A switched system approach to dynamic race modelling

Franco Blanchini[a], Daniele Casagrande[b], Giulia Giordano[a], Umberto Viaro[b]

[a]*Department of Mathematics and Computer Science, University of Udine, Via delle Scienze 206, 33100 Udine, Italy*
[b]*Department of Electrical, Management and Mechanical Engineering, University of Udine, Via delle Scienze 206, 33100 Udine, Italy*

Abstract

The paper presents and analyses some nonlinear continuous–time dynamic models of social systems whose members, groups or individuals, may change partners and/or opponents at any time, according to a greedy criterion. The main structural properties of these models, which belong to the class of positive switching systems, are investigated with particular regard to the existence of solutions, their positivity, boundedness and asymptotic behaviour. Simulations show how the cooperative or hostile attitudes of the participants affect their yield.

Key words: Switching systems, Positive systems, Cooperative/competitive games, Greedy algorithms, Stability

1. Introduction

Considerable effort has recently been devoted to the study and simulation of systems consisting of interacting decision makers [34] and various kinds of models have been adopted for this purpose [5]. A common feature of these models is that each decision maker has its own goal that can depend on the present, past or expected values of the system states. Usually, decisions are made at well–defined time instants on the basis of partial or complete information about the state of the system and its evolution.

In the following, we consider a class of variable–structure continuous–time differential models that account fairly well for the behaviour of multi–agent

Email addresses: blanchini@uniud.it (Franco Blanchini), daniele.casagrande@uniud.it (Daniele Casagrande), giulia.giordano@uniud.it (Giulia Giordano), viaro@uniud.it (Umberto Viaro)

systems characterised by the possibility of switching instantaneously from one configuration to another, whenever an actor finds it more convenient to change its ties, according to a greedy or shortsighted criterion. It is assumed that the system state represents the strength of every participant and that participants' choices can take place at any time, depending on the state of the competitors, and are put immediately into effect (provided a reciprocity condition is satisfied in the case of alliances). This class of models adequately describes the behaviour of many social networks, as well as political, academic and sport contests, such as cycle and motor races. Therefore, in this paper they are collectively called dynamical *race* models, even if they do not refer to a specific application. In fact, the aim of the present contribution is to analyse some important structural properties of the mathematical models, such as the conditions ensuring the formation of coalitions, the positivity and boundedness of the solutions.

The studies of networks whose overall structure evolves depending on processes taking place at their nodes have been reviewed in [16, 28]. These networks are referred to as *coevolutionary* networks, because local and global dynamics interact. These networks play a fundamental role in the study of choices at the basis of social behaviour [11]. Several papers have recently analysed by means of graphs this kind of interactions either in general [35, 26] or with reference to specific problems [14, 21]. In particular, it has ben shown that partner switching from a lower reputation partner to a higher reputation neighbouring partner [19, 20] or breaking bonds with uncooperative partners [31] can promote cooperation. In [1] the division of the payoff among the coalition members is considered and it is shown that, eventually, the coalition structure reaches an equilibrium in the case of *myopic* players' strategies. In [18] the coalition formation is shown to reach an equilibrium, under some hypotheses, also in the case of *farsighted* strategies. In [29] attention has been focused on switching strategies in a two–person zero–sum differential game of finite horizon, and in [27] on the benefits of partner switching among self–interested agents in a resource–exchange environment. It has been pointed out that allowing for assortative mating and defector exclusion in dynamic partner updating [33] can be beneficial to cooperation even when the cost associated with dynamism is taken into account in terms of time and/or investment for finding and establishing new partnerships [6].

All of the aforementioned models can be broadly classified as game–theory models [4, 25] with certain "events" occurring at discrete instants of time and marking the rounds of the game (see also [24], a survey on game-theoretic approaches to the study of coalition formation). Only after each round, the strategy of the participants is updated, even if their interaction between consecutive updates is sometimes accounted for in an aggregated way, e.g., in terms of numbers of cooperators or defectors [14], or in a probabilistic way, e.g., in terms of action and reaction probabilities [17]. Also, the coevolutionary dynamics is investigated almost exclusively by means of numerical simulation (see, e.g., [21] where a model borrowed from [13] is used) and no theoretical results concerning the model structural properties, such as stability and positivity, is provided.

The focus of this paper, instead, is on the rigorous *analysis of the structural properties* of a class of nonlinear continuous–time dynamic models describing the states of a (generic) number of competitors which may update alliances or enmities *at any instant of time* depending on their current states. However, to show the effects of different criteria for choosing partners and/or defectors, a number of simulations will also be reported. To the present purposes, the theory of switching systems proves particularly useful. As is known, switching systems are currently attracting a great interest from the control community [22, 9, 32] with particular regard to *positive* systems which can explain the evolution of various kinds of populations very well [12, 7, 15, 8]. Most research efforts deal primarily with stability and stabilizability. Here, instead, we are mainly concerned with network configuration and state evolution patterns.

The contributions of this paper can be summarized as follows.
(i) A fairly general family of nonlinear state models that describe the contest among racers to improve their rankings is proposed.
(ii) Both the case in which only alliances between pairs of racers are allowed (cooperative model) and the case in which the racers can obstruct the top–ranked racer (competitive model) are considered.
(iii) It is proved that in the cooperative model at least one alliance is always established, provided a reciprocity condition is satisfied.
(iv) It is shown that the system evolution remains positive even in the competitive case.
(v) A mixed model, in which both alliances and obstructions are allowed, is also examined.

The present paper extends and generalizes in many respects the results in [10], where the racers' attitude to partner switching depends *linearly* on the competitors' strength, which limits the applicability of the model to actual systems.

2. Modelling

Consider a set of n independent racers (individuals or groups) whose aim is to prevail over the others, and let the state x_i of the i–th racer denote its strength, whose evolution depends on: (i) its own internal dynamics, (ii) the interactions with the other racers, and (iii) an exogenous input representing environmental resources. A racer may either associate with other racers to increase its own strength or sabotage other racers to decrease their strength and, consequently, increase the possibility of improving its own rank. In principle, alliances or obstructions can involve more than two racers and be either symmetric or asymmetric (in the sense that both racers share the same disposition towards each other or not); however, for the sake of simplicity, in the following it is assumed that: (i) each racer may have, at most, one ally, and (ii) alliances are symmetric, while obstructions are not.

It is assumed that *in the absence of interactions* among racers the dynamics are described by

$$\dot{\mathbf{x}}(t) = -\Lambda \mathbf{x}(t) + \mathbf{b}, \tag{1}$$

where $\mathbf{x} = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$ is the state vector representing the strength of every racer, vector $\mathbf{b} = [b_1, \ldots, b_n]^\top \in \mathbb{R}^n$ represents an exogenous input, and $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ is a *positive–definite* diagonal matrix accounting for a natural decline of the racers' strength. In this case, each x_i autonomously reaches a constant steady–state value dependent only on b_i and λ_i. In a more general context, a time–varying input $\mathbf{b}(t)$ could be considered.

In the sequel, three models that account also for interactions are obtained from the basic model (1). They correspond respectively to the cases in which the dynamics are affected only by alliances (cooperative model), only by obstructions (competitive model), and by both (mixed model). For reasons that will be clear soon, the following standing assumption is made.

