A saturated strategy robustly ensures stability of the cooperative equilibrium for Prisoner's dilemma

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***Abstract*— We study diffusion of cooperation in a two-population game in continuous time. At each instant, the game involves two random individuals, one from each population. The game has the structure of a Prisoner's dilemma where each player can choose either to cooperate (c) or to defect (d), and is reframed within the field of approachability in two-player repeated game with vector payoffs. We turn the game into a dynamical system, which is positive, and propose a saturated strategy that ensures local asymptotic stability of the equilibrium (c, c) for any possible choice of the payoff matrix. We show that there exists a rectangle, in the space of payoffs, which is positively invariant for the system. We also prove that there exists a region in the space of payoffs for which the equilibrium solution (d, d) is an attractor, while all of the trajectories originating outside that region, but still in the positive quadrant, are ultimately bounded in the rectangle and, under suitable assumptions, converge to the solution (c, c).**

I. Introduction

Two large populations of individuals play a game in continuous time. At each instant, a random individual of the first population engages in play with a random opponent extracted from the second population. The resulting payoff, which depends on the action profiles of both players, is a vector. The game is a Prisoner's dilemma [25], [16]: each player can choose either to cooperate (c) or to defect (d). The defection of a single player is most beneficial for the defecting player and most harmful for the other player, while for both players mutual cooperation is preferable to mutual defection. We reframe the problem within the field of approachability in two-player repeated game with vector payoffs [8], [7].

In a two-player repeated game with vector payoffs, a set of payoffs is approachable [10] if the row player has a strategy such that, for any strategy used by her opponent, her average payoff uniformly approaches the set with probability 1. The notion of *approachability* is due to [10]: Blackwell's Theorem, giving conditions for a set to be approachable, is often used to prove convergence in different application domains, *e.g.*, allocation processes in coalitional games [17], regret minimization [19], [15] and weak approachability [28]. A similar concept can be found in adaptive learning and evolutionary games [14], [24]. The original discrete-time formulation of approachability has been adapted to continuous-time repeated games in [15], which also highlights the connection with Lyapunov theory. An extension to infinite-dimensional spaces is due to [18]. Approachability shares striking similarities with differential game theory and, as such, can be studied using differential calculus and stability theory [22], [26]. Approachability and differential inclusions [2] are studied in [22], where it is highlighted that Blackwell's theorem is a generalization of von Neumann's minmax theorem [29]. [26] proposes a set-valued analytical perspective [1], [3]; approachable and discriminating sets can be reframed within the context of set invariance theory [11]. A core concept in approachability is that of nonanticipative strategies, similar to those in differential games [5], [12], [26], [23], [27]; classical feedback strategies in differential games are special nonanticipative strategies. *Excludability* of sets is a complementary notion to approachability [20] and another concept related to approachability is *attainability* [6], [21], useful in application domains such as transportation, distribution and production networks.

In this paper, we set up the approachability problem for the two-player repeated game with vector payoffs in a system-theoretical framework, turning the game into a positive dynamical system and showing that it can be reviewed as a population game. In Smale's *good strategies* for Prisoner's dilemma [25] and subsequent developments [9], [8], [7], the decision of each player is based on the knowledge of the whole current average payoff vector. Here, we propose a novel saturated strategy where each player's decision is based on the exclusive knowledge of her/his own current average payoff and of the diagonal entries of her/his own payoff matrix: information about the other player is not required, which makes this strategy well suited also for games with incomplete information [4]. The main contributions can

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be summarized as follows:

- the saturated strategy, for both players, ensures that the equilibrium (c, c) is locally asymptotically stable for any choice of the payoff matrix;
- there exists a rectangle $\mathcal{R}$, in the space of payoffs, which is a positively invariant set for the system;
- there exists a region $\mathcal{D}$, in the space of payoffs, for which the equilibrium solution (d, d) is an attractor, namely, all of the trajectories originating in the region converge to the equilibrium;
- all of the trajectories originating outside region $\mathcal{D}$, but still in the positive quadrant, are ultimately bounded in the rectangle $\mathcal{R}$: if the equilibrium (d, d) is unstable, they all converge to the equilibrium (c, c);
- under suitable assumptions on the payoff matrix values, the solution (d, d) can be rendered unstable by a proper choice of the saturated strategy.

Numerical simulations illustrate the evolution of the game with the proposed saturated strategy.

