VARIETIES OF MV-ALGEBRAS

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Abstract. We characterize, for every subvariety **V** of the variety of all MV-algebras, the free objects in **V**. We use our results to compute coproducts in **V** and to provide simple single-axiom axiomatizations of all many-valued logics extending the Łukasiewicz one.

1. Preliminaries and Definitions

Subvarieties of MV-algebras have been studied in [Gri77], [Kom81], [DNL]. It is known that any such variety is generated by finitely many algebras, and explicit axiomatizations have been obtained. The techniques used in the above papers are algebraic, and the computations involved relatively complex. In this paper we use geometric techniques, as developed in [Mun94], [Pan95]. Our results are easily visualizable, and the topology of the unit interval allows us to dispose of almost any computation.

We assume familiarity with MV-algebras; we refer to [Cha58], [Cha59], [Mun86, §2], [CDM95] for all unexplained notions and claims. To fix notation, we recall that an *MV-algebra* is an algebra $A = (A, \oplus, \neg, 0)$ such that $A = (A, \oplus, 0)$ is an abelian monoid and the following identities hold:

$$\begin{aligned}\neg\neg a &= a\\ a \oplus (\neg 0) &= \neg 0\\ \neg(\neg a \oplus b) \oplus b &= \neg(\neg b \oplus a) \oplus a\end{aligned}$$

A *lattice-ordered abelian group* (ℓ-group) is an algebra $(\mathfrak{A}, +, -, 0, \vee, \wedge)$ such that $(\mathfrak{A}, +, -, 0)$ is an abelian group, $(\mathfrak{A}, \vee, \wedge)$ is a lattice, and $+$ distributes over $\vee$ and $\wedge$. A *totally-ordered abelian group* (o-group) is an ℓ-group in which the order is total. A *strong unit* of the ℓ-group $\mathfrak{A}$ is an element $u > 0$ of $\mathfrak{A}$ such that, for every $a \in \mathfrak{A}$, there exists $m \in \mathbb{N}$ with $a \leq mu$.

Let $(\mathfrak{A}, u)$ be an ℓ-group equipped with a fixed strong unit u. $\Gamma(\mathfrak{A}, u)$ is the structure

$$\Gamma(\mathfrak{A}, u) = ([0, u], \oplus, \neg, 0)$$

defined as follows:

$$\begin{aligned}[0, u] &= \{a \in \mathfrak{A} : 0 \leq a \leq u\}\\ a \oplus b &= (a + b) \wedge u\\ \neg a &= u - a\\ 0 &= \text{the additive identity } 0 \text{ of } \mathfrak{A}\end{aligned}$$

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It is easy to check that $\Gamma(\mathfrak{A},u)$ is an MV-algebra. The construction of $\Gamma(\mathfrak{A},u)$ from $(\mathfrak{A},u)$ is due to Chang [Cha59] for the totally-ordered case, and to Mundici [Mun86] for the general case. We have the following key properties, first proved in [Mun86]; see [CM98] for a new presentation:

- the lattice-order induced by the MV operations in $\Gamma(\mathfrak{A},u)$ coincides with the order inherited from $\mathfrak{A}$;
- if $\varphi:(\mathfrak{A},u)\to(\mathfrak{B},v)$ is an ℓ-group homomorphism mapping u to v, then the restriction $\Gamma\varphi$ of φ to $[0,u]$ is an MV-algebra homomorphism $\Gamma\varphi:\Gamma(\mathfrak{A},u)\to\Gamma(\mathfrak{B},v)$;
- Γ is a full, faithful, and representative functor (i.e., a categorical equivalence) between the category of ℓ-groups with strong unit and the category of MV-algebras. In particular, for every MV-algebra A, there exists a unique ℓ-group with strong unit $(\mathfrak{A},u)$ such that A is isomorphic to $\Gamma(\mathfrak{A},u)$. If A is countable, then $\mathfrak{A}$ is countable;
- the ideals (i.e., kernels of homomorphisms) of $(\mathfrak{A},u)$ correspond bijectively to the ideals of $\Gamma(\mathfrak{A},u)$ via the inclusion-preserving application $\mathfrak{I}\mapsto\mathfrak{I}\cap[0,u]$, whose inverse is $I\mapsto$ (ideal generated by I in $\mathfrak{A}$). If $I=\mathfrak{I}\cap[0,u]$, then $\Gamma(\mathfrak{A},u)/I$ and $\Gamma(\mathfrak{A}/\mathfrak{I},u/\mathfrak{I})$ are isomorphic via $a/I\mapsto a/\mathfrak{I}$.

Following [Kom81], the MV-algebras S_m and S_m^ω, for $1\le m$, are defined as follows:

$$\begin{aligned} S_m &= \Gamma(\mathbb{Z},m)\\ S_m^\omega &= \Gamma(\mathbb{Z}\otimes\mathbb{Z},(m,0)) \end{aligned}$$

(here $\mathbb{Z}\otimes\mathbb{Z}$ is the lexicographic sum of two copies of the o-group $\mathbb{Z}$ of the integers).

We denote the variety of all MV-algebras by $\mathbf{MV}$. If $\emptyset\ne X\subseteq\mathbf{MV}$, then $\mathbf{V}(X)$ is the subvariety of $\mathbf{MV}$ generated by X. In [Kom81], Komori proved that every subvariety $\mathbf{V}$ of $\mathbf{MV}$ is of the form

$$(*)\qquad \mathbf{V}=\mathbf{V}(S_{m_1},\dots,S_{m_r},S^\omega_{t_1},\dots,S^\omega_{t_s})$$

for some finite sets $I=\{m_1,\dots,m_r\}$ and $J=\{t_1,\dots,t_s\}$, not both empty.

Let $\mathbf{V}$ be as in $(*)$; by [Kom81, Theorems 2.1 and 2.3], $S_m\in\mathbf{V}$ iff m divides some element of $I\cup J$, and $S^\omega_t\in\mathbf{V}$ iff t divides some element of J. Let us call a pair (I,J) as above *reduced* if no $m\in I$ divides any $m'\in(I\setminus\{m\})\cup J$, and no $t\in J$ divides any $t'\in J\setminus\{t\}$ (in particular, $I\cap J=\emptyset$).

