

Properties of Switching–Dynamics Race Models

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Abstract—The paper analyses some continuous–time dynamic models that describe the evolution of social systems characterized by the possibility of changing the alliances among the parties involved or damaging one’s competitors. At any time each participant, either an individual or a coalition, can decide to form or terminate a bond, or to start or stop damaging an opponent (i.e., to switch from a network configuration to another), based on a greedy, or shortsighted, criterion that does not consider the long–term effects of the decision. The proposed models fall within the domain of positive switching systems. Although they can obviously simulate the behaviour of many real–life situations, in which contenders aim at prevailing over one another to achieve supremacy, the paper does not refer to a specific context and concentrates on the main structural properties of the mathematical models, such as positivity and boundedness of the solutions, existence of coalitions, steady–state behaviour. Simulations show how the different cooperative or hostile attitudes of the participants affect their yield.

I. INTRODUCTION

A variety of models have been proposed in the literature to simulate the behaviour of interacting rational decision makers in various contexts, such as economics, political science, psychology, biology and logic. This kind of study pertains to the broad realm of decision science and game theory (see, e.g., [2], [24]). Decision models can be either deterministic or stochastic, static or dynamic, discrete–time or continuous–time [3]. Also, decisions can be made on the basis of partial or complete information about the state of the system and its evolution. An essential feature of decision theory, which makes it different from classic optimal control theory and dynamic programming, where a single (possibly, vector) criterion is to be optimized, is that each decision maker has its own goal that can depend on the present, past or expected values of the system states.

This paper focuses on a particular, yet meaningful, class of linear continuous–time variable–structure dynamic models describing the evolution of the system state whose components represent the “strength” of every participant. The system structure is assumed to switch instantaneously from one configuration to another whenever a participant creates or changes a partnership, or an enmity, according to a greedy criterion, i.e., its instantaneous convenience. The choices of every participant are made to depend linearly on the (known) state of the other participants and are put immediately into effect, provided a reciprocity condition is satisfied.

Despite their simplicity, the considered models account fairly well for the perceived behaviour of many social and political systems (e.g., political campaigns) or sport contests (e.g., cycle racing), consisting of individuals or groups that interact according to selfish interest, each wishing to prevail over the others. For this reason, we call them *race models*. The following analysis, however, does not refer to a specific application. Rather, it aims at pointing out some structural properties enjoyed by the mathematical models, such as positivity, boundedness, coalition formation, steady–state behaviour.

An increasing attention has recently been devoted to the study of choices at the basis of social dynamics [8] and to the analysis of (adaptive) coevolutionary networks [12], [19], a class of networks whose structure evolves depending on the dynamics of processes taking place in the network nodes. A network can be dynamic either because its topology evolves in time or because dynamic processes occur at the network nodes (so that their states evolve in time). In *coevolutionary* networks, these two types of dynamics interplay (the network topology being a function of the system states). Several papers have lately studied dynamic networks of interactions [25], [17] by means of graphs, where agents can change their partnerships [10], [16] in order to join the agent of highest reputation [13] or to break bonds with uncooperative partners [21]. In particular, in [20] attention has been focused on switching strategies in a two–person zero–sum differential game of finite horizon, and in [18] on the benefits of partner switching among self–interested agents in a resource–exchange environment. It has been pointed out that dynamic switching can enhance cooperation [14], while simultaneously excluding defectors [23], even when the cost associated with dynamism is taken into account (in terms of time or resource investments for finding and establishing new partnerships) [4].

In this work, we pursue a rigorous mathematical formulation which allows us to assess the main structural properties of the considered systems, which legitimately belong to the class of positive switching systems (see [7] and references therein). Switching systems are currently attracting a great attention from the control systems community [15], [22], with particular regard to *positive* switching systems, which can account for the evolution of populations very well [6], [9], [11]. Most research efforts, however, are primarily concerned with stability and stabilizability. Instead, here we are mainly concerned with evolution patterns and outcome. Specifically, this paper addresses the following aspects.

(i) We formulate an original race model in which a set of contenders compete in order to improve their ranking,

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evaluated in terms of a positive variable representing the contender's strength.