II. Problem Formulation and Motivation

We consider a Prisoner's dilemma with two players, each striving to maximise its payoff. Each player can choose either to cooperate or to defect; depending on the players' choice, the average payoff vector (x, y) evolves according to the game payoff matrix

$$\begin{bmatrix} (\alpha_1, \alpha_2) & (\gamma_1, \gamma_2) \\ (\beta_1, \beta_2) & (\delta_1, \delta_2) \end{bmatrix}, \tag{1}$$

whose entries represent the payoff vectors when both players cooperate (α), player 1 only defects (β), player 2 only defects (γ) and both players defect (δ). Player 1, whose average payoff is x, chooses the row, while player 2, whose average payoff is y, chooses the column. In the usual formulation, the defection of a single player is beneficial to the highest degree for the defecting player and harmful to the highest degree for the other player, while for both players mutual cooperation is more advantageous than mutual defection.

Assumption 1: In the payoff matrix (1),

$$\begin{cases} \beta_1 > \alpha_1 > \delta_1 > \gamma_1, \\ \gamma_2 > \alpha_2 > \delta_2 > \beta_2. \end{cases} \tag{2}$$

Fig. 1a shows the points corresponding to the payoff vectors in the outcome plane (x, y): $A = (\alpha_1, \alpha_2)$, $B = (\beta_1, \beta_2)$, $C = (\gamma_1, \gamma_2)$ and $D = (\delta_1, \delta_2)$.

We assume that the two players adopt a mixed strategy

$$u = u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}, \qquad v = v(y) = \begin{bmatrix} v_1(y) \\ v_2(y) \end{bmatrix},$$

such that $u_1 + u_2 = 1$, with $u_1, u_2 \geq 0$, and $v_1 + v_2 = 1$, with $v_1, v_2 \geq 0$. Then, defining the matrices

$$F = \begin{bmatrix} \alpha_1 & \gamma_1 \\ \beta_1 & \delta_1 \end{bmatrix}, \quad G = \begin{bmatrix} \alpha_2 & \gamma_2 \\ \beta_2 & \delta_2 \end{bmatrix},$$

the evolution of the average payoff in the repeated game (or, as will be discussed in Section II-A, of the average payoff over the population) is described by the system

$$\begin{cases} \dot{x}(t) = -x(t) + u(x(t))^\top F v(y(t)), \\ \dot{y}(t) = -y(t) + u(x(t))^\top G v(y(t)). \end{cases} \tag{3}$$

Given $h, k \in \mathbb{R}$ such that $h < k$, we define the *saturation function* as

$$\mathrm{sat}_{[h\ k]} f(x) = \begin{cases} k & \text{if} \quad f(x) \geq k, \\ f(x) & \text{if } h < f(x) < k, \\ h & \text{if} \quad f(x) \leq h, \end{cases} \tag{4}$$

and we consider the *saturated strategy*

$$\begin{cases} u_1(x) = 1 - \sigma_1(x), \\ u_2(x) = \sigma_1(x), \\ v_1(y) = 1 - \sigma_2(y), \\ v_2(y) = \sigma_2(y), \end{cases} \tag{5}$$

where

$$\begin{cases} \sigma_1(x) = \left[\mathrm{sat}_{[0\ 1]} \left(\frac{\alpha_1 - x}{\alpha_1 - \delta_1} \right) \right]^p, \\ \sigma_2(y) = \left[\mathrm{sat}_{[0\ 1]} \left(\frac{\alpha_2 - y}{\alpha_2 - \delta_2} \right) \right]^p, \end{cases} \tag{6}$$

for $p \in \mathbb{N}$, $p \geq 1$. Function $\sigma_1(x)$ is illustrated in Fig. 1c for various values of p: the larger is p, the steeper is the saturated function in the interval $[\delta_1\ \alpha_1]$.

For each player, the proposed strategy is exclusively based on her/his own current average payoff and on the diagonal entries of her/his own payoff matrix and information about the other player is not required.

Adopting the strategy (5)–(6), system (3) becomes

$$\begin{cases} \dot{x} = -x + \alpha_1 + (\beta_1 - \alpha_1)\sigma_1(x) + (\gamma_1 - \alpha_1)\sigma_2(y) \\ \qquad + (\alpha_1 + \delta_1 - \beta_1 - \gamma_1)\sigma_1(x)\sigma_2(y), \\ \dot{y} = -y + \alpha_2 + (\beta_2 - \alpha_2)\sigma_1(x) + (\gamma_2 - \alpha_2)\sigma_2(y) \\ \qquad + (\alpha_2 + \delta_2 - \beta_2 - \gamma_2)\sigma_1(x)\sigma_2(y). \end{cases} \tag{7}$$