Proposition 1.1. *The proper subvarieties of* $\mathbf{MV}$ *are in 1-1 correspondence with reduced pairs.*

Proof. Let

$$\mathbf{V}(S_{m_1},\dots,S_{m_r},S^\omega_{t_1},\dots,S^\omega_{t_s})=\mathbf{V}(S_{n_1},\dots,S_{n_p},S^\omega_{v_1},\dots,S^\omega_{v_q})$$

with $(\{m_1,\dots,m_r\},\{t_1,\dots,t_s\})$ and $(\{n_1,\dots,n_p\},\{v_1,\dots,v_q\})$ reduced pairs. m_1 must divide some element of $\{n_1,\dots,n_p,v_1,\dots,v_q\}$. For no $j\in\{1,\dots,q\}$ it can be $m_1\mid v_j$ because, since every such v_j divides some element of $\{t_1,\dots,t_s\}$, it would follow that $\{m_1,\dots,m_r\},\{t_1,\dots,t_s\}$ is not reduced. Without loss of generality $m_1\mid n_1$ and, since n_1 divides some element of $\{m_1,\dots,m_r,t_1,\dots,t_s\}$, we must have $n_1\mid m_1$ and $m_1=n_1$. This proves that $\{m_1,\dots,m_r\}=\{n_1,\dots,n_p\}$. An analogous argument shows that $\{t_1,\dots,t_s\}=\{v_1,\dots,v_q\}$. $\square$

A *McNaughton function over the n-cube* is a continuous functions $f : [0,1]^n \to [0,1]$ for which the following holds:

> There exist finitely many affine linear polynomials $f_1, \dots, f_k$, each f_i of the form $f_i = a_i^0 x_0 + a_i^1 x_1 + \cdots + a_i^{n-1} x_{n-1} + a_i^n$, with $a_i^0, \dots, a_i^n$ integers, such that, for each $v \in [0,1]^n$, there exists $i \in \{1, \dots, k\}$ with $f(v) = f_i(v)$.

If κ is a possibly infinite cardinal, a *McNaughton function over the κ-cube* is a function $f : [0,1]^\kappa \to [0,1]$ which depends on finitely many variables $x_{i_1}, \dots, x_{i_n}$, and such that $f(x_{i_1}, \dots, x_{i_n})$ is a McNaughton function over the n-cube.

We denote by M_κ the MV-algebra of all McNaughton functions over the κ-cube, under pointwise operations. Any M_κ is a subalgebra of a power of the algebra $\Gamma(\mathbb{R}, 1)$, which generates **MV**, and M_κ is indeed the free MV-algebra over κ generators, the latter being the projection functions $x_i : [0,1]^\kappa \to [0,1]$, for $i < \kappa$. We will always identify elements of M_κ with [classes of equivalence of] terms in the language of MV-algebras.

The function $[0,1]^\kappa \to \operatorname{MaxSpec} M_\kappa$ given by $v \mapsto J_v = \{f \in M_\kappa : f(v) = 0\}$ is a homeomorphism between the κ-cube with the standard topology and the set of maximal ideals of M_κ, endowed with the hull-kernel topology [Mun86, Lemma 8.1].

If $v \in [0,1]^\kappa$, we denote by $M_\kappa \restriction v \simeq M_\kappa / J_v$ the MV-algebra of restrictions of McNaughton functions over the κ-cube to v.

We need two algebras of germs: $M_\kappa \restriction (v)$ is the algebra of equivalence classes of pairs (f, U), with $f \in M_\kappa$ and U an open set in $[0,1]^\kappa$ containing v. Two such pairs (f, U) and (g, V) are equivalent if $f = g$ on $U \cap V$; operations are inherited from M_κ. Similarly, given $v, w \in [0,1]^\kappa$, $M_\kappa \restriction [v, w)$ is the MV-algebra of equivalence classes of pairs (f, γ), with $f \in M_\kappa$ and $0 < \gamma < 1$ a real number. (f, γ) and (g, δ) are equivalent if $f = g$ on the line segment whose endpoints are v and $v + \min(\gamma, \delta)(w - v)$. Operations are inherited from M_κ (or, as it is tantamount, from $M_\kappa \restriction (v)$); if $w = v$, then $M_\kappa \restriction [v, w) \simeq M_\kappa \restriction v$. If $J_{(v)}$ and $J_{[v,w)}$ are the ideals of functions vanishing —respectively— in a neighborhood of v and in a line segment starting from v in the direction of w, then $M_\kappa \restriction (v) \simeq M_\kappa / J_{(v)}$ and $M_\kappa \restriction [v, w) \simeq M_\kappa / J_{[v,w)}$.

A final definition: a *rational point* of the κ-cube is a point $v \in [0,1]^\kappa$ such that $x_i(v) \in \mathbb{Q}$ for every $i < \kappa$ and, moreover, $x_i(v) = 0$ for all but finitely many i's. If v is a rational point, then there exists a uniquely determined sequence $\{a_i : i \leq \kappa\}$ of positive integers such that:

- $a_\kappa > 0$;
- $x_i(v) = a_i / a_\kappa$, for every $0 \leq i < \kappa$;
- the greater common divisor of the a_i's is 1.

We say that the a_i's are the *homogeneous coordinates* of v, and that a_κ is the *denominator* of v, $\operatorname{den}(v)$. Observe that the set of rational points is dense in $[0,1]^\kappa$.

2. Free MV-algebras

Let A be an MV-algebra; the *radical* of A, written $\operatorname{Rad} A$, is the intersection of all maximal ideals of A. If A is totally ordered, $\operatorname{Rad} A$ is the unique maximal ideal of A.

Definition 2.1. A subalgebra A of S_m^ω is *full* if the homomorphism $A \to S_m$ given by the composition of the natural mappings

$$A \twoheadrightarrow \frac{A}{\operatorname{Rad} A} \hookrightarrow \frac{S_m^\omega}{\operatorname{Rad} S_m^\omega} \simeq S_m$$

is surjective, but not injective.

Lemma 2.2. *Up to isomorphism, there are exactly m full subalgebras of S_m^ω. These are the algebras $A_0, \dots, A_{m-1}$, where A_i is the subalgebra generated by $\{(0,m),(1,-i)\}$.*

Proof. The algebras $A_0, \dots, A_{m-1}$ are pairwise non-isomorphic. This can be easily checked by embedding them in their enveloping o-groups $\Gamma^{-1}A_0, \dots, \Gamma^{-1}A_{m-1}$, and observing that the element $(m,-mi) \in \Gamma^{-1}A_0 \cap \dots \cap \Gamma^{-1}A_{m-1}$ is divisible by m in $\Gamma^{-1}A_i$ only.
Let B be a full subalgebra of S_m^ω, and let $(0,r)$ be the only atom of B. Let j be the least positive integer such that $(1,-j) \in B$. Then $0 \le j \le r-1$ and $r \mid mj$. Let $\mathfrak{B}$ be the o-group with strong unit enveloping B, i.e., $B = \Gamma(\mathfrak{B},(m,0))$. Then $\mathfrak{B}$, as an o-group, is isomorphic to $\mathbb{Z} \otimes \mathbb{Z}$, with generators $(0,r),(1,-j)$. Let ψ be the mapping : $\mathfrak{B} \to \mathbb{Z} \otimes \mathbb{Q}$ that fixes the x axis and contracts all vertical line segments by a factor of m/r; in cartesian coordinates, $\psi : (x,y) \mapsto (x, my/r)$. Then ψ maps $(0,r),(1,-j)$ into $(0,m),(1,-i)$, where $i = mj/r \in \mathbb{Z}$ and $0 \le i \le m-1$. Hence ψ maps $\mathfrak{B}$ isomorphically into an o-subgroup of $\mathbb{Z} \otimes \mathbb{Z}$, and fixes the strong unit $(m,0)$. By the properties of the Γ functor, B is isomorphic to $\Gamma(\psi(\mathfrak{B}),(m,0))$; the latter is the algebra A_i. □

It is not difficult to prove that A_i is isomorphic to $\Gamma(\mathbb{Z} \otimes \mathbb{Z},(m,i))$ (compare with [DNGP98, Lemma 1.3 and Corollary 1.4]).