(ii) We analyse both the case in which only alliances can be established between pairs of contenders (cooperative model) and the case in which the competitors obstruct or undermine the top-ranked competitor (competitive model).

(iii) We show that in the cooperative model, in which each competitor looks for the most profitable partnership, at least one alliance is always established provided a certain utility matrix is symmetric (reciprocity condition).

(iv) We show that the system evolution remains positive even in the competitive case.

(v) We put forth a mixed model, in which both alliances and obstructions are possible.

Theoretical results are complemented by simulations that show how the cooperative and/or hostile behaviour of the contenders can affect their yield.

The present account is a preliminary version of a more thorough paper that will be available soon [5].

II. MODEL DESCRIPTION

We begin by describing the mathematical model of the interactions among n independent contenders (individuals or groups) whose aim is to prevail over the others, and let the state x_i of the i -th contender represent its strength, whose evolution depends not only on its own internal dynamics, but also on an exogenous input, as well as on the interactions with the other contenders. Two kinds of interactions, called respectively "alliances" and "obstructions", are considered. The former refer to a situation in which contenders may cooperate to increase their own strength; the latter accounts for a situation in which a contender may sabotage other contenders so as to decrease their strength and, consequently, increase the possibility of improving its own rank. In principle, alliances or obstructions can involve more than two contenders and be either symmetric or asymmetric (in the sense that both contenders share the same disposition towards each other or not); however, for the sake of simplicity, in the following it is assumed that: (i) each contender may have, at most, one ally and/or one enemy, and (ii) alliances are symmetric, while obstructions are not.

We also assume that, in the absence of interactions among contenders, the dynamics are linear and described by

$$\dot{x} = -\Lambda x + b, \quad (1)$$

where $x = [x_1, \dots, x_n]^\top \in \mathbb{R}^n$ is the state vector representing the strength of every contender, $b = [b_1, \dots, b_n]^\top \in \mathbb{R}^n$ represents an exogenous input, and $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$ is a *positive-definite* diagonal matrix accounting for a natural decline of the contenders' strength.

Clearly, in this simple case, each x_i autonomously reaches a steady-state value dependent only on b_i and on λ_i . A time-varying input $b(t)$ could be considered; this case, however, is outside the scope of the present contribution.

In the sequel, we consider three variants of the basic model (1), corresponding respectively to the cases in which the dynamics are affected only by alliances (cooperative

model), only by obstructions (competitive model), and by both (mixed model). For reasons that will be clear soon, the following standing assumption is made.

Assumption 1: The system has n distinct eigenvalues and $i \neq j \Rightarrow \frac{b_i}{\lambda_i} \neq \frac{b_j}{\lambda_j}$.

The implication in this assumption means that each individual (or group) has unique characteristics and, therefore, the steady-state values of all the contenders considered separately are different.

A. Cooperative model

Consider now the case in which two contenders can associate to increase their strength. The alliance takes place only when both contenders are willing to make it and the strength increment of either ally is proportional to the strength of the other according to a proportionality factor that expresses the "degree of commitment" or "attitude" of an ally towards the other. The attitude of the two partners towards each other is assumed to be the same (symmetry or reciprocity condition).

We model the *intention* of the contenders to make alliances at time t by means of a Boolean matrix $V(t)$, whose generic entry $V_{ij}(t)$ is 1 if, at time t , contender i is willing to make an alliance with contender j , and 0 otherwise. Since every contender is allowed to make one alliance only, every row of $V(t)$ contains a single 1. An alliance between contender i and contender j takes place only if $V_{ij}(t) = V_{ji}(t) = 1$.

To account for the gains afforded by alliances, we introduce a matrix $M \in \mathbb{R}^{n \times n}$ whose diagonal entries M_{ii} are all zero and whose off-diagonal entries M_{ij} are strictly positive and represent the proportionality factor relating x_j to the incremental tendency \dot{x}_i when x_i and x_j are allied.

The state equations of such a cooperative model are

$$\dot{x}_i = -\lambda_i x_i + \sum_{j=1}^n V_{ij}(t) V_{ji}(t) M_{ij} x_j + b_i, \quad i = 1, \dots, n. \quad (2)$$

Since M is a Metzler matrix, the sum in (2) is non-negative; therefore, the optimal choice for contender i corresponds to

$$V_{ij}(t) = \begin{cases} 1 & \text{if } M_{ij} x_j(t) > M_{ik} x_k(t), \text{ for all } k \neq j, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

The i -th contender thus chooses as its possible ally the contender that currently maximizes the increase of its strength.