Assumption 1.

$$i \neq j \Rightarrow \frac{b_i}{\lambda_i} \neq \frac{b_j}{\lambda_j}. \tag{2}$$

Implication (2) means that, in the absence of interactions, different racers reach different steady–state values.

2.1. Cooperative model

Consider first the case in which two racers can associate to increase their strength. Obviously, the alliance takes place only if both racers are willing to make it; in addition, it is assumed that the strength increment of either ally is given by a function $\varphi : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+$ that depends on the current strengths of both. The attitude of the racers to make alliances at time t is modelled by means of a Boolean matrix $V(t)$, whose generic entry $V_{ij}(t)$ is 1 if, at time t, racer i is willing to make an alliance with racer j, and 0 otherwise. For simplicity, every racer is allowed to make one alliance only. Hence, every row of $V(t)$ contains a single 1. Moreover, an alliance between racer i and racer j takes place only if $V_{ij}(t) = V_{ji}(t) = 1$ (reciprocity condition). This means that a racer willing to establish a collaboration with another racer k may be rejected if racer k finds it more profitable to collaborate with a different racer.

The state equations of such a cooperative model can then be written as

$$\dot{x}_i(t) = -\lambda_i x_i(t) + \sum_{j=1}^{n} V_{ij}(t)V_{ji}(t)\varphi(x_i(t), x_j(t)) + b_i\,, \qquad i = 1, \ldots, n\,. \tag{3}$$

Since $\varphi(x_i, x_j) \geq 0$, the sum in (3) is non–negative; therefore, based on a greedy (but shortsighted) criterion, the optimal attitude of racer i towards every other racer $j \neq i$ is given by

$$V_{ij}(t) = \begin{cases} 1 & \text{if } \varphi(x_i(t), x_j(t)) > \varphi(x_i(t), x_k(t)) \text{ for all } k \neq j\,, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

In other words, the i–th racer would like to choose as its partner the racer that currently maximizes the increase of its strength.

Remark 1. *If for a pair of indices, say j_1 and j_2, and a time–instant t^* we have*

$$\varphi(x_i(t^*), x_{j_1}(t^*)) = \varphi(x_i(t^*), x_{j_2}(t^*)) > \varphi(x_i(t^*), x_k(t^*))\,, \tag{5}$$

for all $k \notin \{j_1, j_2\}$, then, due to the strict inequality in (4), the i–th racer will not be willing to make any alliance at time t^*, even though, given the positivity of φ, any alliance would be profitable. However, due to Assumption 1, the evolutions of x_{j_1} and x_{j_2} are different. Hence, equation (5) is no longer satisfied immediately after t^*, and a possible ally will be chosen without delay.

In view of Remark 1, it is assumed that no undecidable situation (stall) occurs.

2.2. Competitive model

Model (3) applies to a context in which cooperation is always beneficial, e.g., because unlimited resources prevent the outbreak of conflicts. To describe a situation in which resources are limited and the strength of a racer is seen as a menace by the other racers, resort must be made to a different model. Here, attention is limited to a competitive model in which the strongest racer (which may not be unique) is sabotaged by all the other racers and the rate of change of its strength contains a sum of negative terms that depend on the current strengths of every other racer.

To this purpose, let $\mathcal{H}(t)$ denote the set indexing the strongest racers at time t, i.e.,

$$\mathcal{H}(t) = \arg\max_k x_k(t), \tag{6}$$

and let $S : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+$ be a function accounting for the extent of the damage that x_j can inflict on x_i. The state equations of the aforementioned competitive model can thus be written as

$$\dot{x}_i(t) = -\lambda_i x_i(t) - \sum_{j \neq i} S(x_i(t), x_j(t)) + b_i \tag{7}$$

for $i \in \mathcal{H}(t)$, and

$$\dot{x}_i(t) = -\lambda_i x_i(t) + b_i \tag{8}$$

for all $i \notin \mathcal{H}(t)$.

2.3. Mixed model

Consider finally a situation in which alliances and obstructions coexist. Again, let $\mathcal{H}(t)$ denote the set indexing the strongest racer at time t, and introduce the symbol $\mathcal{F}_i$ to indicate the set of indices associated with the racers that do not obstruct racer $i \in \mathcal{H}(t)$. Obviously, if at time t the j–th racer is allied with i, it will not make any obstruction. Recalling the assumption that each racer can enter at most one pairwise alliance, we have

$$\mathcal{F}_i(t) = \begin{cases} \{i, j\} & \text{if } V_{ij}(t) = V_{ji}(t) = 1\,, \\ \{i\} & \text{otherwise.} \end{cases} \tag{9}$$

The dynamic model of this system can thus be written as

$$\dot{x}_i(t) = -\lambda_i x_i(t) - \sum_{j \notin \mathcal{F}_i(t)} S(x_i(t), x_j(t)) + \sum_{j=1}^{n} V_{ij}(t) V_{ji}(t) \varphi(x_i(t), x_j(t)) + b_i \tag{10}$$

for $i \in \mathcal{H}(t)$, while, for all $i \notin \mathcal{H}(t)$,

$$\dot{x}_i(t) = -\lambda_i x_i(t) + \sum_{j=1}^{n} V_{ij}(t) V_{ji}(t) \varphi(x_i(t), x_j(t)) + b_i\,. \tag{11}$$

3. Theoretical Results

This section analyses the main properties of the models described in Section 2, which are all full–fledged switching systems. The switching nature of the cooperative model depends on the choice of alliances and, in particular, on the *attitude* of agent i towards agent j, i.e., the Boolean variable $V_{ij}(t)$ defined in (4). Since the state vector evolves in time, V_{ij} may switch at any instant from the value 1 to the value 0, or vice versa. As a consequence, the topology of the network of alliances changes too. Analogous considerations apply to both the competitive and the mixed model, since the set of strongest racers may change in time.

Remark 2. *The right–hand sides of the differential equations of all of the aforementioned models are discontinuous. Hence, their solution must be intended in the sense of Filippov by resorting to a differential inclusion formulation (see for instance [2]). Consequently, chattering phenomena and sliding trajectories may occur, as shown by the examples in the following sections.*

3.1. System positivity

It is proved now that the entire state trajectories starting from a positive initial state lie in the positive orthant for all the models considered in Section 2, which qualifies them as positive systems.