Lemma 2.3. *Let $v \neq w$ be rational points of the κ-cube, with $\operatorname{den}(v) = m$. Then $M_\kappa \restriction [v,w)$ is isomorphic to a full subalgebra of S_m^ω.*

Proof. Let $\{a_i\}_{i \le \kappa}$, $\{b_j\}_{j \le \kappa}$ be the homogeneous coordinates of v, w, respectively, and let c_κ be the least common multiple of a_κ and b_κ. We claim that, for every $f \in M_\kappa$, the one-sided directional derivative

$$f'(v;w) = \lim_{\lambda \to 0^+} \frac{f(v + \lambda(w-v)) - f(v)}{\lambda}$$

(see [Mun88, Proposition 2.3]) at v in the direction of w is an integral multiple of $1/c_\kappa$. Indeed, choose a positive integer c so big that f is linear on the line segment $[v, v + c^{-1}(w-v)]$. Then

$$f'(v;w) = c\big[f(v + c^{-1}(w-v)) - f(v)\big].$$

Without loss of generality, f has the form $d^0x_0 + d^1x_1 + \dots + d^{n-1}x_{n-1} + d^n$ over $[v, v+c^{-1}(w-v)]$, with $d^0, \dots, d^n$ integers. Then

$$\begin{aligned} f(v + c^{-1}(w-v)) - f(v) &= \sum_{i=0}^{n-1} d^i \left(\frac{a_i}{a_\kappa} + c^{-1} \left(\frac{c_\kappa b_\kappa^{-1} b_i - c_\kappa a_\kappa^{-1} a_i}{c_\kappa} \right) - \frac{a_i}{a_\kappa} \right) \\ &= \frac{c^{-1}}{c_\kappa} \sum_{i=0}^{n-1} d^i e_i \end{aligned}$$

where $e_i = c_\kappa b_\kappa^{-1} b_i - c_\kappa a_\kappa^{-1} a_i \in \mathbb{Z}$. This proves our claim.
Consider the mapping

$$\psi_w : M_\kappa \restriction [v, w) \to \Gamma(\mathbb{Z} \otimes \mathbb{Z}, (m, 0))$$

given by $\psi_w(f) = (mf(v), c_\kappa f'(v; w))$. It is obvious that ψ_w is an injective MV-algebra homomorphism, and hence that $M_\kappa \restriction [v, w)$ is isomorphic to a subalgebra of S_m^ω. This subalgebra is full because, on the one hand, for every $0 \leq j \leq m$, there is some $f \in M_\kappa$ with $f(v) = j/m$. On the other hand, let $i < \kappa$ be such that $x_i(v) \neq x_i(w)$. It is easy to find a one-variable McNaughton function g that has value 0 in $x_i(v)$ and whose derivative at $x_i(v)$ in the direction of $x_i(w)$ is nonzero. Then the germ of $g \circ x_i$ in $M_\kappa \restriction [v, w)$ witnesses that $\mathrm{Rad}(M_\kappa \restriction [v, w))$ is not trivial. □

The embedding ψ_w constructed in Lemma 2.3 depends on the particular w we choose. If w' is another rational point along the half line from v to w, the embeddings ψ_w and $\psi_{w'}$ may be different, but their images are isomorphic. This ambiguity is removed by Lemma 2.2; although we do not need uniqueness, we state our conclusions as a corollary.

Corollary 2.4. *Let $v \neq w$ be rational points of the κ-cube, with* $\mathrm{den}(v) = m$. *Then there exists a unique $0 \leq i \leq m-1$ and a unique isomorphism of $M_\kappa \restriction [v, w)$ onto A_i, where A_i is the full subalgebra of S_m^ω defined in Lemma 2.2.*

We may now prove our main theorem.

Theorem 2.5. *Fix $\kappa > 0$, and let $\mathbf{V} = \mathbf{V}(S_{m_1}, \ldots, S_{m_r}, S_{t_1}^\omega, \ldots, S_{t_s}^\omega)$ be a proper subvariety of* **MV**. *Let X be the set of rational points of the κ-cube whose denominator divides at least one of $m_1, \ldots, m_r$, and let Y be the set of rational points of the κ-cube whose denominator divides at least one of $t_1, \ldots, t_s$. Consider the MV-algebra A defined by*

$$A = \prod_{u \in X \setminus Y} M_\kappa \restriction u \times \prod_{v \in Y} M_\kappa \restriction (v)$$

and let $\bar{x}_i$ be the image in A of the ith projection $x_i \in M_\kappa$. Then the subalgebra $M_\kappa^{\mathbf{V}}$ of A generated by $\{\bar{x}_i : i < \kappa\}$ is the free algebra over κ generators in $\mathbf{V}$, *the $\bar{x}_i$'s being free generators.*

Proof. We first show that each factor in the definition of A belongs to $\mathbf{V}$; it will follow that both A and $M_\kappa^{\mathbf{V}}$ are in $\mathbf{V}$.
Let $u \in X \setminus Y$, with $\mathrm{den}(u) = k \mid m$, for some $m \in \{m_1, \ldots, m_r\}$. Then $M_\kappa \restriction u$ is isomorphic to S_k, which is a homomorphic image of S_m; hence $M_\kappa \restriction u \in \mathbf{V}$.
Let $v \in Y$, with $\mathrm{den}(v) = k \mid t$, for some $t \in \{t_1, \ldots, t_s\}$, and let $w \neq v$ be any rational point of the κ-cube. By Lemma 2.3, $M_\kappa \restriction [v, w)$ is isomorphic to a subalgebra of S_k^ω. Since S_k^ω is a homomorphic image of S_t^ω, it follows that $M_\kappa \restriction [v, w) \in \mathbf{V}$. If $w = v$, then $M_\kappa \restriction [v, w) \simeq M_\kappa \restriction v \simeq S_k \in \mathbf{V}$. Since the set of rational points is dense in $[0, 1]^\kappa$, the natural mapping

$$M_\kappa \restriction (v) \to \prod_{\substack{w \text{ a rational point} \\ \text{of the } \kappa\text{-cube}}} M_\kappa \restriction [v, w)$$

is injective (compare with [Mun88, Propositions 2.3]), and hence $M_\kappa \restriction (v)$, being a subdirect product of the $M_\kappa \restriction [v, w)$'s, belongs to $\mathbf{V}$. This concludes the proof that $M_\kappa^{\mathbf{V}} \in \mathbf{V}$.

Let now $p(x_1, \dots, x_n) = q(x_1, \dots, x_n)$ be an n-variable identity in the language of MV-algebras that fails in at least one of $S_{m_1}, \dots, S_{m_r}, S^\omega_{t_1}, \dots, S^\omega_{t_s}$. Choose n generators $\bar{x}_{i_1}, \dots, \bar{x}_{i_n}$ of $M_\kappa^{\mathbf{V}}$; for simplicity's sake, we write y_j for $\bar{x}_{i_j}$. We claim that $p(y_1, \dots, y_n) = q(y_1, \dots, y_n)$ fails in $M_\kappa^{\mathbf{V}}$.