Remark 1: If for a pair of indices, say j_1 and j_2 , and a time-instant t^* we have

$$M_{i,j_1} x_{j_1}(t^*) = M_{i,j_2} x_{j_2}(t^*) > M_{ik} x_k(t^*), \quad (4)$$

for all $k \notin \{j_1, j_2\}$, then, due to the strict inequality in (3), the i -th contender will not be willing to make any alliance at time t^* , even though, given the positivity of all off-diagonal entries of M , any alliance would be profitable. However, according to Assumption 1, the evolutions of x_{j_1} and x_{j_2} are different. Hence the equality in (4) will be no longer satisfied for $t > t^*$, and an ally will be chosen immediately after t^* .

In view of Remark 1, in the following it is reasonably assumed that no undecidable situation (stall) occurs.

Equations (2) can be written more compactly as

$$\dot{x} = A_A(t)x + b, \quad (5)$$

where $A_A(t)$ is a time-varying matrix, whose subscript A has been introduced to distinguish this cooperative model, involving *alliances* only, from the models considered later on. Since $x(t)$ evolves continuously between consecutive switchings, $V(t)$ and $A_A(t)$ are piecewise-constant matrices, typical of a switching model.

B. Competitive model

A model like (5) is suitable for a context in which cooperation is always beneficial, e.g., because unlimited resources prevent the outbreak of conflicts. To describe a situation in which resources are limited and the strength of a contender is seen as a menace by the other contenders, resort must be made to a different model. Here, we limit attention to a simple competitive model in which the strongest contenders are sabotaged by all of the other contenders, and the variation in their strength is the sum of negative terms that are proportional to the strengths of the other contenders. Precisely, if $\mathcal{H}(t)$ denotes the set of the strongest contenders at time t , i.e.,

$$\mathcal{H}(t) = \arg \max_k x_k(t), \quad (6)$$

and $S \in \mathbb{R}^{n \times n}$ is a matrix whose entries S_{ij} account for the severity of the damage that x_j can inflict on x_i , then the state equations of the model can be written as

$$\dot{x}_i = -\lambda_i x_i - \sum_{j \notin \mathcal{H}(t)} S_{ij} x_j + b_i \quad (7)$$

for $i \in \mathcal{H}(t)$, and

$$\dot{x}_i = -\lambda_i x_i + b_i \quad (8)$$

for all $i \notin \mathcal{H}(t)$.

Again, equations (7) and (8) can be written in a compact form as

$$\dot{x} = A_O(t)x + b, \quad (9)$$

where the subscript O of the piecewise-constant matrix $A_O(t)$ refers to the presence of *obstructions*.

C. Mixed model

Consider finally a situation in which alliances and obstructions coexist. Let again $\mathcal{H}(t)$ denote the set of the strongest contenders at time t and introduce the symbol $\mathcal{F}_i(t)$ to denote the set of indices associated with the contenders that do not obstruct contender $i \in \mathcal{H}(t)$. Obviously, if the j -th contender is allied with i , it does not make any obstruction. Therefore, assuming that $V_{ij} = 1$, namely that one of the strongest contenders at time t is willing to make an alliance with the j -th contender, and recalling the assumption that each contender can enter at most one pairwise alliance, we have

$$\mathcal{F}_i(t) = \begin{cases} \{i, j\} & \text{if } V_{ij} = 1, \\ \{i\} & \text{otherwise.} \end{cases} \quad (10)$$

The resulting dynamic model is

$$\dot{x}_i = -\lambda_i x_i - \sum_{j \notin \mathcal{F}_i(t)} S_{ij} x_j + \sum_{j=1}^n V_{ij} V_{ji} M_{ij} x_j + b_i \quad (11)$$

for $i \in \mathcal{H}(t)$, while, for all $i \notin \mathcal{H}(t)$,

$$\dot{x}_i = -\lambda_i x_i + \sum_{j=1}^n V_{ij} V_{ji} M_{ij} x_j + b_i. \quad (12)$$

Equations (11) and (12) can be written more compactly as

$$\dot{x} = A_M(t)x + b, \quad (13)$$

where the subscript M of the time-varying matrix $A_M(t)$ refers to the aforementioned *mixed* behaviour in which both alliances and obstructions are present.