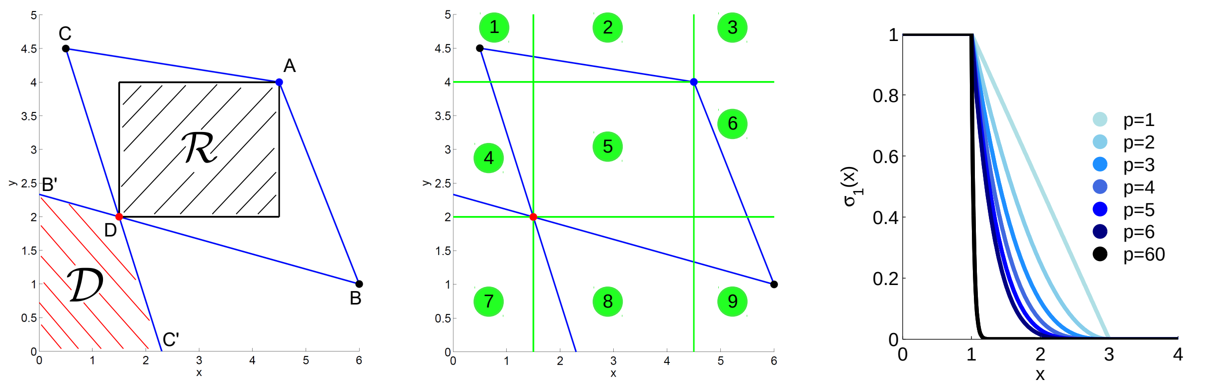
A. A Population-Game Perspective

Equation (3) is in the same spirit as in Hart and Mas-Colell's paper [15] on continuous-time approachability. To see this, consider the time-average expected (over opponent's play) payoff defined as

$$\Gamma(s) = \frac{1}{s} \int_0^s \left[\begin{array}{c} u^\top F v \\ u^\top G v \end{array} \right] d\tau \in \mathbb{R}^2.$$

If we rescale the time window using $s = e^t$, take $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \Gamma(e^t)$ and differentiate with respect to t, we obtain the differential equation (3). Note that, after rescaling the time window, we have

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \int_0^1 \left[\begin{array}{c} u^\top F v \\ u^\top G v \end{array} \right] d\tau \in \mathbb{R}^2.$$



(a) The outcome plane (x,y). (b) The outcome plane (x,y). (c) Function $\sigma_1(x)$, $\alpha_1 = 3$, $\delta_1 = 1$.

Fig. 1: The outcome plane (x,y): (a) with the indication of sets $\mathcal{D}$ and $\mathcal{R}$; (b) divided in nine regions. (c) Plot of function $\sigma_1(x)$, with $\alpha_1 = 3$ and $\delta_1 = 1$, for various values of p.

Adopting a *population-game dynamics* perspective, the state $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \in \mathbb{R}^2$ represents the current average payoff over the population.

III. Main Results

We begin by showing that both the cooperative point A and the non-cooperative point D are equilibria for the dynamical system (7), and that A is always locally asymptotically stable if a saturated strategy with $p > 1$ is chosen, regardless of parameter values.

Proposition 1: For any $p \geq 1$, both $A = (\alpha_1, \alpha_2)$ and $D = (\delta_1, \delta_2)$ are equilibria for system (7). Furthermore, for $p > 1$, $A = (\alpha_1, \alpha_2)$ is locally asymptotically stable for any choice of the values in the payoff matrix (1).

Proof: If $(x, y) = (\alpha_1, \alpha_2)$, then $\sigma_1(x) = \sigma_2(y) = 0$, while, if $(x, y) = (\delta_1, \delta_2)$, then $\sigma_1(x) = \sigma_2(y) = 1$; in both cases, as can be computed by substitution in system (7), $\dot{x} = \dot{y} = 0$, hence both of the points are equilibria for the system.

The system Jacobian matrix in the non-saturated region is shown in Table I. When computed at the cooperative equilibrium, for $p > 1$, it becomes $J_{(\alpha_1,\alpha_2)} = -I$, where I is the identity (and analogously for the saturated regions 2, 3 and 6 in Fig. 1b). Hence, the equilibrium $A = (\alpha_1, \alpha_2)$ is locally asymptotically stable, for any choice of the payoff matrix values. ∎

The equilibrium A is not necessarily stable if $p = 1$, while the equilibrium D can be either stable or unstable, depending on the payoff matrix values and on p.

Example 1: Consider system (7) with payoff matrix

$$\begin{bmatrix} (\alpha_1, \alpha_2) & (\gamma_1, \gamma_2) \\ (\beta_1, \beta_2) & (\delta_1, \delta_2) \end{bmatrix} = \begin{bmatrix} (4.5, 4) & (0.5, 4.5) \\ (6, 1) & (1.5, 2) \end{bmatrix}. \quad (8)$$

If $p = 1$, the equilibrium $A = (\alpha_1, \alpha_2)$ is unstable, while the equilibrium $D = (\delta_1, \delta_2)$ is asymptotically stable.