Case 1. $p = q$ fails in some S_m, for $m \in \{m_1, \dots, m_r\}$, in the elements $a_1, \dots, a_n \in \Gamma(\mathbb{Z}, m)$. Let u be the rational point of the κ-cube defined by

$$x_i(u) = \begin{cases} a_j/m, & \text{if } i = i_j \text{ for some } j \in \{1, \dots, n\}; \\ 0, & \text{otherwise.} \end{cases}$$

Then u belongs either to $X \setminus Y$ or to Y. Let $\psi : M_\kappa^{\mathbf{V}} \to S_m$ be the homomorphism defined as follows:

- if $u \in X \setminus Y$, then ψ is the composition of the projection $M_\kappa^{\mathbf{V}} \to M_\kappa \restriction u$ followed by the unique monomorphism $M_\kappa \restriction u \to S_m$;
- if $u \in Y$, then ψ is the composition of the projection $M_\kappa^{\mathbf{V}} \to M_\kappa \restriction (u)$, followed by the retraction $M_\kappa \restriction (u) \to M_\kappa \restriction u$, and again the monomorphism $M_\kappa \restriction u \to S_m$.

As trivially $\psi(y_j) = a_j$, for all $j = 1, \dots, n$, our claim is settled.

Case 2. $p = q$ fails in some S^ω_t, for $t \in \{t_1, \dots, t_s\}$, in the elements $(a_1, b_1), \dots, (a_n, b_n) \in \Gamma(\mathbb{Z} \otimes \mathbb{Z}, (t, 0))$. As in Case 1, define $u \in [0, 1]^\kappa$ by

$$x_i(u) = \begin{cases} a_j/t, & \text{if } i = i_j \text{ for some } j \in \{1, \dots, n\}; \\ 0, & \text{otherwise.} \end{cases}$$

Define $\nu \in \mathbb{Z}^\kappa$ to be the vector whose i_jth component is b_j, for $j = 1, \dots, n$, and that has all other components equal to 0. Choose a positive integer c so large that $w = u + c^{-1}\nu$ is a point of the κ-cube; note that w is rational. For u, w so defined, the embedding

$$\psi_u : M_\kappa \restriction [u, w) \to \Gamma(\mathbb{Z} \otimes \mathbb{Z}, (t, 0))$$

defined in the proof of Lemma 2.3 has the form $\psi_u(f) = (tf(u), df'(u; w))$, where d is the least common multiple of $\operatorname{den}(u)$ and $\operatorname{den}(w)$. Consider the homomorphisms

$$M_\kappa \restriction (u) \xrightarrow{\mu} M_\kappa \restriction [u, w) \xrightarrow{\psi_u} \Gamma(\mathbb{Z} \otimes \mathbb{Z}, (t, 0))$$

where μ is the natural retraction. Then clearly, for every $j = 1, \dots, n$, it is $(\psi_u \circ \mu)(y_j) = (a_j, dc^{-1}b_j)$. Since $p = q$ fails in $\Gamma(\mathbb{Z} \otimes \mathbb{Z}, (t, 0))$ over the (a_j, b_j)'s, and dc^{-1} is nonzero positive, it follows that $p = q$ fails in $\Gamma(\mathbb{Z} \otimes \mathbb{Z}, (t, 0))$ over the $(a_j, dc^{-1}b_j)$'s, too. Pulling back along $\psi_u \circ \mu$, we see that $p = q$ fails in $M_\kappa \restriction (u)$ over the y_j's, as was to be shown. □

Lemma 2.6. *Let $\kappa = n > 0$ be finite, and let $T_1, \dots, T_l$ be finitely many pairwise disjoint closed subsets of $[0, 1]^n$. Let $f_1, \dots, f_l \in M_n$. Then there exists $g \in M_n$ such that, for every $i = 1, \dots, l$, we have $g = f_i$ over T_i.*

Proof. Induction over l, using [Mun88, Corollary 3.4(ii)]. □

Corollary 2.7. *Assume the hypotheses of Theorem 2.5, and let $\kappa = n > 0$ be finite. Then the free algebra over n generators in $\mathbf{V}$ is the finite product*

$$M_n^{\mathbf{V}} = \prod_{u \in X \setminus Y} M_n \restriction u \times \prod_{v \in Y} M_n \restriction (v)$$

3. First application: coproducts

Coproducts of MV-algebras have been considered, and explicit computations have been given, in [Mun88]. While products of abstract algebras are independent of the ambient variety, this is not the case for coproducts. As an example, the coproduct of S_2 and S_3 in $\mathbf{MV}$ is S_6 [Mun88, Theorem 4.2] but, as we shall see, is the one-element algebra in $\mathbf{V}(S_2, S_3)$. Free products always exist in $\mathbf{MV}$, but may not exist in proper subvarieties. We recall the basic definitions for the case of two algebras, the extension to the general case being straightforward.

Let $\mathbf{V}$ be any variety of abstract algebras, whose signature contains at least one constant. Taking as morphisms the homomorphisms, $\mathbf{V}$ becomes a concrete category with initial object the algebra generated by the constants, and terminal object the one-element algebra. For $A_1, A_2 \in \mathbf{V}$, their *coproduct* in $\mathbf{V}$ is some $B \in \mathbf{V}$, together with morphisms $\iota_i : A_i \to B$ such that, for every $C \in \mathbf{V}$ and every pair of morphisms $\varphi_i : A_i \to C$, there exists a unique $\psi : B \to C$ with $\varphi_i = \psi \circ \iota_i$. Coproducts are unique up to isomorphism; a coproduct is called a *free product* if the maps ι_i are injective. Coproducts always exist in a variety $\mathbf{V}$, and can be constructed as follows: for $i = 1, 2$, represent the algebras A_i as F_{κ_i}/Θ_i, where F_{κ_i} is the free algebra over κ_i generators in $\mathbf{V}$, and Θ_i is a congruence in F_{κ_i}. Embed canonically F_{κ_1} and F_{κ_2} into $F_{\kappa_1+\kappa_2}$, and let Θ be the congruence in $F_{\kappa_1+\kappa_2}$ generated by the images of Θ_1 and Θ_2. Then $F_{\kappa_1+\kappa_2}/\Theta$ is the coproduct of A_1 and A_2 in $\mathbf{V}$. When one has sufficient information about the free objects in $\mathbf{V}$, this construction can be carried out explicitly. In the case of MV-algebras, congruences are in 1-1 correspondence with ideals, and one works better with ideals. In this section we give a few coproduct computations. We intend them mainly as specimens of the above technique; once the latter is understood, the computation of similar examples becomes an exercise.