Remark 2: Since the differential equations of the proposed models have a discontinuous right-hand side, their solution must be intended in the sense of Filippov by resorting to a differential inclusion formulation (see for instance [1]).

III. THEORETICAL RESULTS

This section analyses the main properties of the models described in the previous section.

A. System positivity

The positivity of the system trajectories for the model involving alliances only is obvious, since at any time instant the state matrix $A_A(t)$ can be chosen in a set of matrices that are all Metzler. Yet, we can show that the same property holds also for the other models, even though the related state matrix can be non-Metzler (since, in the case of obstructions, there are negative off-diagonal entries).

Proposition 1: If $x_i(0) \geq 0$ and $b_i \geq 0$, $i = 1, \dots, n$, then all the models considered in Section II are positive, i.e., $x_i(t) \geq 0$ for all $t \geq 0$. \square

Proof: The proposition is clearly true for the cooperative model (2). Also, if the above statement holds for the competitive model (7)–(8), it holds for the mixed model (11)–(12) as well. Therefore, it is sufficient to prove the proposition for the model (7)–(8). By contradiction, assume that, for some $x(0)$ such that $x_i(0) \geq 0$ for all $i = 1, \dots, n$, there exists \bar{t} such that, for some i , $x_i(\bar{t}) = 0$ and $x_i(t) < 0$ in a right neighborhood of \bar{t} . Let $\mathcal{Z}(\bar{t})$ denote the set of all indices i for which $x_i(\bar{t}) = 0$ and let $|\mathcal{Z}(\bar{t})|$ denote its cardinality, obviously different from zero. If $|\mathcal{Z}(\bar{t})| < n$, then the obstructed contenders are associated to indices not belonging to $\mathcal{Z}(\bar{t})$, while the dynamic equation associated to each $i \in \mathcal{Z}(\bar{t})$ (see(8)) is $\dot{x}_i(\bar{t}) = b_i$; hence, since $b_i > 0$, no zero crossing is possible. On the other hand, if $|\mathcal{Z}(\bar{t})| = n$, then $x(\bar{t}) = 0$. In this case no contender is obstructed and all contender dynamics are described by (8). Therefore $\dot{x}_i(\bar{t}) = b_i > 0$ for all $i = 1, \dots, n$ and, again, no zero crossing is possible. \blacksquare

B. Existence of alliances

Proposition 2: If the matrix M corresponding to system (2) (cooperative case) is symmetric, then at least one alliance is established.¹ \square

Proof: Consider, for simplicity, the case of three contenders and suppose, by contradiction, that no alliance takes place. Excluding undecidable situations (see Remark 1), which, in view of Assumption 1, can occur only for some time instants and have previously been ruled out, this means that, if a contender i wishes to associate with contender j , the latter will reject the proposal because an alliance with the third contender k would be more profitable. The same consideration applies to the other two possible pairs of contenders, j and k and, respectively, k and i . According to (3), the aforementioned sequence of intentions correspond to

$$M_{ij}x_j(t) > M_{ik}x_k(t), \quad (14)$$

$$M_{jk}x_k(t) > M_{ji}x_i(t), \quad (15)$$

$$M_{ki}x_i(t) > M_{kj}x_j(t). \quad (16)$$

Multiplying side by side all of the above inequalities and recalling that the system is positive, we obtain

$$M_{ij}M_{jk}M_{ki}x_j(t)x_k(t)x_i(t) > M_{ik}M_{ji}M_{kj}x_k(t)x_i(t)x_j(t),$$

which, given the symmetry of M , is a contradiction. Hence, at least one alliance must be formed. The same procedure applies to the case of more than three contenders. \blacksquare

Observe that, if the utility matrix M is not symmetric, then the thesis of Proposition 2 does not hold, as shown by the following counter-example.