Proposition 1. *If $x_i(0) \geq 0$ and $b_i \geq 0$, $i = 1, \ldots, n$, then model (3), model (7)–(8) and model (10)–(11) are positive systems, i.e., $x_i(t) \geq 0$ for all $t \geq 0$.* □

Proof. Consider first the cooperative model (3) and suppose that $x_i(t_1) < 0$ for some i and some $t_1 > 0$. Since $x_i(0) \geq 0$ and the trajectories are continuous, $x_i(t_1)$ may be negative only if there exists a time instant $t_2 \in [0, t_1)$ such that $x_i(t_2) = 0$ and $\dot{x}_i(t_2) < 0$. However, for $x_i(t_2) = 0$, equation (3) gives

$$\dot{x}_i(t_2) = \sum_{j=1}^{n} V_{ij}(t_2) V_{ji}(t_2) \varphi(x_i(t_2), x_j(t_2)) + b_i \geq b_i \geq 0. \tag{12}$$

Then the proposition is proved for the cooperative model (3). Also, it is immediately seen that, if it holds for the competitive model (7)–(8), it holds for the mixed model (10)–(11) as well. Therefore, it is sufficient to prove it for the model (7)–(8). By contradiction, assume that (starting from an initial condition such that $x_i(0) \geq 0$ for $i = 1, \ldots, n$) there exists $\bar{t}$ such that, for some i, $x_i(\bar{t}) = 0$ and $x_i(t) < 0$ in a right neighborhood of $\bar{t}$. Let $\mathcal{Z}(\bar{t})$ denote the set of all indices i for which $x_i(\bar{t}) = 0$ and let $|\mathcal{Z}(\bar{t})|$ denote its cardinality (greater than 0 by assumption). If $|\mathcal{Z}(\bar{t})| < n$, then the obstructed contenders are associated with indices not belonging to $\mathcal{Z}(\bar{t})$, while the dynamic equation associated with each $i \in \mathcal{Z}(\bar{t})$ (see (8)) is $\dot{x}_i(\bar{t}) = b_i$; hence, since $b_i \geq 0$, no zero crossing is possible. On the other hand, if $|\mathcal{Z}(\bar{t})| = n$, then $x(\bar{t}) = 0$. In this case no contender is obstructed and the dynamics of all contenders is described by (8). Therefore $\dot{x}_i(\bar{t}) = b_i \geq 0$ for all $i \in \{1, \dots, n\}$; again, no zero crossing is possible and the state variables cannot become negative.

■

3.2. Existence of alliances

Proposition 2. *Consider equations (3) and (4). If there exist functions $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, symmetric with respect to both its arguments, $\psi : \mathbb{R}_+ \to \mathbb{R}$ and $\xi : \mathbb{R}_+ \to \mathbb{R}$ such that*

$$\varphi(x_i, x_j) = \theta(x_i, x_j)\psi(x_i)\xi(x_j), \tag{13}$$

then for almost all[1] *t there exist i and j such that $V_{ij}(t) = 1$.* □

Proof Consider first the simple case of three racers and suppose, by contradiction, that no alliance takes place in a whole time-interval $[t_1, t_2]$, so that $V_{ij}(t) = 0$, $\forall i = 1 \dots n$, $\forall j = 1 \dots n$, and $\forall t \in [t_1, t_2]$. In this case, excluding undecidable situations (see Remark 1) which, in view of Assumption 1, can occur only for isolated time instants and have previously been ruled out, if a racer i wishes to associate with racer j, the latter will reject the proposal because an alliance with the third racer k would be more profitable. The same consideration applies to the other two possible pairs of racers, i.e., j and k and, respectively, k and i. According to (4), the aforementioned sequence of intentions correspond to

$$\begin{aligned} \theta(x_i, x_j)\psi(x_i)\xi(x_j) &> \theta(x_i, x_k)\psi(x_i)\xi(x_k)\,, & (14)\\ \theta(x_j, x_k)\psi(x_j)\xi(x_k) &> \theta(x_j, x_i)\psi(x_j)\xi(x_i)\,, & (15)\\ \theta(x_k, x_i)\psi(x_k)\xi(x_i) &> \theta(x_k, x_j)\psi(x_k)\xi(x_j)\,. & (16) \end{aligned}$$

Without loss of generality, we may assume θ, ψ and ξ to be positive. Hence, multiplying side by side all of the above inequalities we obtain

$$\begin{aligned} &\theta(x_i, x_j)\theta(x_j, x_k)\theta(x_k, x_i)\psi(x_i)\psi(x_j)\psi(x_k)\xi(x_i)\xi(x_k)\xi(x_j) > \\ &\quad > \theta(x_i, x_k)\theta(x_j, x_i)\theta(x_k, x_j)\psi(x_i)\psi(x_j)\psi(x_k)\xi(x_k)\xi(x_i)\xi(x_j)\,, \end{aligned} \tag{17}$$

[1]Except, possibly, for some isolated time instants (see Remark 1).

which, given the symmetry of θ, is a contradiction. It follows that at least one alliance must be formed.
The same argument holds in the case of more than three racers. Indeed, assume, by contradiction, that each racer seeks alliance with (for short, s.a.w.) another racer who declines, according to

$$x_{i_1} \text{ s.a.w. } x_{i_2} \text{ s.a.w. } x_{i_3} \ \ldots \ \text{s.a.w. } x_{i_n} \text{ s.a.w. } x_{i_{n+1}}.$$

Now, this chain *must* exhibit a cycle, since the number of nodes is finite. Without loss of generality, assume that the cycle has length k and the racers involved are

$$x_{i_1} \text{ s.a.w. } x_{i_2} \ \ldots \ \text{s.a.w. } x_{i_k} \text{ s.a.w. } x_{i_1}$$

Considering inequalities similar to those written above for the three–node case, we would again arrive at a contradiction. ■

Observe that, if θ is not symmetric, then the thesis of Proposition 2 does not hold (a counterexample in the linear case is reported in [10]).

3.3. Boundedness of the trajectories

To ensure the boundedness of the trajectories, it is necessary to assume that the natural decline of every racer (clearly related to the negative values $-\lambda_i$) is sufficiently large to compensate for the positive contributions afforded by the possible alliances, which indeed occurs under a mild assumption.

Definition 1. *A function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is* underlinear of degree α *if, for all $x \in \mathbb{R}_+$ and all $y \in \mathbb{R}_+$,*

$$\varphi(x, y) \leq \mu x + \nu y + c\,, \tag{18}$$

for some $\mu \geq 0$, $\nu \geq 0$ and $c \geq 0$ such that $\mu + \nu \leq \alpha$.

Proposition 3. *Consider system (10)–(11). Let $\beta \triangleq \min_i \lambda_i$. If φ is underlinear of degree β, then the trajectory $\mathbf{x}(t)$ is bounded, i.e., there exists a compact set $\mathcal{S}$ such that $\mathbf{x}(t) \in \mathcal{S}$, $\forall t \geq 0$.* □

Proof In view of the positivity of the system trajectories, consider the function $V(x) = 1^\top \mathbf{x}$, which is copositive [23]. Its time-derivative along the trajectory of the system is

$$\dot{V} = -\Lambda^\top \mathbf{x} + 1^\top \mathbf{b} + \sum_{i,j=1}^{n} V_{ij} V_{ji} \varphi(x_i, x_j) - \sum_{i \in \mathcal{H}(t)} \sum_{j \notin \mathcal{F}_i(t)} S(x_i, x_j). \tag{19}$$

If racers i_1 and i_2 are allied, their alliance affects both the time–variation of x_{i_1} and the time–variation of x_{i_2}. Hence, the first summation in (19) must contain both the term $\varphi(x_{i_1}, x_{i_2})$ and the term $\varphi(x_{i_2}, x_{i_1})$. Moreover, in view of the underlinearity of φ we have

$$\varphi(x_{i_1}, x_{i_2}) + \varphi(x_{i_2}, x_{i_1}) \leq (\mu + \nu)(x_{i_1} + x_{i_2}) + 2c\,. \tag{20}$$