Our building blocks being the various S_m and S_t^ω, we compute the coproducts $S_m \coprod S_t$, $S_m \coprod S_t^\omega$, $S_m^\omega \coprod S_t^\omega$. In each case, we compute the coproduct with respect to the smallest variety in which it makes sense, i.e., in $\mathbf{V}(S_m, S_t)$, $\mathbf{V}(S_m, S_t^\omega)$, $\mathbf{V}(S_m^\omega \coprod S_t^\omega)$, respectively. In order to avoid burdening of notation, when we write $A \coprod B$ in the following, we always intend the coproduct of A and B in the subvariety $\mathbf{V}(A, B)$ of $\mathbf{MV}$. We need a name for the one-element MV-algebra, and we call it S_0. Note that S_m and S_t^ω have easy presentations:

- $S_m \simeq M_1 \restriction p \simeq M_1/J_p$, with $p = 1/m \in [0, 1]$;
- $S_t^\omega \simeq M_2 \restriction [q, r) \simeq M_2/J_{[q,r)}$, with $q = (1/t, 1/t)$ and $r = (1/t, 1)$.

Let $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ be the first cartesian quadrant, and let $\mathfrak{H}$ be the ℓ-group of positively homogeneous piecewise-linear continuous functions with integer coefficients : $\mathbb{R}^2_+ \to \mathbb{R}$. Explicitly: $h \in \mathfrak{H}$ iff h is a finite sup of finite infs of functions of the form $ax + by$ with a, b integers [Bey77, §1].

Lemma 3.1. *Let $\mathfrak{A}$ be an ℓ-group, $\mathfrak{I}$ an ideal of $\mathfrak{A}$. Suppose that there is a unique maximal ideal $\mathfrak{K}$ of $\mathfrak{A}$ that extends $\mathfrak{I}$. Suppose that $\varphi : \mathfrak{A} \to \mathbb{Z}$ is an epimorphism with kernel $\mathfrak{K}$, and choose $e \in \mathfrak{A}$ with $\varphi(e) = 1$. Then the mapping $\psi : \mathfrak{A} \to \mathbb{Z} \otimes (\mathfrak{K}/\mathfrak{I})$ given by*

$$\psi(f) = \left(\varphi(f), \frac{f - \varphi(f)e}{\mathfrak{I}}\right)$$

is an epimorphism with kernel $\mathfrak{I}$.

Proof. Since $\mathfrak{A}/\mathfrak{K}$ and $\mathfrak{A}/\mathfrak{I}/\mathfrak{K}/\mathfrak{I}$ are isomorphic [BKW77, p. 43], we may assume that $\mathfrak{I}$ is the zero ideal of $\mathfrak{A}$. It is straightforward to show that ψ is injective and distributes over the group operations. Let $f, g \in \mathfrak{A}$, $\varphi(f \vee g) = a$, $\varphi(f) = b$, $\varphi(g) = c$; without loss of generality, $a = b \geq c$.
Case 1. $b = c$. Then $\psi(f \vee g) = (a, (f \vee g) - ae) = (a, (f - ae) \vee (g - ae)) = (b, f - be) \vee (c, g - ce) = \psi(f) \vee \psi(g)$.
Case 2. $b > c$. Then $f > g$. Indeed, if not, then there exists a prime ideal $\mathfrak{P}$ of $\mathfrak{A}$ with $f/\mathfrak{P} < g/\mathfrak{P}$. But since $\mathfrak{P} \subseteq \mathfrak{K}$, this implies $f/\mathfrak{K} \leq g/\mathfrak{K}$, which is contrary to our assumption. Hence $\psi(f \vee g) = \psi(f) = (b, f - be) = (b, f - be) \vee (c, g - ce) = \psi(f) \vee \psi(g)$.
Finally, for every $a \in \mathbb{Z}$ and every $f \in \mathfrak{K}$, we have $\psi(f + ae) = (a, f)$. □

Theorem 3.2. (Compare with [Mun88, Theorems 4.2 and 4.6]) *Let $0 < m, t$. If $m \nmid t$ and $t \nmid m$, then $S_m \coprod S_t$, $S_m \coprod S_t^\omega$, $S_m^\omega \coprod S_t^\omega$ are all equal to S_0. Assume $m \mid t$. Then:*

(i) *$S_m \coprod S_t = S_t$;*
(ii) *$S_m \coprod S_t^\omega = S_t^\omega$;*
(iii) *if $m \neq t$, then $S_m^\omega \coprod S_t = S_t$;*
(iv) *$S_m^\omega \coprod S_t^\omega = \Gamma(\mathbb{Z} \otimes \mathfrak{H}, (t, 0))$.*

Proof. We prove our assertions concerning $S_m \coprod S_t^\omega$ and $S_m^\omega \coprod S_t^\omega$, the other cases being similar. Let $\mathbf{V} = \mathbf{V}(S_m, S_t^\omega)$. Represent S_m as $M_1^\mathbf{V}/I_p$, where I_p is the ideal of germs of one-variable McNaughton functions vanishing at $p = 1/m$; represent S_t^ω as $M_2^\mathbf{V}/I_{[q,r)}$, with $I_{[q,r)}$ the ideal of two-variable germs vanishing at $q = (1/t, 1/t)$ along the direction of $r = (1/t, 1)$. Embed $M_1^\mathbf{V}$ and $M_2^\mathbf{V}$ canonically in $M_3^\mathbf{V}$, and let I be the ideal of $M_3^\mathbf{V}$ generated by the images of I_p and $I_{[q,r)}$. In the notation of Theorem 2.5 and Corollary 2.7, denote by $\bar{f}$ the image of $f \in M_3$ under the natural epimorphism

$$M_3 \to M_3^\mathbf{V} = \prod_{u \in X \setminus Y} M_3 \restriction u \times \prod_{v \in Y} M_3 \restriction (v)$$

Then it is clear that $\bar{f} \in I$ iff f vanishes at $q' = (1/m, 1/t, 1/t)$ along the direction of $r' = (1/m, 1/t, 1)$. We have $\mathrm{den}(q') = \mathrm{lcm}(m, t)$. If $m \nmid t$ and $t \nmid m$, then $q' \notin X \cup Y$ and, by Lemma 2.6, we can find $f \in M_3$ such that $\bar{f} \in I$ and $\bar{f} = \bar{1}$. Hence I is the improper ideal of $M_3^\mathbf{V}$ and $S_m \coprod S_t^\omega = S_0$. If $m \mid t$, then $\mathrm{den}(q') = t$, $q' \in Y$, and $M_3^\mathbf{V}/I$ is isomorphic to $M_3 \restriction [q', r')$. Since the images of the generators x_0, x_1, x_2 in the monomorphism

$$\psi_{r'} : M_3 \restriction [q', r') \to \Gamma(\mathbb{Z} \otimes \mathbb{Z}, (t, 0))$$

of Lemma 2.3 are $(t/m, 0), (1, 0), (1, t-1)$, the range of $\psi_{r'}$ is the full subalgebra of $\Gamma(\mathbb{Z} \otimes \mathbb{Z}, (t, 0))$ generated by $(0, t-1), (1, 0)$, which is isomorphic to S_t^ω.