To state our main theorem, we need to consider two sets in the outcome plane: the quadrilateral

$$\begin{aligned} \mathcal{D} = \{(x, y) : x \geq 0,\ y \geq 0, \\ (\beta_1 - \delta_1)y + (\delta_2 - \beta_2)x \leq \beta_1\delta_2 - \beta_2\delta_1, \\ (\gamma_1 - \delta_1)y + (\delta_2 - \gamma_2)x \geq \gamma_1\delta_2 - \gamma_2\delta_1\} \end{aligned} \quad (9)$$

and the rectangle corresponding to the non-saturated region,

$$\mathcal{R} = \{(x, y) : \delta_1 \leq x \leq \alpha_1 \text{ and } \delta_2 \leq y \leq \alpha_2\}, \quad (10)$$

both shown in Fig. 1a.

Theorem 1: Given system (7) under Assumption 1, the following statements hold.

(a) System (7) is positive.[1]
(b) The rectangle $\mathcal{R}$ is a positively invariant set for the system.
(c) All of the trajectories originating in $\mathcal{D}$ converge to the equilibrium $D = (\delta_1, \delta_2)$.
(d) All of the trajectories originating in $\mathbb{R}^2_+ \setminus \mathcal{D}$ are ultimately bounded in $\mathcal{R}$.

Proof: The lines $x = \delta_1$, $y = \delta_2$, $x = \alpha_1$ and $y = \alpha_2$ divide the positive orthant $\mathbb{R}^2_+ = \{(x, y) : x \geq 0 \text{ and } y \geq 0\}$ in nine regions, as shown in Fig. 1b. According to the values of the saturation functions inside each region, we can compute the derivatives $\dot{x}$ and $\dot{y}$.

• Region 1: $x < \delta_1$ and $y > \alpha_2$, hence $\sigma_1(x) = 1$ and $\sigma_2(y) = 0$, $\dot{x} = -x + \beta_1 > 0$ and $\dot{y} = -y + \beta_2 < 0$.
• Region 2: $\delta_1 \leq x \leq \alpha_1$ and $y > \alpha_2$, hence $\sigma_2(y) = 0$, $\dot{x} = -x + \alpha_1 + (\beta_1 - \alpha_1)\sigma_1(x) \geq 0$ (strictly positive if $x \neq \alpha_1$) and $\dot{y} = -y + \alpha_2 + (\beta_2 - \alpha_2)\sigma_1(x) < 0$.
• Region 3: $x > \alpha_1$ and $y > \alpha_2$, hence $\sigma_1(x) = \sigma_2(y) = 0$, $\dot{x} = -x + \alpha_1 < 0$ and $\dot{y} = -y + \alpha_2 < 0$.
• Region 4: $x < \delta_1$ and $\delta_2 \leq y \leq \alpha_2$, hence $\sigma_1(x) = 1$, $\dot{x} = -x + \beta_1 + (\delta_1 - \beta_1)\sigma_2(y) \geq (\beta_1 - \delta_1)[1 - \sigma_2(y)] > 0$

[1] A system is *positive* if the positive orthant is a positively invariant set for the system: the state variables are always positive in value.

$$J_{(x,y)} = \begin{bmatrix} -1 - p(\alpha_1 - x)^{p-1}\left[\frac{\beta_1-\alpha_1}{(\alpha_1-\delta_1)^p} + \frac{(\alpha_1+\delta_1-\beta_1-\gamma_1)(\alpha_2-y)^p}{(\alpha_1-\delta_1)^p(\alpha_2-\delta_2)^p}\right] & -p(\alpha_2 - y)^{p-1}\left[\frac{\gamma_1-\alpha_1}{(\alpha_2-\delta_2)^p} + \frac{(\alpha_1+\delta_1-\beta_1-\gamma_1)(\alpha_1-x)^p}{(\alpha_1-\delta_1)^p(\alpha_2-\delta_2)^p}\right] \\ -p(\alpha_1 - x)^{p-1}\left[\frac{\beta_2-\alpha_2}{(\alpha_1-\delta_1)^p} + \frac{(\alpha_2+\delta_2-\beta_2-\gamma_2)(\alpha_2-y)^p}{(\alpha_1-\delta_1)^p(\alpha_2-\delta_2)^p}\right] & -1 - p(\alpha_2 - y)^{p-1}\left[\frac{\gamma_2-\alpha_2}{(\alpha_2-\delta_2)^p} + \frac{(\alpha_2+\delta_2-\beta_2-\gamma_2)(\alpha_1-x)^p}{(\alpha_1-\delta_1)^p(\alpha_2-\delta_2)^p}\right] \end{bmatrix}$$

TABLE I: Jacobian of system (7) for $\delta_1 \leq x \leq \alpha_1$ and $\delta_2 \leq y \leq \alpha_2$.

and $\dot{y} = -y + \beta_2 + (\delta_2 - \beta_2)\sigma_2(y) \leq (\beta_2 - \delta_2)[1 - \sigma_2(y)] \leq 0$ (strictly negative if $y \neq \delta_2$).