Taking (20) into account, from (19) we get

$$\begin{aligned} \dot{V} &\leq -\Lambda^\top \mathbf{x} + 1^\top \mathbf{b} + \sum_{i,j=1}^{n} V_{ij} V_{ji} \varphi(x_i, x_j) \leq \\ &\leq -\Lambda^\top \mathbf{x} + 1^\top \mathbf{b} + \mu 1^\top \mathbf{x} + \nu 1^\top \mathbf{x} + nc \leq -\rho 1^\top \mathbf{x} + 1^\top \mathbf{b} + nc \end{aligned} \tag{21}$$

for some $\rho > 0$. If $\gamma > 0$ is large enough to guarantee that $-\rho\gamma + 1^\top \mathbf{b} + nc < 0$, then the trajectories of the switching system are ultimately globally bounded in the set $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n_+ \,:\, 1^\top \mathbf{x} \leq \gamma\}$. ■

3.4. Profitability of obstructions

A natural question arising from the previous considerations is whether obstructions can be profitable. We have seen that the contribution of an alliance to the dynamics of the allies is always positive; on the other hand, an obstruction provides a negative contribution to the obstructed racer, but no direct positive contribution to the obstructing one. Hence, one might ask whether, being adverse to a racer (in this case, to the strongest racer), an obstruction could be profitable to another racer. The answer is yes, as shown by the simulations in Figure 1 that refer to a pool of 5 racers: the two scenarios are characterised by the same matrix $\Lambda = 2I$, vector $b = [0.4387 \quad 0.3816 \quad 0.7655 \quad 0.7952 \quad 0.1869]^\top$ and initial condition $x(0) = [0.4898 \quad 0.4456 \quad 0.6463 \quad 0.7094 \quad 0.7547]^\top$. The difference between the two lies in the fact that in the scenario of Figure 1 (a) only alliances are allowed, while in that of Figure 1 (b) both alliances and obstructions are possible. It is seen that, at least for racer 3, whose trajectory is plotted with a bold line, obstructing the strongest racer is indeed profitable. In fact, not only the ranking of the racer improves with respect to its competitors, but also its own strength increases: after six time units it is below 0.4 in the case with alliances only and above 0.6 in the case with both alliances and obstructions. This happens because, while in the cooperative case racer 3 is never involved in any alliance, in the mixed case the topology of the network of alliances switches at a certain time instant: due to the obstructions, which decrease the strength of the top–ranked racer, the racer that was initially allied with it suddenly finds the alliance with racer 3 more profitable. Thanks to this partnership, the strength of racer 3 starts increasing and, eventually, racer 3 is fighting for supremacy (while in the cooperative case it comes third).

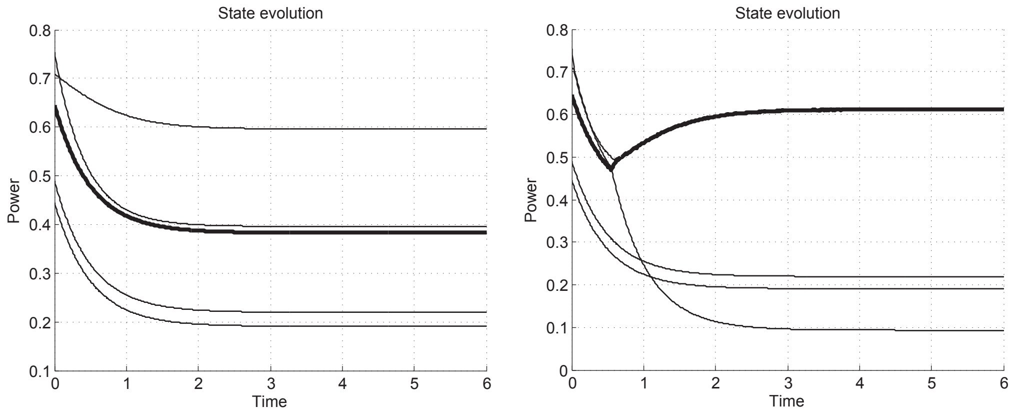


Figure 1: Evolution of the states of the five racers considered in Section 3.4 when: (a) only alliances are allowed; (b) both alliances and obstructions are allowed.

4. Examples of applications

Three possible applications of the nonlinear models described in Section 2 are outlined next.

4.1. Commercial partnerships

Consider the case in which two companies ally with each other to consolidate their presence on the market. Usually, in this kind of agreement the amount of money that the i-th company invests in the alliance is proportional to its own economic strength x_i. Similarly, also the amount of money (strength) that it receives back from the alliance increases with its own strength. Instead, the total profit afforded by the alliance is proportional to the total investment. This situation can be modelled by choosing $G = \gamma(x_i + x_j)$ with $\gamma > 0$ and

$$\varphi(x_i, x_j) = x_i \, G = x_i \, \gamma(x_i + x_j) \,. \tag{22}$$

Note that this model belongs to the class of systems for which Proposition 2 holds. In fact we may choose $\theta(x_i, x_j) = x_i + x_j$, $\psi(x_i) = \gamma x_i$ and $\xi(x_j) = 1$.

4.2. Market competition

Suppose that the market of a specific product is monopolistic until when a new competitor enters the market, causing a loss of profits to the formerly monopolistic company. This loss increases both with the strength of the new competitor (the bigger it is, the larger the percentage of market it gains) and with the strength of the formerly monopolistic company. It seems reasonable to make this loss depend on the product of these two quantities. Hence, the state x_1 of the stronger previously monopolistic company can be modelled as

$$\dot{x}_1 = -\lambda_1 x_1 - x_1 x_2 + b_1 \,, \tag{23}$$

where x_2 denotes the state of the newly entered company.

4.3. Buffer systems

Consider a production system and let x_i represent the "work to be done" by racer x_i. A strategy for distributing the work among the racers (a well–known problem in the context of hybrid systems [30]) can be described by means of the competitive model presented in Subsection 2.2. For the sake of simplicity, we consider only a two–racer system and let x_i denote the content of the buffer associated with the i-th racer. In many practical situations, the time–decay of the buffer content is proportional to the content itself [2]. Such a decay can be modelled with a linear term $-\lambda_i x_i$ for some positive constant λ_i. The input to the buffer can be accounted for by a further constant term b_i. Finally, it is reasonable to assume that, when the buffer content x_i is larger than x_j, the j-th racer will help the i-th racer in doing his job; this behaviour could be modelled by adding to the rate of change of x_i a negative term of the form

$$-\frac{\sigma_{ij}}{1+x_j} \tag{24}$$

with

$$\sigma_{ij} = \begin{cases} \bar{\sigma}_i \,, & \text{if } x_i > x_j \,, \\ 0 \,, & \text{otherwise,} \end{cases}$$

for some positive $\bar{\sigma}_i$. The maximum of (24) occurs when $x_j = 0$, i.e., when the j-th racer has nothing to do. Function (24) decreases when x_j increases, and is zero when $x_j > x_i$ since in this case the j-th racer is the one who needs help. This scenario can be represented as in Figure 2: Σ_1 denotes the region where $\sigma_{12} = \bar{\sigma}_1 > 0$ and $\sigma_{21} = 0$, while Σ_2 denotes the region where $\sigma_{12} = \bar{\sigma}_2 > 0$ and $\sigma_{12} = 0$. The model resulting from the above considerations is then

$$\dot{x}_1 = -\lambda_1 x_1 - \frac{\sigma_{12}}{1+x_2} + b_1 \,, \tag{25}$$

$$\dot{x}_2 = -\lambda_2 x_2 - \frac{\sigma_{21}}{1+x_1} + b_2 \,. \tag{26}$$

---

[2] For example, in the process of hiring workers during the harvesting of a particular crop, the number of hired workers, and hence the work done, is proportional to the amount of crop to be harvested.