We now compute $S_m^\omega \coprod S_t^\omega$ in $\mathbf{V}(S_m^\omega, S_t^\omega)$. If $m \nmid t$ and $t \nmid m$, then the same argument as in the preceding case shows that the coproduct is S_0. Assume $m \mid t$, let $w = (1/m, 1/m, 1/t, 1/t) \in [0,1]^4$, $W = \{(1/m, b, 1/t, d) \in [0,1]^4 : b \geq 1/m \text{ and } d \geq 1/t\}$, and let I be the ideal of all four-variable McNaughton functions vanishing in some set of the form $U \cap W$, where U is an open set containing w. Then the standard construction shows that $A = M_4/I$ is the coproduct of S_m^ω and S_t^ω in $\mathbf{V}(S_m^\omega, S_t^\omega)$ (as well as in $\mathbf{MV}$, by the way). Applying Γ^{-1} to the exact sequence

$$I \hookrightarrow M_4 \twoheadrightarrow A$$

yields a well-defined exact sequence

$$\mathfrak{J} \hookrightarrow (\mathfrak{M}_4, 1) \twoheadrightarrow (\mathfrak{M}_4/\mathfrak{J}, 1/\mathfrak{J})$$

of ℓ-groups ($1 \in \mathfrak{M}_4$ is the function whose value is identically 1). By taking $\mathfrak{K}$ to be the kernel of the epimorphism $\varphi : \mathfrak{M}_4 \to \mathbb{Z}$ given by $\varphi(f) = tf(w)$, and by choosing $e = x_2$, Lemma 3.1 applies and we conclude that the function

$$\bar{\psi} : \mathfrak{M}_4/\mathfrak{J} \to \mathbb{Z} \otimes (\mathfrak{K}/\mathfrak{J})$$

given by

$$\bar{\psi}(f/\mathfrak{J}) = \big(tf(w), (f - tf(w)x_2)/\mathfrak{J}\big)$$

is an isomorphism. Since $A = \Gamma(\mathfrak{M}_4/\mathfrak{J}, 1/\mathfrak{J})$ and $\bar{\psi}(1/\mathfrak{J}) = \bar{\psi}(tx_2/\mathfrak{J}) = (t^2/t, (tx_2 - tx_2)/\mathfrak{J}) = (t, 0)$, we know that A and $\Gamma(\mathbb{Z} \otimes (\mathfrak{K}/\mathfrak{J}), (t, 0))$ are isomorphic.

It remains to be proved that $\mathfrak{K}/\mathfrak{J}$ and $\mathfrak{H}$ are isomorphic as ℓ-groups. $\mathfrak{M}_4/\mathfrak{J}$ is generated by the germs $x_0/\mathfrak{J}$, $x_1/\mathfrak{J}$, $x_2/\mathfrak{J}$, $x_3/\mathfrak{J}$, $1/\mathfrak{J}$. Set

$$\begin{aligned}
\bar{z}_1 &= x_1/\mathfrak{J} - (t/m)x_2/\mathfrak{J} \\
\bar{z}_2 &= x_3/\mathfrak{J} - x_2/\mathfrak{J} \\
\bar{z}_3 &= x_2/\mathfrak{J}
\end{aligned}$$

Then $\mathfrak{M}_4/\mathfrak{J}$ is generated by $\bar{z}_1, \bar{z}_2, \bar{z}_3$, too. Note that $\bar{z}_1 > 0$ and $\bar{z}_2 > 0$. Since $\mathfrak{H}$ is the free ℓ-group over two generators (the projection functions $x, y : \mathbb{R}^2_+ \to \mathbb{R}$), subject to the relations $x \geq 0$ and $y \geq 0$ [Bey77, Example 2], we can define a homomorphism

$$\varphi : \mathfrak{H} \to \mathfrak{M}_4/\mathfrak{J}$$

by $\varphi(x) = \bar{z}_1$, $\varphi(y) = \bar{z}_2$. It is clear that φ is injective, and we only need to show that $\varphi(\mathfrak{H}) = \mathfrak{K}/\mathfrak{J}$. The inclusion $\subseteq$ being trivial, let $\bar{f} \in \mathfrak{K}/\mathfrak{J}$. By repeated applications of the distributive and De Morgan laws [BKW77, Proposition 2.1.4], we can write $\bar{f}$ as

$$(*) \qquad \bar{f} = \bigvee_{i \in I} \bigwedge_{j \in J_i} (a_{ij}\bar{z}_1 + b_{ij}\bar{z}_2 + c_{ij}\bar{z}_3)$$

with I and every J_i, for $i \in I$, finite index sets, and all the coefficients of the $\bar{z}$'s integer numbers. If $c < c'$, then $a\bar{z}_1 + b\bar{z}_2 + c\bar{z}_3 < a'\bar{z}_1 + b'\bar{z}_2 + c'\bar{z}_3$; hence we can drop superfluous conjuncts in $(*)$, writing

$$\bar{f} = \bigvee_{i \in I} \left(c_i\bar{z}_3 + \bigwedge_{r \in R_i} (a_{ir}\bar{z}_1 + b_{ir}\bar{z}_2) \right)$$

Analogously, setting $c = \bigvee_i c_i$, we write

$$\bar{f} = c\bar{z}_3 + \bigvee_{s \in S} \bigwedge_{t \in T_s} (a_{st}\bar{z}_1 + b_{st}\bar{z}_2)$$

Since $\bar{f}(w) = 0$, we must have $c = 0$, and hence $\bar{f}$ is in the range of φ. $\square$

4. Second application: axiomatizations

For the rest of this paper we fix a reduced pair $(\{m_1,\dots,m_r\},\{t_1,\dots,t_s\})$, and we consider the proper subvariety $\mathbf{V}=\mathbf{V}(S_{m_1},\dots,S_{m_r},S^\omega_{t_1},\dots,S^\omega_{t_s})$ of $\mathbf{MV}$. We will construct a one-variable identity $\alpha(a)=1$ that, together with the MV-algebra identities, axiomatize $\mathbf{V}$ (as customary, 1 denotes the MV-term $\neg 0$). Similar finite axiomatizations have been obtained in [DNL], but we think that our geometric approach is anyhow interesting, because it is easily visualizable, and exploits the compactness properties of the n-cube.

We need a few basics on [a variant of] Farey sequences; we just sketch the construction, and refer to [HW85, §6.10], [MP94] for unproved claims.

A *Farey sequence* is a finite increasing set of reduced fractions in the interval $[0,1]$, defined by recursion as follows:

- the set $\{0/1,1/1\}$ is a Farey sequence;
- if $\mathfrak{F}$ is a Farey sequence and $c/d, c'/d'$ are two consecutive terms in $\mathfrak{F}$, then the set obtained from $\mathfrak{F}$ by inserting $(c+c')/(d+d')$ between c/d and c'/d' is a Farey sequence (this insertion process is called a *starring*).