Example 1: Consider the case of three contenders x_1 , x_2 and x_3 , and suppose that $[10, 8, 4]^\top$ is an equilibrium for the system (1) without interactions. Let also

$$M = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix}, \quad (17)$$

and suppose that $x(t^*) = [10, 8, 4]^\top$ at some time t^* . Since $M_{12}x_2(t^*) = 16 > M_{13}x_3(t^*) = 12$, according to (3) contender 1 is willing to make an alliance with contender 2. Similarly, contender 2 is willing to make an alliance with contender 3 because $M_{23}x_3(t^*) = 12 > M_{21}x_1(t^*) = 10$. Finally, contender 3 is willing to make an alliance with contender 1 as $M_{31}x_1(t^*) = 10 > M_{32}x_2(t^*) = 8$. Consequently, no alliance is established at t^* . Since $V_{ij}V_{ji} = 0$ for all i and all j , the cooperative system (2) with such a utility matrix M is at equilibrium like system (1). Hence, no alliance will be established for all $t \geq t^*$.

C. Boundedness of the trajectories

To ensure the boundedness of the trajectories, we need to assume that the natural decline of every contender (clearly related to the negative diagonal entries $-\lambda_i$ of the system matrix A_A , A_O , or A_M) is sufficiently large to compensate for the positive contributions afforded by the possible alliances, which indeed occurs under the following assumption.

¹Except, possibly, for some time instants (see Remark 1).

Assumption 2: At each time instant, the state matrix $A_A(t)$, $A_O(t)$, or $A_M(t)$ is strictly diagonally dominant.

In fact, the subsequent proposition holds.

Proposition 3: Under Assumption 2, the system trajectories are bounded. \square

Proof: In light of the positivity of the system trajectories, consider the co-positive function $V(x) = 1^\top x$. To show that it is a co-positive Lyapunov-like function outside a bounded non-empty set, decompose the system state matrix as $A_k(t) = -\Lambda + \tilde{A}_k(t)$, where $k \in \{A, O, M\}$, and write the Lyapunov derivative of $V(x)$ as

$$\begin{aligned} \dot{V}(x) &= 1^\top (-\Lambda + \tilde{A}_k)x + 1^\top b \\ &= \sum_{i=1}^n \left[\left(-\lambda_i + \sum_{j=1}^n [\tilde{A}_k]_{ji} \right) x_i \right] + 1^\top b. \end{aligned} \quad (18)$$

Diagonal dominance guarantees that, for a suitable $\rho > 0$,

$$\dot{V}(x) \leq -\rho 1^\top x + 1^\top b. \quad (19)$$

Let $\mu > 0$ be large enough to guarantee that $-\rho\mu + 1^\top b < 0$. Then, the trajectories of the switching system are ultimately globally bounded in the set $\mathcal{S} = \{x \geq 0 : 1^\top x \leq \mu\}$. \blacksquare

D. Profitability of obstructions

A natural question arising from the previous considerations is whether obstructions can be profitable. We have seen that the contribution of an alliance to the dynamics of the allies is always positive; on the other hand, an obstruction provides a negative contribution to the obstructed contender, but no direct positive contribution to the obstructing one. Hence, one might ask whether, being adverse to a contender (in this case, to the strongest contender), an obstruction could be profitable to another contender. The answer is yes, as shown by the simulations in Figures 1 and 2 that refer to a pool of 10 contenders. The two scenarios are characterized by the same matrix Λ , vector b and initial condition $x(0)$; in the scenario of Fig. 1 only alliances are allowed, while in that of Fig. 2 both alliances and obstructions are possible. It is seen that, at least for the contender whose trajectory is plotted with a bold line, obstructing the strongest contender is indeed profitable. In fact, not only the ranking of the contender improves with respect to its competitors, but also its own strength increases: after five time units it is about 0.4 in the case with alliances only and about 0.5 in the case with both alliances and obstructions.

IV. NUMERICAL SIMULATIONS

In this section, we simulate the behaviour of switching dynamic models describing the interactions among five contenders in the presence of (i) alliances only, (ii) obstructions only, and (iii) both alliances and obstructions. In all of the three cases, the system is started from the same initial condition $x(0) = [0.0855 \ 0.2625 \ 0.8010 \ 0.0292 \ 0.9289]^\top$ and exhibits the same matrix $\Lambda = 2I$ and vector $b = [0.7303 \ 0.4886 \ 0.5785 \ 0.2373 \ 0.4588]^\top$.