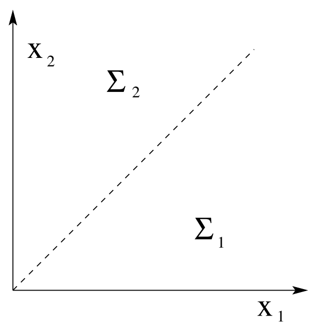


Figure 2: The case of two racers processing the content of two buffers.

Note that for $x_2 = 0$ equation (25) reduces to $\dot{x}_1 = -\lambda_1 x_1 - \sigma_{12} + b_1$. Therefore, to prevent x_1 from becoming negative, $\bar{\sigma}_1$ must be less than b_1. For the same reason, $\bar{\sigma}_2$ must be less than b_2.

5. Simulations

This section shows the simulation results of three nonlinear switching models that describe the interactions among five racers in the presence of (i) alliances only (cooperative model), (ii) obstructions only (competitive model), and (iii) both alliances and obstructions (mixed model). In all models $\Lambda = 2I$.

In the cooperative and mixed cases, the alliances are modelled by means of one of the following functions:

1. $\varphi(x_i, x_j) = \dfrac{x_i}{x_i + x_j}$: "selfish partnership", in which each of the allies increases its strength according to the fraction it has invested in the partnership;

2. $\varphi(x_i, x_j) = \dfrac{x_j}{x_i + x_j}$: "altruistic partnership", in which each of the allies increases its strength according to the fraction invested by its ally;

3. $\varphi(x_i, x_j) = x_i(x_i + x_j)$: "commercial partnership" as in Section 4.1.

Note that Proposition 2 holds in all cases so that at least one alliance always forms. Since functions 1 and 2 are underlinear of degree 2 (and $\lambda_i = 2$, $\forall i$), the system trajectories are expected to be bounded in the first two cases. Instead, the third choice leads to a finite escape time, as shown next.

The obstructions in the competitive and mixed cases are modelled by means of the function:

$$S(x_i, x_j) = \frac{x_j}{x_i + x_j},$$

which is increasing with the strength of the obstructing racer and decreasing with the strength of the obstructed one.

Figures 3, 4 and 5 show the system evolution for different initial conditions and vectors b (defined in the corresponding captions) in the case of alliances of type 1, 2 and 3 above, with or without obstructions. Figure 6 shows the system evolution when only obstructions are present. As already observed, the trajectories are always bounded, except for the case of alliances of type 3 in which the system exhibits a finite escape time.

The switching nature of the systems is revealed by the discontinuities in the derivative of the trajectories, which occur whenever alliances or enmities change. As observed in Remark 2, in some cases two or more racers' states evolve along a sliding surface characterised by a high switching frequency, since a group of racers is continually fighting for supremacy. This behaviour can arise not only in the presence of obstructions, as is the case for linear systems [10]. Indeed, Figures 4 (b) and 5 (b) show that chattering can appear also when only alliances are permitted, provided they are of type 1 (selfish partnerships). This fact has a simple explanation. When the strength of the ally increases, functions 2 and 3 increase, while function 1 decreases. Hence, if the strength of the ally exceeds a certain threshold, breaking the bond and finding a different ally may be more profitable; however, after the end of the alliance, the previous ally gets weaker and soon the alliance becomes profitable again. Note, however, that, even in the case of alliances of type 1, sustained chattering phenomena do not *necessarily* arise, as shown by Figure 3 (b).

Usually, the racers' strength is higher in the case of alliances only, lower in the case of both alliances and obstructions, and even lower in the case of obstructions only. However, this is not a general rule, as already observed in Section 3.4. Figures 4 (c) and (d) show that, if only alliances are allowed as in case (d), racer 1 (blue) has the third position in the ranking and its strength is around 0.4 after 6 time units; when also obstructions are allowed, as in case (c), racer 1 improves both its ranking (it is eventually fighting for supremacy with racer 5) and its strength which goes above 0.6.

6. Conclusions

A family of nonlinear switching models describing the behaviour of a set of greedy racers striving for supremacy has been presented. Both the case in which only alliances are allowed and the case in which obstructing actions towards the strongest competitor are exerted, have been considered. A mixed model, in which both alliances and obstructions are allowed, has been examined as well.

It has been proved that: (i) all systems are positive, (ii) at least one alliance is always established under mild assumptions, and (iii) the solutions are ultimately bounded for some reasonable choice of the functions modelling the alliance criterion. It has also been shown that obstructions may be profitable to obstructing racers.

Simulations have pointed out unexpected effects of the competition rules on the race outcome. For instance, not only the rank of an obstructing racer may improve due to obstructions, but also its strength.

Several extensions of this work can be conceived, with particular regard to the allowed maximal number of allies and rivals and to the kind of (linear and nonlinear) interactions. The possibility of negotiating an alliance in the absence of a reciprocal advantage could be considered as well, since a racer might be willing to sacrifice a part of its profit to obtain a strategic partnership. In addition, different criteria for deciding which are the most convenient rivals could be taken into account: for instance, each racer could obstruct the competitor which is coming immediately before in the ranking. Also, the decision about possible allies could be based on a less myopic criterion: for instance, a racer's goal could be that of maximising its steady–state, or long–term, strength instead of its current strength variation.

References

[1] T. Arnold and U. Schwalbe, "Dynamic coalition formation and the core", *Journal of Economic Behavior & Organization*, vol. 49, no. 3, pp. 363–380, 2002.

[2] J. P. Aubin and A. Cellina, *Differential Inclusions. Set-Valued Maps and Viability Theory*, volume 264 of *Grundlehren der mathematischen Wissenschaften [Foundations of Mathematical Sciences]*. Springer, Berlin, Germany, 1984.

[3] R. M. Axelrod, *The Complexity of Cooperation: Contender-Based Models of Competition and Collaboration*. Princeton University Press, Princeton, NJ, USA, 1997.

[4] R. J. Aumann, "Game theory", in S.N. Durlauf and L.E. Blume, Eds., *The New Palgrave Dictionary of Economics*, 2nd edition. Palgrave Macmillan, Houndmills, Basingstoke, Hampshire, RG21 6XS, England, 2008.

[5] T. Başar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, 2nd edition. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1999.

[6] P. Bednarik, K. Fehl, and D. Semmann, "Costs for switching partners reduce network dynamics but not cooperative behaviour", *Proc. Royal Society B: Biological Sciences*, vol. 281, no. 1792, 2014.

[7] F. Blanchini, P. Colaneri, and M. E. Valcher, "Co–positive Lyapunov functions for the stabilization of positive switched systems", *IEEE Trans. Automat. Contr.*, vol. 57, no. 12, pp. 3038–3050, 2012.

[8] F. Blanchini, P. Colaneri, and M. E. Valcher, "Switched Linear Positive Systems", Foundations and Trends in Systems and Control. Now Publishers, Boston, MA, USA, submitted 2015.