For every Farey sequence $\mathfrak{F}$ and every reduced fraction $0\le c/d\le 1$, there exists a sequence of starrings that leads from $\mathfrak{F}$ to a sequence $\mathfrak{F}'$ that includes c/d. Given two consecutive terms $c/d<c'/d'$ in any $\mathfrak{F}$, the determinant

$$\begin{vmatrix} c' & d' \\ c & d \end{vmatrix}$$

has value 1. Owing to this property, for every $0\le e\le d$ and every $0\le e'\le d'$, the affine line $y=ax+b$ passing through $(c/d,e/d)$ and $(c'/d',e'/d')$ has integer coefficients; indeed, $a=de'-d'e$ and $b=c'e-ce'$. Let

$$0=\frac{c_1}{d_1}<\frac{c_2}{d_2}<\cdots<\frac{c_u}{d_u}=1$$

display a Farey sequence, and choose $0\le e_p\le d_p$, for every $1\le p\le u$. Then the function $f:[0,1]\to[0,1]$ that assumes value e_p/d_p on c_p/d_p, for every $1\le p\le u$, and is linear on each Farey interval, is a McNaughton function, and we identify it with a one-variable term —also denoted by f— in the language of MV-algebras.

Lemma 4.1. *Let u be a rational point of the n-cube, with* $\mathrm{den}(u)=d$, *and choose $0\le e\le d$. Then there exist $g,h\in M_n$ such that $g(u)=h(u)=e/d$ and, for every $v\ne u$, we have $g(v)<e/d<h(v)$.*

Proof. This is trivial by the theory of Schauder hats [Mun94], [Pan95]. □

Definition 4.2. Let $I=\{m_1,\dots,m_r\}$, $J=\{t_1,\dots,t_s\}$ be a reduced pair. An (I,J)-*comb* is any $\alpha\in M_1$ such that the following conditions hold:

(i) for every $t\in J$ and every $0\le k\le t$, there exists a neighborhood U of k/t such that $\alpha\restriction U$ is identically 1;

(ii) for every $m\in I$ we have:

(ii′) for every $0\le h\le m$, it is $\alpha(h/m)=1$;

(ii″) there exists $0\le h\le m$ and some $h/m\ne u\in[0,1]$ such that α never takes value 1 either on the open interval $(u,h/m)$ (if $u<h/m$), or on the open interval $(h/m,u)$ (if $h/m<u$);

(iii) if $d>1$ is any integer that does not divide any $m\in I\cup J$, then there exists $1\le l\le d-1$ with $\alpha(l/d)\ne 1$.

It is easy to construct an (I, J)-comb: let X be the set of all rational points in $[0, 1]$ whose denominator divides one of $m_1, \dots, m_r$, and let Y be the set of all points whose denominator divides one of $t_1, \dots, t_s$. Choose a Farey sequence $\mathfrak{F}$ that includes all points of $X \cup Y$ and such that, moreover, for every $u, v \in X \cup Y$ with $u < v$, there exist $w_1, w_2, w_3 \in \mathfrak{F}$ with $u < w_1 < w_2 < w_3 < v$. Let Z be the set of points of $\mathfrak{F}$ that are immediately to the left or immediately to the right of some point of Y. Define $\alpha_{\mathfrak{F}} \in M_1$ to be the function that is linear on each interval of $\mathfrak{F}$ and whose values on $\mathfrak{F}$ are given by

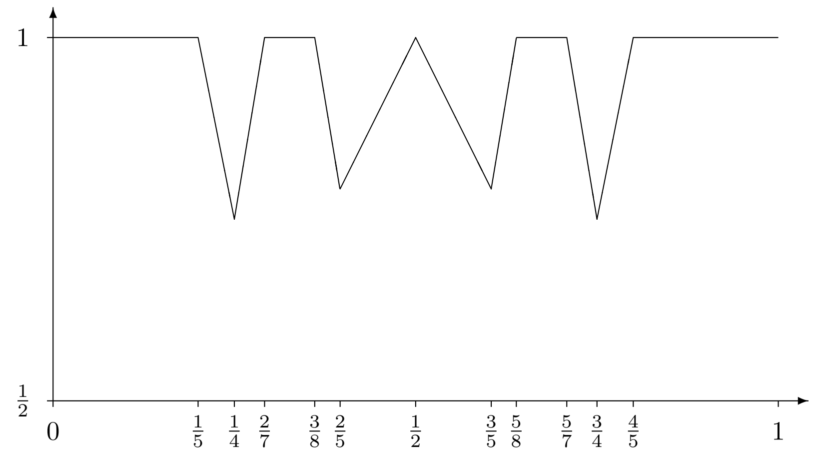
$$\alpha_{\mathfrak{F}}(u) = \begin{cases} 1, & \text{if } u \in X \cup Y \cup Z; \\ (\operatorname{den}(u) - 1)/\operatorname{den}(u), & \text{if } u \in \mathfrak{F} \setminus (X \cup Y \cup Z). \end{cases}$$

Then $\alpha_{\mathfrak{F}}$ automatically satisfies conditions (i) and (ii) in Definition 4.2. Let $r \in \mathbb{Q}$ be the length of the largest open subinterval (u, v) of $[0, 1]$ such that $\alpha_{\mathfrak{F}} \restriction (u, v)$ never takes value 1, and let d' be the smallest positive integer with $r > 1/d'$. Then $\alpha_{\mathfrak{F}}$ may fail condition (iii) only for finitely many d's, because if $d \geq d'$ then the interval (u, v) contains a point of the form l/d. Once the list of all d's for which (iii) fails is written down, it is easy to refine $\mathfrak{F}$ by successive starrings to a Farey sequence $\mathfrak{F}'$ in such a way that the resulting $\alpha_{\mathfrak{F}'}$ satisfies (iii). Since (i) and (ii) are not affected, $\alpha_{\mathfrak{F}'}$ is an (I, J)-comb.

Example 4.3. Let $I = \{2\}$, $J = \{3\}$. We have $X = \{0, 1/2, 1\}$, $Y = \{0, 1/3, 2/3, 1\}$, and an appropriate $\mathfrak{F}$ is

$$0, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, 1$$

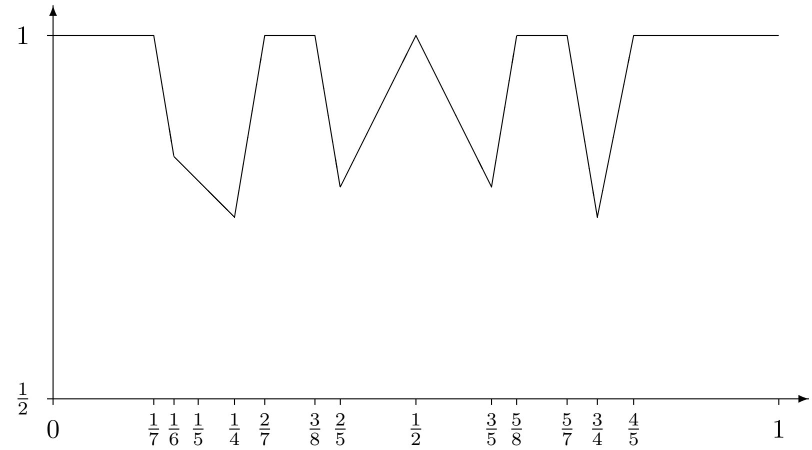
The graph of $\alpha_{\mathfrak{F}}$ is



The largest interval in which $\alpha_{\mathfrak{F}}$ never takes value 1 has length $(1/2) - (3/8) = 1/8$; hence condition (iii) may fail only for $d \leq 8$. The only possibilities are $d = 4, 5, 6, 7, 8$. Since $\alpha(1/4) = \alpha(2/8), \alpha(2/5), \alpha(3/7)$ are all different from 1, we have to take care of the case $d = 6$ only. By starring $\mathfrak{F}$ twice, we obtain the sequence $\mathfrak{F}'$ given by

$$0, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, 1$$

whose corresponding $\alpha_{\mathfrak{F}'}$ is



Since $\alpha_{\mathfrak{F}'}(1/6) \neq 1$, this $\alpha_{\mathfrak{F}'}$ is an (I, J)-comb.