- Region 5, namely region $\mathcal{R}$ in (10), is positively invariant: in fact, if $x = \delta_1$, $\sigma_1(x) = 1$ and $\dot{x} = (\beta_1 - \delta_1)[1 - \sigma_2(y)] \geq 0$; if $x = \alpha_1$, $\sigma_1(x) = 0$ and $\dot{x} = (\gamma_1 - \alpha_1)\sigma_2(y) \leq 0$; if $y = \delta_2$, $\sigma_2(y) = 1$ and $\dot{y} = (\gamma_2 - \delta_2)[1 - \sigma_1(x)] \geq 0$; if $y = \alpha_2$, $\sigma_2(y) = 0$ and $\dot{y} = (\beta_2 - \alpha_2)\sigma_1(x) \leq 0$. This proves statement (b).
- Region 6: $x > \alpha_1$ and $\delta_2 \leq y \leq \alpha_2$, hence $\sigma_1(x) = 0$, $\dot{x} = -x + \alpha_1 + (\gamma_1 - \alpha_1)\sigma_2(y) < 0$ and $\dot{y} = -y + \alpha_2 + (\gamma_2 - \alpha_2)\sigma_2(y) \geq 0$ (null only if $y = \alpha_2$).
- Region 7: $x < \delta_1$ and $y < \delta_2$, hence $\sigma_1(x) = \sigma_2(y) = 1$, $\dot{x} = -x + \delta_1 > 0$ and $\dot{y} = -y + \delta_2 > 0$.
- Region 8: $\delta_1 \leq x \leq \alpha_1$ and $y < \delta_2$, hence $\sigma_2(y) = 1$, $\dot{x} = -x + \gamma_1 + (\delta_1 - \gamma_1)\sigma_1(x) \leq (\gamma_1 - \delta_1)[1 - \sigma_1(x)] \leq 0$ (strictly negative if $x \neq \delta_1$) and $\dot{y} = -y + \gamma_2 + (\delta_2 - \gamma_2)\sigma_1(x) > (\gamma_2 - \delta_2)[1 - \sigma_1(x)] \geq 0$.
- Region 9: $x > \alpha_1$ and $y < \delta_2$, hence $\sigma_1(x) = 0$ and $\sigma_2(y) = 1$, $\dot{x} = -x + \gamma_1 < 0$ and $\dot{y} = -y + \gamma_2 > 0$.

Statement (a) follows from the fact that, for any point of the regions 1, 4 and 7, including $x = 0$, $\dot{x} > 0$, while for any point of the regions 7, 8 and 9, including $y = 0$, $\dot{y} > 0$, hence the positive orthant is positively invariant.

Consider now the segments $B'D$ and DC' in Fig. 1a: segment $B'D$ lies on the line $(\beta_1 - \delta_1)y + (\delta_2 - \beta_2)x = \beta_1\delta_2 - \beta_2\delta_1$, while segment DC' lies on the line $(\gamma_1 - \delta_1)y + (\delta_2 - \gamma_2)x = \gamma_1\delta_2 - \gamma_2\delta_1$. If we compute the normal component of the derivative, for all points belonging to these segments, and we recall that $\sigma_1(x) = 1$ on $B'D$ and $\sigma_2(y) = 1$ on DC', we obtain $(\beta_1 - \delta_1)\dot{y} + (\delta_2 - \beta_2)\dot{x} = 0$ and $(\gamma_1 - \delta_1)\dot{y} + (\delta_2 - \gamma_2)\dot{x} = 0$. Hence, no trajectory can cross these lines (actually, the two segments are invariant sets). As a consequence, the trajectories originating in $\mathcal{D}$ are bounded in $\mathcal{D}$ and, due to the sign of the derivatives in regions 4, 7 and 8, converge to the equilibrium D for large enough time, thus proving statement (c).

The signs of the derivatives computed above also show that any trajectory starting in $\mathbb{R}^2_+ \setminus \mathcal{D}$ (which is bounded in $\mathbb{R}^2_+ \setminus \mathcal{D}$) converges to $\mathcal{R}$ for large enough time, hence proving statement (d). ∎

The proof of Theorem 1 entails the following result.

Corollary 1: System (7) does not admit equilibrium points outside the rectangle $\mathcal{R}$.