[9] F. Blanchini and S. Miani, *Set–Theoretic Methods in Control*, 2nd edition. Systems and Control: Foundations and Applications. Birkhäuser, Boston, MA, USA, 2015.

[10] F. Blanchini, D. Casagrande, G. Giordano, and U. Viaro, "Properties of Switching–Dynamics Race Models", *Proc. European Control Conference*, Linz, Austria, 2015, pp. 2907–2912.

[11] F. Brandt, V. Conitzer, and U. Endriss, "Computational social choice", in G. Weiss, Ed., *Multiagent Systems*, 2nd edition. MIT Press, Cambridge, MA, USA, 2013.

[12] E. De Santis and G. Pola, "Positive Switching Systems", in C. Commault and N. Marchand, Eds., *Positive Systems: Theory and Applications (POSTA 2006).* Lecture Notes in Control and Information Sciences, vol. 341, Springer, Berlin, Germany, pp. 49–56, 2006.

[13] F. Fu, C. Hauert, M.A. Nowak, and L. Wang, "Reputation–based partner choice promotes cooperation in social networks", *Phys. Rev. E*, vol. 78, no. 2, pp. 026117-1 – 026117-8, 2008.

[14] F. Fu, T. Wu, and L. Wang, "Partner switching stabilizes cooperation in coevolutionary prisoner's dilemma", *Phys. Rev. E*, vol. 79, no. 3, pp. 036101-1 – 036101-7, 2009.

[15] E. Hernandez–Vargas, P. Colaneri, R. Middleton, and F. Blanchini, "Discrete–time control for switched positive systems with application to mitigating viral escape", *Int. J. Robust Nonlinear Control*, vol. 21, no. 10, pp. 1093–1111, 2011.

[16] T. Gross and B. Blasius, "Adaptive coevolutionary networks: a review", *J. Royal Society–Interface*, vol. 5, no. 20, pp. 259–271, 2008.

[17] A. Kianercy and A. Galstyan, "Coevolutionary networks of reinforcement–learning agents", *Phys. Rev. E*, vol. 88, no. 1, pp. 012815-1 – 012815-8, 2013.

[18] H. Konishi and D. Ray, "Coalition formation as a dynamic process", *Journal of Economic Theory*, vol. 110, no. 1, pp. 1–41, 2003.

[19] Y. Li, "The evolution of reputation–based partner–switching behaviors with a cost", *Scientific Reports*, vol. 4, available online: `http://dx.doi.org/10.1038/srep05957`, 2014.

[20] Y. Li, Y. Min, X. Zhu, and J. Cao, "Partner switching promotes cooperation among myopic agents on a geographical plane", *Phys. Rev. E*, vol. 87, no. 2, pp. 022823-1 – 022823-9, 2013.

[21] Y. Li and B. Shen, "The coevolution of partner switching and strategy updating in non–excludable public goods game", *Physica A*, vol. 392, no. 20, pp. 4956–4965, 2013.

[22] D. Liberzon, *Switching in Systems and Control*. Birkhäuser, Boston, MA, USA, 2003.

[23] O. Mason and R.N. Shorten. "On linear copositive Lyapunov functions and the stability of switched positive linear systems", *IEEE Trans. Automat. Contr.*, vol. 52, no. 7, pp. 1346–1349, 2007.

[24] M. Mesterton-Gibbons, S. Gavrilets, J. Gravner, and E. Akçay, "Models of coalition or alliance formation", *Journal of Theoretical Biology*, vol. 274, no. 1, pp. 187–204, 2011.

[25] R. G. Myerson, *Game Theory: Analysis of Conflict*. Harvard University Press, Cambridge, MA, USA, 1997.

[26] J. M. Pacheco, A. Traulsen, and M. A. Nowak, "Coevolution of strategy and structure in complex networks with dynamical linking", *Phys. Rev. Lett.*, vol. 97, no. 25, pp. 258103–258106, 2006.

[27] A. Peleteiro, J. C. Burguillo, J. Ll. Arcos, and J. A. Roidriguez–Aguilar, "Fostering cooperation through dynamic coalition formation and partner switching", *AMC Trans. Autonomous and Adaptive Systems*, vol. 9, no. 1, pp. 1–31, 2014.

[28] M. Perc and A. Szolnoki, "Coevolutionary games - a mini review", *Biosystems*, vol. 99, no. 2, pp. 109–125, 2010.

[29] F. L. Pereira and J. B. Sousa, "A differential game with graph constrained dynamic switching strategies", *Proc. 40th IEEE Conf. Decision and Control*, Orlando, FL, USA, Dec. 2001, pp. 4394–4399.

[30] A. S. Matveev and A. V. Savkin, *Qualitative Theory of Hybrid Dynamical Systems*. Springer Science+Business Media, New York, USA, 2000.

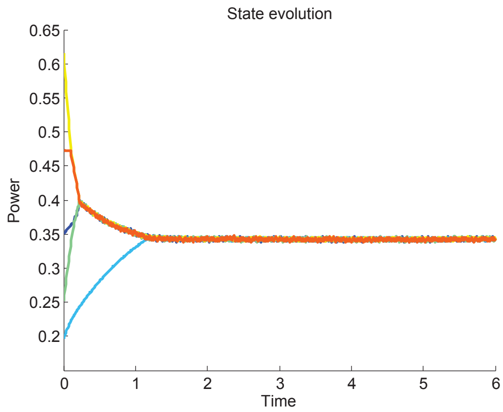
[31] P. L. Schwagmeyer, "Partner switching can favour cooperation in a biological market", *J. Evolutionary Biology*, vol. 27, no. 9, pp. 1765–1774, 2014.

[32] Z. Sun and S. S. Ge, *Stability Theory of Switched Dynamical Systems*. Springer, London, UK, 2011.

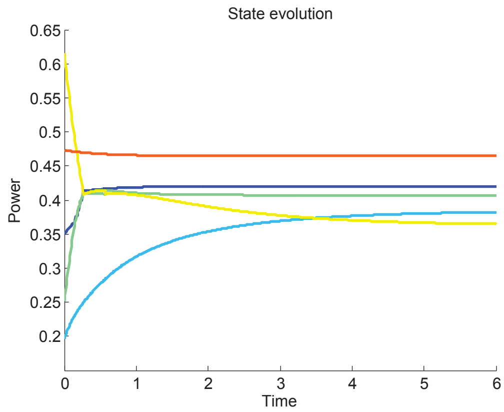
[33] J. Wang, S. Suri, and D. J. Watts, "Cooperation and assortativity with dynamic partner updating", *Proc. Natl. Acad. Sci.*, vol. 109, no. 36, pp. 14363–14368, 2012.

[34] J. M. Webb, *Game Theory: Decisions, Interaction and Evolution*. Springer, London, UK, 2007.

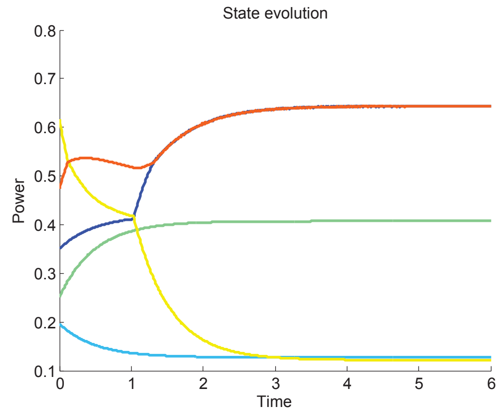
[35] M. G. Zimmermann, V. M. Eguiluz, and M. San Miguel, "Coevolution of dynamical states and interactions in dynamic networks", *Phys. Rev. E*, vol. 69, no. 6, pp. 065102-1 – 065102-4(R), 2004.