For $\alpha \in M_1$ and $f \in M_n$, we continue to identify the term $\alpha(f)$, obtained by substituting the term f for the propositional variable x_0 in α, with the function $\alpha \circ f \in M_n$.

Theorem 4.4. *Let $I = \{m_1, \dots, m_r\}$, $J = \{t_1, \dots, t_s\}$ be a reduced pair, and let $\alpha \in M_1$. Then the identity $\alpha(a) = 1$, together with the MV-algebra axioms, axiomatize $\mathbf{V} = \mathbf{V}(S_{m_1}, \dots, S_{m_r}, S^{\omega}_{t_1}, \dots, S^{\omega}_{t_s})$ if and only if α is an (I, J)-comb.*

Proof. Assume that $\alpha(a) = 1$ axiomatize $\mathbf{V}$. By setting $n = 1$ in Corollary 2.7, we see that conditions (i) and (ii$'$) of Definition 4.2 are satisfied. Assume by contradiction that (ii$''$) fails for some $m \in I$, and let $u = (1/m, 1/m) \in [0, 1]^2$. Since $M_2^{\mathbf{V}}$ is the quotient of M_2 by the filter generated by $\{\alpha(f) : f \in M_2\}$, it follows that $M_2 \restriction (u)$ is a quotient of $M_2^{\mathbf{V}}$. Set $v = (1/m, 1)$; since S^{ω}_m is isomorphic to $M_2 \restriction [u, v)$, which is a quotient of $M_2 \restriction (u)$, it follows that $S^{\omega}_m \in \mathbf{V}$. By Komori's results cited before Proposition 1.1, m divides some element of J, and this is absurd, because I, J is reduced. The same argument shows that if (iii) fails for some d, then S_d is a quotient of $M_1^{\mathbf{V}}$, and hence belongs to $\mathbf{V}$; again, this is a contradiction.
For the reverse direction, assume that α is an (I, J)-comb. For every n and every $f \in M_n$, denote the image of f under the canonical epimorphism $M_n \to M_n^{\mathbf{V}}$ by $\bar{f}$. We must show that, for every such f:

(i) $\overline{\alpha(f)} = \bar{1}$;
(ii) if $\bar{f} = \bar{1}$, then there exist $f_1, \dots, f_k \in M_n$ such that f belongs to the filter generated by $\alpha(f_1), \dots, \alpha(f_k)$.

Now, (i) follows immediately from Corollary 2.7 and the observation that, for every rational point $u \in [0, 1]^n$, $f(u)$ is an integral multiple of $1/\operatorname{den}(u)$. Let us introduce the Euclidean metric over the n-cube; we write $B(u, \varepsilon)$ for the open ball of center u and radius ε. For $g \in M_n$, let $Og \subseteq [0, 1]^n$ be the set of points in which g attains value 1; it is well known that g belongs to the filter generated by $g_1, \dots, g_k$ iff $Og \supseteq Og_1 \cap \dots \cap Og_k$. Let $X, Y \subseteq [0, 1]^n$ be defined as in Theorem 2.5. We work in two steps.
Step 1. If $X \setminus Y = \emptyset$, go to Step 2. Otherwise, let $X \setminus Y = \{u_1, \dots, u_p\}$. Consider u_1; without loss of generality, $\operatorname{den}(u_1) = d \mid m_1$. By condition (ii$''$) of Definition 4.2,

there exists $0 \leq l \leq m_1$ and some $w \in [0,1]$ such that $\alpha(l/m_1) = 1$ and α is never 1 either on the interval $(w, l/m_1)$ or on the interval $(l/m_1, w)$ (according if $w < l/m_1$ or $l/m_1 < w$). By Lemma 4.1, there exists $g_1 \in M_n$ and a neighborhood U of u_1 such that $\alpha(g_1) \restriction U$ attains value 1 exactly in u_1. Repeat the same construction for every $u_i \in X \setminus Y$, obtaining $g_1, \ldots, g_p$. This ends Step 1.

Step 2. Choose $\varepsilon > 0$ such that:

- for every $u_i \in X \setminus Y$, $\alpha(g_i) \restriction B(u_i, \varepsilon)$ assumes value 1 exactly in u_i;
- for every $v \in Y$, $f \restriction B(v, \varepsilon)$ is identically 1.

Denote by T the closed set

$$T = [0,1]^n \setminus \bigcup_{u \in X \cup Y} B(u, \varepsilon)$$

We claim that, for every $w \in T$, there exists $h_w \in M_n$ such that $w \notin O\alpha(h_w)$, i.e., $\alpha(h_w(w)) \neq 1$. Indeed, there are two possibilities:

Case a. $w \in T$ is rational, with $\mathrm{den}(w) = d'$. Then d' does not divide any element of $I \cup J$. By condition (iii) of Definition 4.2, there exists $1 \leq l \leq d' - 1$ with $\alpha(l/d') \neq 1$. By Lemma 4.1, one can find h_w with $h_w(w) = l/d'$; such an h_w does the job.

Case b. $w \in T$ is not rational. Then the set $\{h(w) : h \in M_n\}$ is dense in $[0,1]$ and, since α is not identically 1, we are through.

The family $\{[0,1]^n \setminus O\alpha(h_w) : w \in T\}$ is an open covering of T. By compactness, there exist $h_1, \ldots, h_q \in M_n$ such that

$$T \subseteq \big([0,1]^n \setminus O\alpha(h_1)\big) \cup \cdots \cup \big([0,1]^n \setminus O\alpha(h_q)\big)$$

and hence

$$O\alpha(h_1) \cap \cdots \cap O\alpha(h_q) \subseteq \bigcup_{u \in X \cup Y} B(u, \varepsilon).$$

Since, for $1 \leq i \leq p$, it is

$$O\alpha(g_i) \cap B(u_i, \varepsilon) = \{u_i\}$$

we obtain

$$O\alpha(g_1) \cap \cdots \cap O\alpha(g_p) \cap O\alpha(h_1) \cap \cdots \cap O\alpha(h_q) \subseteq (X \setminus Y) \cup \bigcup_{v \in Y} B(v, \varepsilon) \subseteq Of.$$

This concludes Step 2 and the proof of the Theorem. □

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