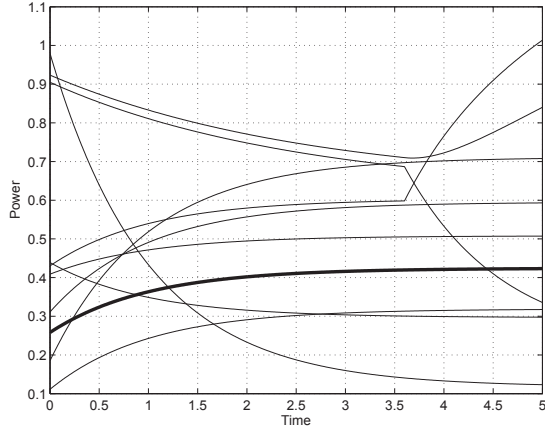


Fig. 1. Evolution of the states of the considered ten contenders when only alliances are allowed.

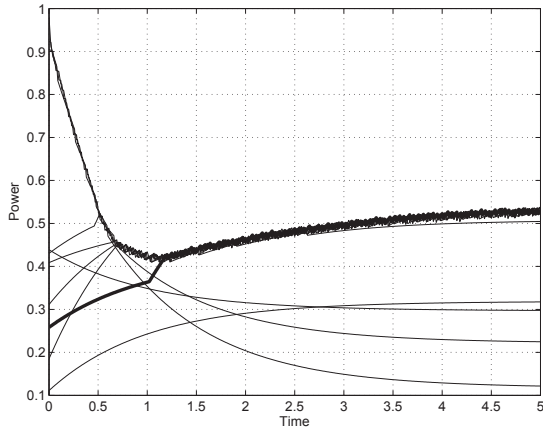


Fig. 2. Evolution of the states of the same ten contenders considered in Fig. 1 when both alliances and obstructions are allowed.

The following symmetric M matrix is considered in the cooperative and mixed cases:

$$M = \begin{bmatrix} 0 & 0.5854 & 0.2794 & 0.2467 & 0.2978 \\ 0.5854 & 0 & 0.6403 & 0.3514 & 0.8209 \\ 0.2794 & 0.6403 & 0 & 0.7380 & 0.2964 \\ 0.2467 & 0.3514 & 0.7380 & 0 & 0.7501 \\ 0.2978 & 0.8209 & 0.2964 & 0.7501 & 0 \end{bmatrix},$$

while the matrix S appearing in the competitive and mixed cases is chosen as:

$$S = \begin{bmatrix} 0 & 0.0305 & 0.6099 & 0.1829 & 0.1679 \\ 0.8909 & 0 & 0.6177 & 0.2399 & 0.9787 \\ 0.3342 & 0.5000 & 0 & 0.8865 & 0.7127 \\ 0.6987 & 0.4799 & 0.8055 & 0 & 0.5005 \\ 0.1978 & 0.9047 & 0.5767 & 0.4899 & 0 \end{bmatrix}.$$

Since $0 \leq M_{ij}, S_{ij} < 1$ and in the present simulations each contender can have at most one ally and one enemy, the state matrix at every time instant is strictly diagonally dominant and the trajectories are bounded in all cases.

Fig. 3 shows the evolution of the system states when only alliances are permitted. Contender 5 is initially allied with contender 3, but then, at time $t = 0.11$, the alliance with

contender 2 becomes more profitable so that contender 5 changes ally. Correspondingly, the system matrix switches too. This change results in a sudden increase in the strength of contender 2, which eventually becomes stronger than contender 5 itself (even if the latter started from the most favourable initial condition).

Fig. 4 shows instead the state evolution in the case of obstructions only. Now, the final values are lower than those in the previous case. Moreover, at the end contenders 1 and 3 keep competing for supremacy and their state evolves along a sliding surface characterized by a high switching frequency. The presence of this chattering phenomenon is evident also when considering which is the strongest contender at every time instant: contender 5 has the supremacy until $t = 0.13$, then it starts fighting for supremacy with contender 3 until $t = 0.7$, when the strength of contender 1 grows above that of 5. For $t > 0.7$, contenders 1 and 3 compete endlessly for supremacy.