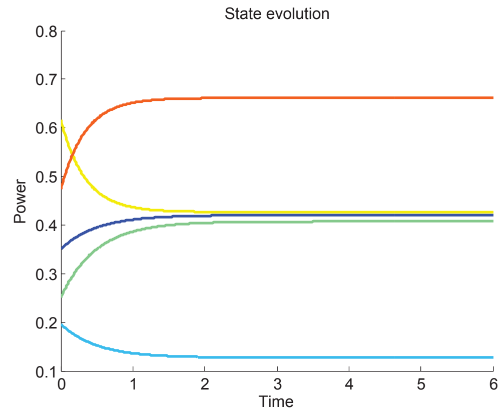
(a) Obstructions and alliances 1.



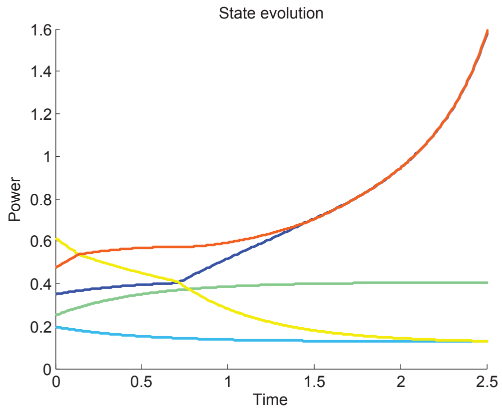
(b) Only alliances 1.



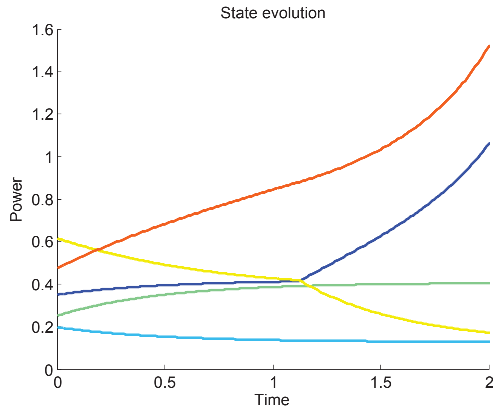
(c) Obstructions and alliances 2.



(d) Only alliances 2.

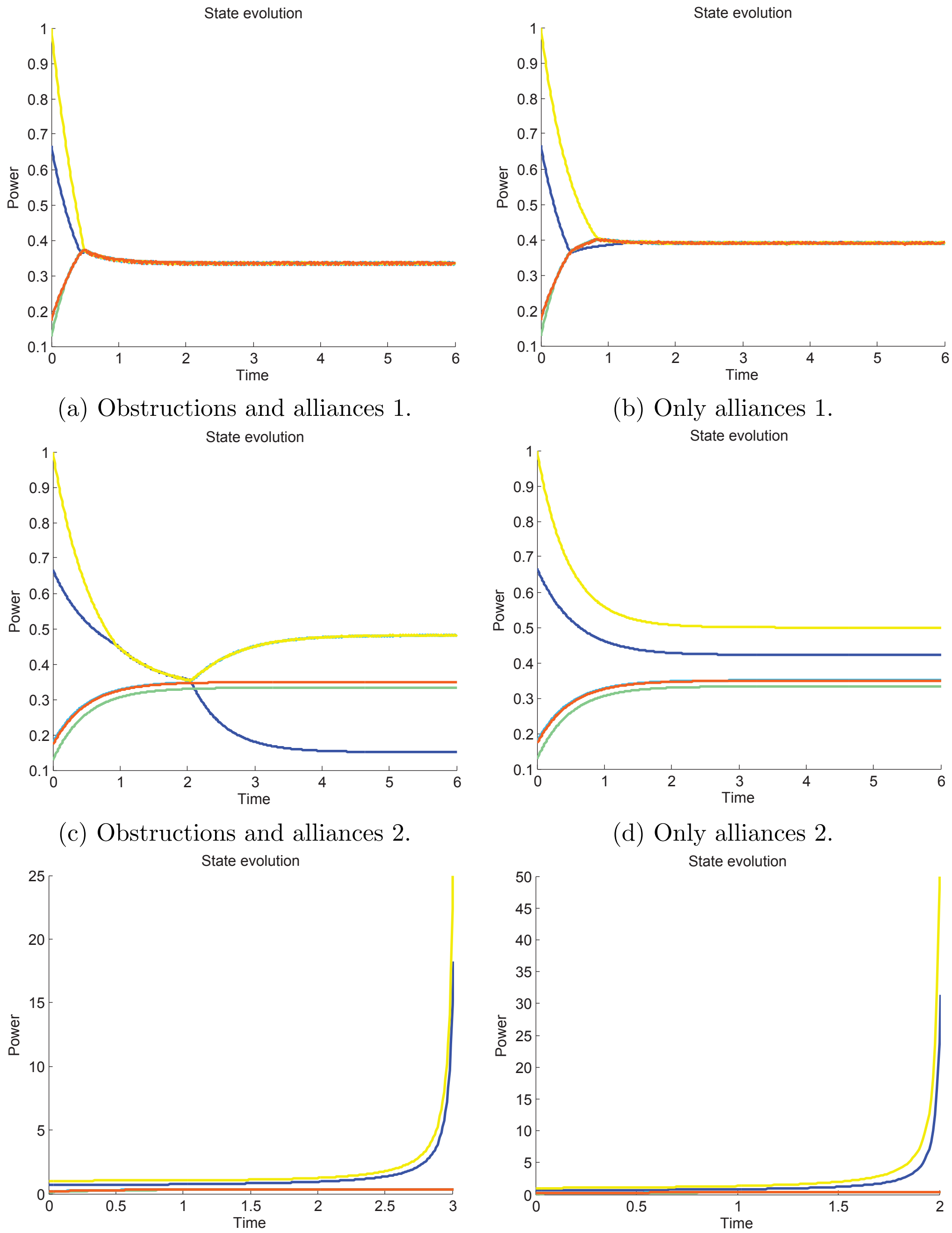


(e) Obstructions and alliances 3.



(f) Only alliances 3.

Figure 3: Evolution of the states of the five racers considered in Section 5: 1-blue, 2-cyan, 3-green, 4-yellow, 5-red. The system is started from $x(0) = [0.3500 \quad 0.1966 \quad 0.2511 \quad 0.6160 \quad 0.4733]^\top$ with $b = [0.8407 \quad 0.2543 \quad 0.8143 \quad 0.2435 \quad 0.9293]^\top$.



(a) Obstructions and alliances 1.

(b) Only alliances 1.