Does the proposed saturated strategy, with p large enough, ensure that all trajectories originating in $\mathbb{R}^2_+ \setminus \mathcal{D}$ converge to the equilibrium $A = (\alpha_1, \alpha_2)$, when the equilibrium D is unstable? To investigate this problem, we just need to consider all of the trajectories originating in $\mathcal{R} \setminus D$ (since all trajectories in $\mathbb{R}^2_+ \setminus \mathcal{D}$ are ultimately bounded in $\mathcal{R}$ and $D = (\delta_1, \delta_2)$ is an equilibrium point).

The system Jacobian computed at $D = (\delta_1, \delta_2)$ is

$$J_{(\delta_1,\delta_2)} = \begin{bmatrix} -1 - p\frac{\delta_1-\gamma_1}{\alpha_1-\delta_1} & p\frac{\beta_1-\delta_1}{\alpha_2-\delta_2} \\ p\frac{\gamma_2-\delta_2}{\alpha_1-\delta_1} & -1 - p\frac{\delta_2-\beta_2}{\alpha_2-\delta_2} \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + p\begin{bmatrix} -\frac{\delta_1-\gamma_1}{\alpha_1-\delta_1} & \frac{\beta_1-\delta_1}{\alpha_2-\delta_2} \\ \frac{\gamma_2-\delta_2}{\alpha_1-\delta_1} & -\frac{\delta_2-\beta_2}{\alpha_2-\delta_2} \end{bmatrix} = -I + p\Theta.$$

Both $J_{(\delta_1,\delta_2)}$ and Θ are irreducible Metzler matrices (their off-diagonal entries are nonnegative) with negative diagonal entries. It is worth recalling that a Metzler matrix has a real dominant eigenvalue, associated with a positive eigenvector (the Frobenius eigenvector) [13].

Lemma 1: There exists a finite $p \geq 1$ for which the equilibrium D of system (7) is exponentially unstable if and only if matrix Θ has a positive eigenvalue.

Proof: It is immediate, since, denoting by θ_i (with $i = 1, 2$) the eigenvalues of Θ, which are real, the eigenvalues of $J_{(\delta_1,\delta_2)}$ are $\lambda_i = -1 + p\theta_i$. ∎

Then, we can state the following.

Theorem 2: There exists a finite $p \geq 1$ such that all of the trajectories of system (7) originating in $\mathcal{R} \setminus (\delta_1, \delta_2)$ converge to the equilibrium A if and only if matrix Θ has a positive eigenvalue.

Proof: **Necessity.** By contradiction, if matrix Θ does not have positive eigenvalues, then the equilibrium D is locally asymptotically stable and for any p there exists a neighborhood $\mathcal{N}$ of D, having a nonempty intersection with $\mathcal{R} \setminus (\delta_1, \delta_2)$, such that the trajectories originating in $\mathcal{N}$ converge to the equilibrium D, and not to the equilibrium A.

Sufficiency. If matrix Θ has a positive eigenvalue, then the equilibrium D is unstable for p large enough: the corresponding Jacobian $J_{(\delta_1,\delta_2)}$ has a real positive eigenvalue associated with a positive Frobenius eigenvector. Consider, for simplicity, the new variables $z = \frac{\alpha_1-x}{\alpha_1-\delta_1}$ and $w = \frac{\alpha_2-y}{\alpha_2-\delta_2}$. In $\mathcal{R}$, both z and w take values between 0 and 1. This change of variables, applied to system (7), gives the system

$$\begin{cases} \dot{z} = -z - \mu_1 z^p + \nu_1 w^p + (1 + \mu_1 - \nu_1)z^p w^p, \\ \dot{w} = -w + \mu_2 z^p - \nu_2 w^p + (1 - \mu_2 + \nu_2)z^p w^p, \end{cases} \tag{11}$$

where $\mu_1 = \frac{\beta_1-\alpha_1}{\alpha_1-\delta_1}$, $\nu_1 = -\frac{\gamma_1-\alpha_1}{\alpha_1-\delta_1}$, $\mu_2 = -\frac{\beta_2-\alpha_2}{\alpha_2-\delta_2}$, $\nu_2 = \frac{\gamma_2-\alpha_2}{\alpha_2-\delta_2}$ are positive values. The equilibrium points are transformed as $A' = (0, 0)$ and $D' = (1, 1)$ and the

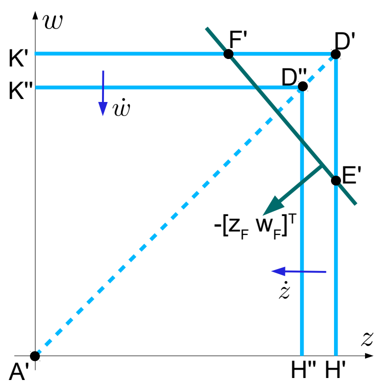


Fig. 2: Plane (z, w), construction for the proof of Theorem 2.

square $A'H'D'K'$ in Fig. 2 is a positively invariant set for the system. Our aim is to show convergence to A'.