In the mixed case, whose state evolution is depicted in Fig. 5, contender 5 is first allied with contender 3 and then with contender 2, as in the case of alliances only. Now, however, the final value of the strongest contenders is limited by the presence of obstructions and no contender benefits from them. Again, a sliding mode behaviour characterized by frequent switchings arises, as contenders 1, 2 and 5 (and 3 over a very short time interval) fight continually for supremacy. At the beginning, 5 is the strongest contender; then it has to compete for supremacy (leading to chattering) first with contender 3 (from $t = 0.3$ to $t = 0.7$), then with 2 only (from $t = 0.7$ to $t = 1$), and finally with both 1 and 2. The possibility of making alliances, besides obstructions, favours contenders 2 and 5, which are now competing for supremacy, and penalizes contender 3, which is eventually excluded from the competition for top rank.

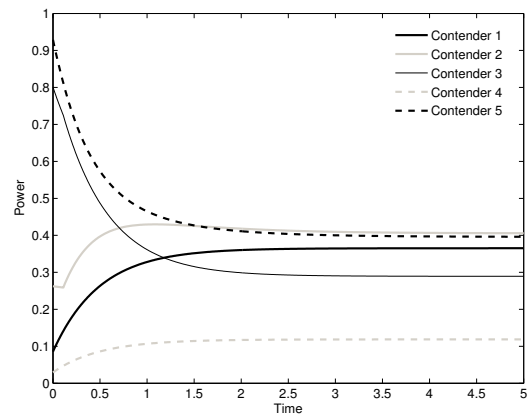


Fig. 3. Evolution of the states of the five contenders in the case of alliances only.

V. CONCLUSIONS

A switching model that describes the dynamic race of a set of competitors striving for supremacy has been suggested. In

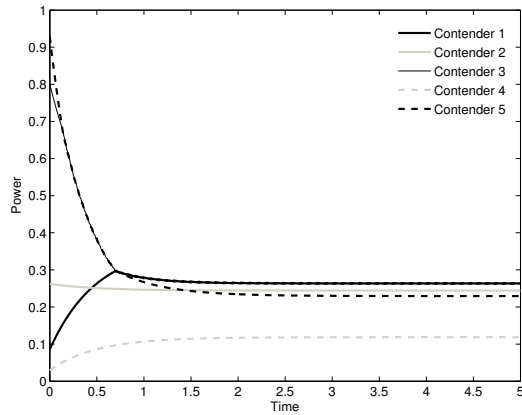


Fig. 4. Evolution of the states of the five contenders in the case of obstructions only.

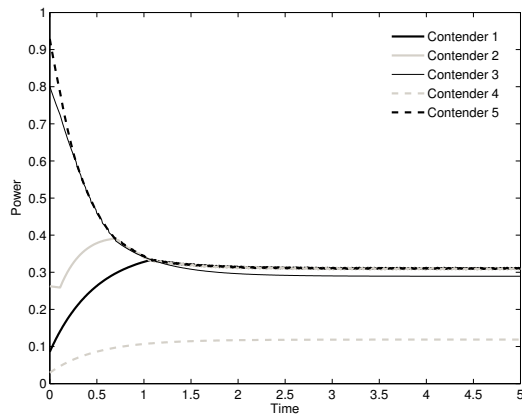


Fig. 5. Evolution of the states of the five contenders in the mixed alliance-obstruction case.

particular, three variants of this model have been considered. The first accounts for pairwise alliances only, the second for obstructions exerted on the most redoubtable competitor(s), and the third for both alliances and obstructions.

It has been shown that the presence of at least one alliance is always guaranteed if the utility matrix M is symmetric; otherwise alliances are not ensured. The positivity of the competitive and mixed switching models has also been proved.

Simulations have pointed out some unexpected behaviours of the the aforementioned models. It turns out, in particular, that obstructing the strongest competitor can be beneficial to some contender.

There are several possible extensions of this work and open problems. For instance, alliances involving more than two partners could be considered. The criteria for choosing allies and/or enemies could be made more complex, too. For brevity, this paper has limited attention to systems characterized by a bounded evolution; it is believed, however, that a ranking can be established even when some variables diverge.

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