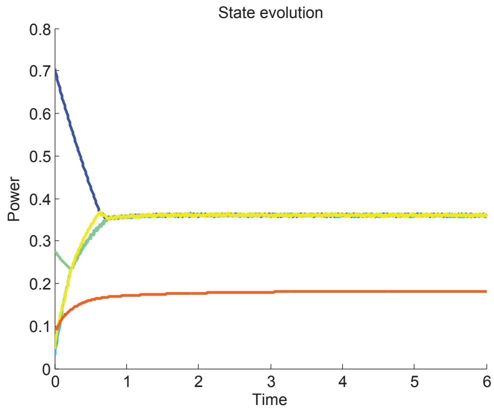
(c) Obstructions and alliances 2.

(d) Only alliances 2.

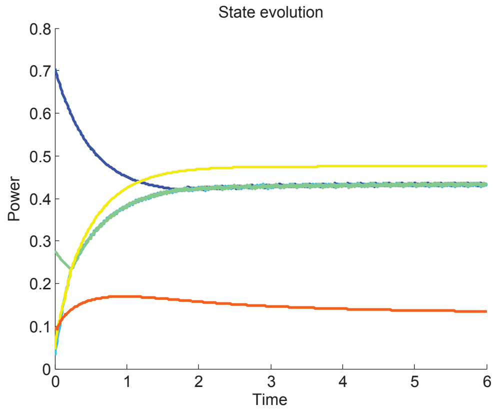
(e) Obstructions and alliances 3.

(f) Only alliances 3.

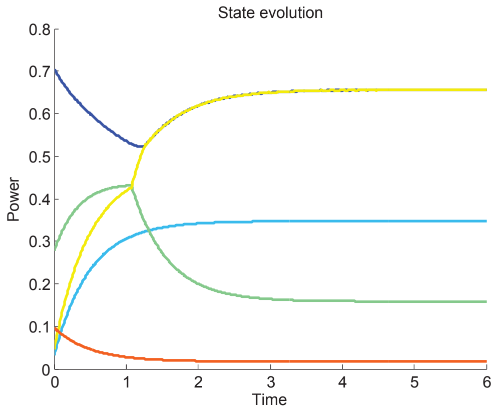
Figure 4: Evolution of the states of the five racers considered in Section 5: 1-blue, 2-cyan, 3-green, 4-yellow, 5-red. The system is started from $x(0) = [0.6665 \quad 0.1781 \quad 0.1280 \quad 0.9991 \quad 0.1711]^\top$ with $b = [0.3015 \quad 0.7011 \quad 0.6663 \quad 0.5391 \quad 0.6981]^\top$.



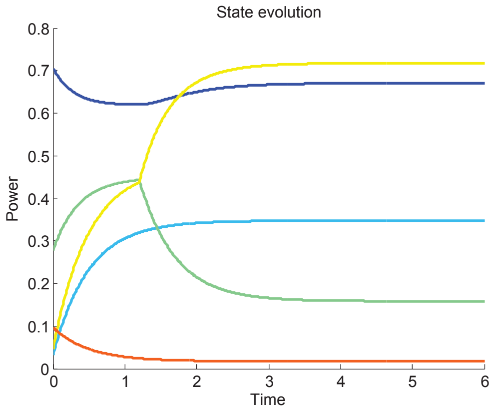
(a) Obstructions and alliances 1.



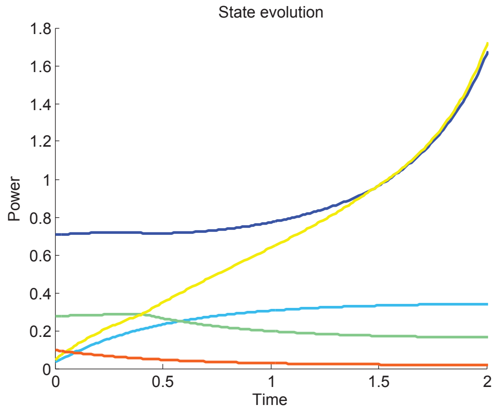
(b) Only alliances 1.



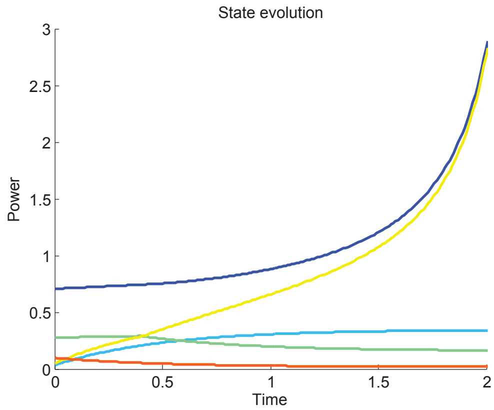
(c) Obstructions and alliances 2.



(d) Only alliances 2.

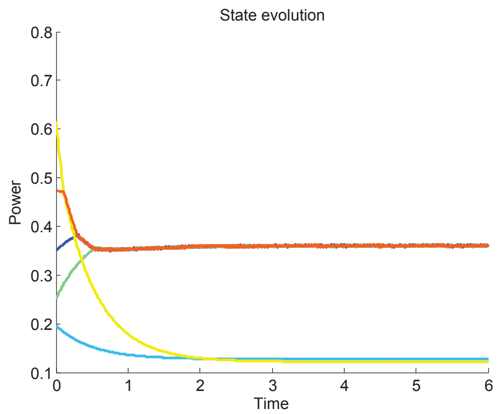


(e) Obstructions and alliances 3.

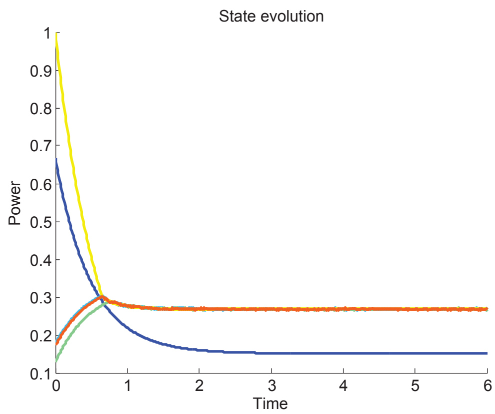


(f) Only alliances 3.

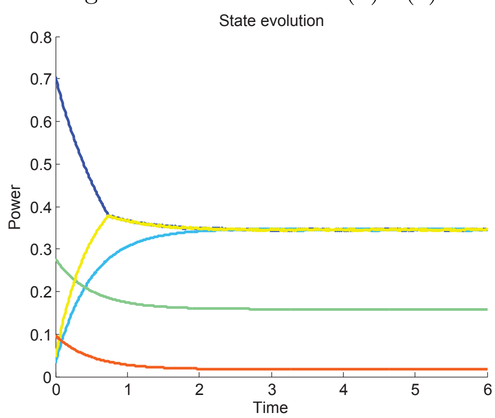
Figure 5: Evolution of the states of the five racers considered in Section 5: 1-blue, 2-cyan, 3-green, 4-yellow, 5-red. The system is started from $x(0) = [0.7060 \quad 0.0318 \quad 0.2769 \quad 0.0462 \quad 0.0971]^\top$ with $b = [0.8235 \quad 0.6948 \quad 0.3171 \quad 0.9502 \quad 0.0344]^\top$.



(a) $x(0)$ and b as in Figure 3



(b) $x(0)$ and b as in Figure 4



(c) $x(0)$ and b as in Figure 5

Figure 6: Evolution of the states of the five racers considered in Section 5, in the case of obstructions only: 1-blue, 2-cyan, 3-green, 4-yellow, 5-red.