transfer function $W = FG$: (a) does not enter an LHP sector with vertex in the origin and axis of symmetry lying on the negative real semi-axis, for bidirectional networks; (b) lies inside an RHP sector with vertex in the origin and axis of symmetry lying on the positive real semi-axis, for unidirectional networks.

Further research directions along this line include the case in which the uncertainties are not homogeneous.

REFERENCES

S. Amari, "Dynamics of pattern formation in lateral-inhibition type neural fields", *Biological Cybern.*, vol. 27, no. 2, pp. 77–87, 1977.

M. Arcak, "Spatially uniform behavior in reaction-diffusion PDE and compartmental ODE systems", *Automatica*, vol. 47, no. 6, pp. 1219–1229, 2011.

M. Arcak, "Pattern formation by lateral inhibition in large-scale networks of cells", *IEEE Trans. Automat. Control*, vol. 58, no. 5, pp. 1250–1262, 2013.

F. Blanchini, D. Casagrande, G. Giordano, and U. Viaro, "A bounded complementary sensitivity function ensures topology-independent stability of homogeneous dynamical networks ", submitted, 2017.

F. Blanchini, E. Franco, and G. Giordano, "Network-decentralized control strategies for stabilization", *IEEE Trans. Automat. Control*, vol. 60, no. 2, pp. 491–496, 2015.

F. Blanchini, E. Franco, G. Giordano, V. Mardanlou, and P. L. Montessoro, "Compartmental flow control: decentralization, robustness and optimality", *Automatica*, vol. 64, no. 2, pp. 18–28, 2016.

Y. Cao, W. Yu, W. Ren, and G. Chen, "An overview of recent progress in the study of distributed multi-agent coordination", *IEEE Trans. Ind. Informatics*, vol. 9, no. 1, pp. 427–438, 2013.

P. De Leenheer and D. Aeyels, "Stabilization of positive linear systems", *Systems & Control Letters*, vol. 44, pp. 259–271, 2001.

D. Del Vecchio, A. J. Ninfa, and E. D. Sontag, "Modular cell biology: retroactivity and insulation", *Molecular Systems Biology*, vol. 4, no. 1, pp. 161 ff., 2008.

A. Gierer and H. Meinhardt, "A theory of biological pattern formation", *Kybernetic*, vol. 12, no. 1, pp. 30–39, 1972.

G. Giordano, *Structural analysis and control of dynamical networks.* Ph.D. thesis, University of Udine, Italy, 2016.

G. Giordano, F. Blanchini, E. Franco, V. Mardanlou, and P. L. Montessoro, "The smallest eigenvalue of the generalized Laplacian matrix, with application to network-decentralized estimation for homogeneous systems", *IEEE Trans. Network Science and Engineering*, vol. 3, no. 4, pp. 312–324, 2016.

A. A. Golovin, B. J. Matkowsky, and V. A. Volpert, "Turing pattern formation in the Brusselator model with superdiffusion", *SIAM J. Appl. Math.*, vol. 69, no. 1, pp. 251–272, 2008.

J. B. A. Green and J. Sharpe, "Positional information and reaction-diffusion: two big ideas in developmental biology combine", *Development*, vol. 142, no. 7, pp. 1203–1211, 2015.

Y. Hori, H. Miyazako, S. Kumagai, and S. Hara, "Coordinated spatial pattern formation in biomolecular communication networks", *IEEE Trans. Mol. Biol. Multi-Scale Commun.*, 2015.

A. Jadbabaie, A. S. Morse and J. Lin, "Coordination of groups of mobile autonomous agents using nearest neighbor rules", *IEEE Trans. Automat. Control*, vol. 48, no. 6, pp. 988–1001, 2003.

N. Le Novère, "Quantitative and logic modelling of molecular and gene networks", *Nature Reviews Genetics*, vol. 16, no. 3, pp. 146–158, 2015.

I. Lestas and G. Vinnicombe, "Scalable decentralized robust stability certificates for networks of interconnected heterogeneous dynamical systems", *IEEE Trans. Automat. Control*, vol. 51, no. 10, pp. 1613–1625, 2006.

Z. Lin, L. Wang, Z. Han, and M. Fu, "A graph Laplacian approach to coordinate-free formation stabilization for directed networks", *IEEE Trans. Automat. Control*, vol. 61, no. 5, pp. 1269–1280, 2016.

P. K. Maini, T. E. Woolly, R. E. Baker, E. A. Gaffney, and S. S. Lee, "Turing's model for biological pattern formation and the robustness problem", *Interface Focus*, vol. 2, no. 4, pp. 487–492, 2012.

R. Merris, "Laplacian Matrices of Graphs: A Survey", *Linear Algebra and its Applications*, vol. 197, 198, pp. 143–176, 1994.

H. Miyazako, Y. Hori, and S. Hara, "Turing instability in reaction-diffusion systems with a single diffuser: characterization based on root locus", *Proc. 52nd IEEE Conf. Decision and Control*, Florence, Italy, 10–12 December, 2013, pp. 2671–2676.

J.R. Munkres, *Topology.* Prentice Hall, NJ, USA, 2000.

C. Nicolaides, R. Juanes, and L. C. Cueto-Felgueroso, "Self-organization of network dynamics into local quantized states", *Sci Rep.*, vol. 6, 21360, 2016.

G. Nicolis and I. Prigogine, *Self Organization in Nonequilibrium Systems.* Wiley, New York, USA, 1977.

R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time delays", *IEEE Trans. Automat. Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

R. Pates and G. Vinnicombe, "Stability certificates for networks of heterogeneous linear systems", *Proc. 51st IEEE Conf. Decision and Control*, Maui (HI), USA, Dec. 10–13, 2012, pp. 6915–6920.

R. S. Smith and F. Y. Hadaegh, "Closed-loop dynamics of cooperative vehicle formations with parallel estimators and communication", *IEEE Trans. Automat. Control*, vol. 52, no. 8, pp. 1404–1414, 2007.

A. M. Turing, "The chemical basis of morphogenesis", *Phil. Trans. Royal Soc. London, Series B, Biological Sciences*, vol. 237, no. 641, pp. 37–72, 1952.

S. Wang, W. Ren, and Z. Li, "Information-driven fully distributed Kalman filter for sensor networks in presence of naive nodes", *arXiv:1410.0411*, 2014.

O. Wolkenhauer, B. K. Ghosh, and K. H. Cho, Eds., *Biochemical Networks and Cell Regulation. IEEE Control Syst. Mag.*, Special Section on Systems Biology, vol. 24, no. 4, pp. 30–102, 2004.

L. Wolpert, "Positional information and the spacial pattern of cellular differentiation", *J. Theor. Biol.*, vol. 25, no. 1, pp. 1–47, 1969.