Since D' is unstable, we can consider the direction of the Frobenius eigenvector $[z_F \;\; w_F]^\top$, with $z_F,\ w_F > 0$, and the Lyapunov-like function

$$U(z, w) = z_F z + w_F w,$$

which is zero at A' and positive for any other point in $\mathbb{R}^2_+$. We can always find a triangle, $D'E'F'$ in Fig. 2, corresponding to the set

$$\mathcal{T}_\sigma = \{(z, w) : U(z, w) > \sigma,\, 0 \le z \le 1,\, 0 \le w \le 1\},$$

such that all the trajectories originating in $\mathcal{T}_\sigma \setminus D'$ are repelled, namely $\dot{U}(z, w) = z_F \dot{z} + w_F \dot{w} < 0$ for all $(z, w) \in \mathcal{T}_\sigma \setminus D'$. Given a point $D'' = (\bar{z}, \bar{w})$ on the diagonal $A'D'$ ($\bar{z} = \bar{w}$), we can always find a value of p such that the inequalities $\dot{z} < 0$ and $\dot{w} < 0$ are satisfied for all $0 < z \le \bar{z}$ and all $0 < w \le \bar{w}$ such that $z = w$. By picking p large enough, we can choose such a point D'' arbitrarily close to D' and, in particular, inside the triangle $\mathcal{T}_\sigma$. Note that, if $\dot{U}(z, w) < 0$ holds for $p = \bar{p}$, it holds for any $p \ge \bar{p}$: in fact, denoting $\zeta = z^p$, $\omega = w^p$, $w_F\mu_2 - z_F\mu_1 = q$, $z_F\nu_1 - w_F\nu_2 = r$ and $[z_F(1 + \mu_1 - \nu_1) + w_F(1 - \mu_2 + \nu_2)] = s$,

$$\begin{aligned}\dot{U} &= -(z_F z + w_F w) + q z^p + r w^p + s z^p w^p \\ &= -(z_F \sqrt[p]{\zeta} + w_F \sqrt[p]{\omega}) + q\zeta + r\omega + s\zeta\omega\end{aligned}$$

is decreasing in p for $0 \le z, w \le 1$ ($0 \le \zeta, \omega \le 1$). Then, we consider the Lyapunov-like function

$$V(z, w) = \max\{z, w\},$$

which is zero at A' and positive for any other point in $\mathbb{R}^2_+$. If $z > w$, $\dot{V}(z, w) = \dot{z}$, while if $z < w$, $\dot{V}(z, w) = \dot{w}$. Now, fix $z = \bar{z}$ and consider the segment $D''H''$ in Fig. 2, with $0 \le w < \bar{w}$ (hence, $z > w$), where

$$\dot{\bar{z}} = -(\bar{z} + \mu_1 \bar{z}^p) + w^p[\nu_1 + (1 + \mu_1 - \nu_1)\bar{z}^p].$$

Since $\dot{\bar{z}} < 0$ for $w = \bar{w}$ (by construction) and for $w = 0$ (because $\bar{z} + \mu_1 \bar{z}^p > 0$), it is $\dot{\bar{z}} < 0$ for all w in the segment, in view of linearity with respect to w^p. Then, fix $w = \bar{w}$ and consider the segment $D''K''$ in Fig. 2, with $0 \le z < \bar{z}$ (hence, $z < w$), where

$$\dot{\bar{w}} = -(\bar{w} + \nu_2 \bar{w}^p) + z^p[\mu_2 + (1 - \mu_2 + \nu_2)\bar{w}^p].$$

Since $\dot{\bar{w}} < 0$ for $z = \bar{z}$ (by construction) and for $z = 0$ (because $\bar{w} + \nu_2 \bar{w}^p > 0$), it is $\dot{\bar{w}} < 0$ for all z in the segment, in view of linearity with respect to z^p. Therefore, $\dot{V}(z, w) < 0$ for $z = \bar{z}$, $0 \le w \le \bar{w}$ and for $w = \bar{w}$, $0 \le z \le \bar{z}$, hence trajectories originating in the square $\mathcal{S} = \{(z, w) : 0 \le z \le \bar{z}$ and $0 \le w \le \bar{w}\}$ ($A'H''D''K''$ in Fig. 2) cannot escape $\mathcal{S}$ and will eventually converge to $A' = (0, 0)$, because, by construction, the same reasoning holds for any point $(\tilde{z}, \tilde{w})$ such that $\tilde{z} = \kappa\bar{z}$ and $\tilde{w} = \kappa\bar{w}$, with $0 < \kappa \le 1$.

Then, the trajectories escaping from $\mathcal{T}_\sigma$ can enter either the square $\mathcal{S}$ (and converge to A), or the region $\mathcal{H}$ where $\bar{z} < z \le 1$ ($H''H'D'D''$ in Fig. 2), or the region $\mathcal{K}$ where $\bar{w} < w \le 1$ ($K''K'D'D''$ in Fig. 2). In region $\mathcal{H}$, $\dot{z} < 0$; in region $\mathcal{K}$, $\dot{w} < 0$. Hence, the trajectories from both $\mathcal{H}$ and $\mathcal{K}$ converge to the square $\mathcal{S}$, and eventually to A'. ∎

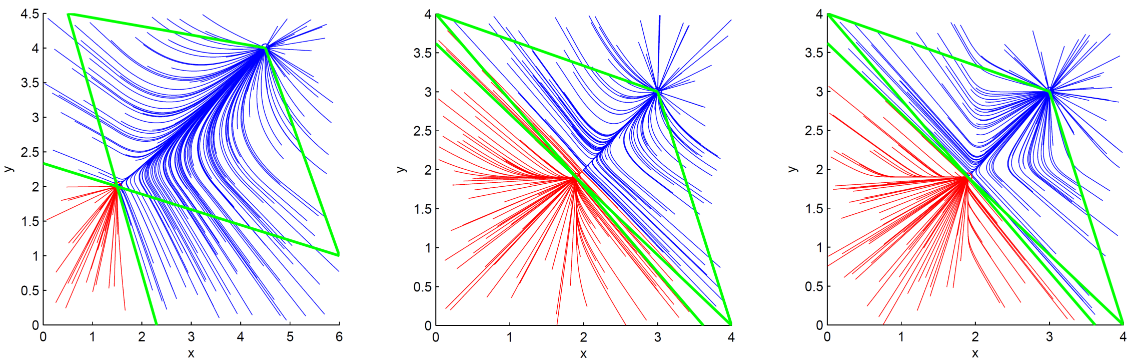
It is worth noting that, in view of Lemma 1, a value of p for which D is unstable, hence the cooperative equilibrium A attracts all of the trajectories originating in $\mathcal{R} \setminus (\delta_1, \delta_2)$, can always be found under suitable assumptions on the payoff matrix values (precisely, for all values such that $\det \Theta < 0$).

Remark 1: The necessity part of the proof claims the local stability of D based on linearization, which is an abuse since the system is not continuously differentiable everywhere. However, the argument can be made rigorous by means of the same Lyapunov-like function $U(z, w) = z_F z + w_F w$ used in the sufficiency part, since $\dot{U}(z, w) > 0$. Details are omitted for space reasons.

IV. Numerical Examples

For some choices of the payoff matrix and of p, we have simulated the system evolution, generating trajectories that start from random initial conditions in the positive orthant. Trajectories converging to D are in red, while trajectories converging to A are in blue. The quadrilateral $ABDC$ and the segments $B'D$ and DC' in Fig. 1a are drawn in green in the plots. Fig. 3a corresponds to the game evolution with payoff matrix (8): the choice $p = 3$ renders D unstable and, as expected, the proposed saturated strategy guarantees convergence to the cooperative equilibrium A of all trajectories originating in $\mathbb{R}^2_+ \setminus \mathcal{D}$. Conversely, in Fig. 3b, showing the system evolution with payoff matrix

$$\begin{bmatrix} (\alpha_1, \alpha_2) & (\gamma_1, \gamma_2) \\ (\beta_1, \beta_2) & (\delta_1, \delta_2) \end{bmatrix} = \begin{bmatrix} (3, 3) & (0, 4) \\ (4, 0) & (1.9, 1.9) \end{bmatrix}, \qquad (12)$$



(a) Payoff matrix (8), $p = 3$. (b) Payoff matrix (12), $p = 3$. (c) Payoff matrix (12), $p = 10$.

Fig. 3: System evolution with the saturated strategy: trajectories with random initial conditions in the positive quadrant.

and with $p = 3$, some of the trajectories originating in $\mathbb{R}^2_+ \setminus \mathcal{D}$ converge to D, which is a stable equilibrium. A new unstable equilibrium appears in between, which is the turning point between trajectories converging to A and trajectories converging to D. However, the choice $p = 10$ destabilizes D: then, all of the trajectories originating in $\mathbb{R}^2_+ \setminus \mathcal{D}$ converge to A, as shown in Fig. 3c, consistently with our results.

V. Conclusions

We have shown that a saturated strategy robustly ensures stability of the cooperative equilibrium of a two-population game having the structure of a Prisoner's dilemma. Future directions involve the analysis of the n-dimensional case and the extension of the analysis to a larger set of games including the *coordination game*.

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