Classification of dimension groups over $\mathbf{Z}^3$

Giovanni Panti
Department of Mathematics
University of Siena
via del Capitano, 15
53100 Siena
ITALY
panti@sivax.cineca.it

Elliott introduced dimension groups in [8], in order to classify approximately finite C^*-algebras. In [6], Effros and Shen showed that dimension groups whose underlying group is $\mathbf{Z}^2$ give rise to interesting examples of C^*-algebras. Dimension groups whose underlying group is a finite product of integers have also been studied in [3],[4],[5],[13],[14],[15], but up to now there is no general classification result. In this paper we classify all dimension groups whose underlying group is $\mathbf{Z}^3$.

1. Preliminaries and Definitions

A *partially ordered abelian group* (p.o. group) is an additive abelian group G, together with a *positive cone* $G^+ \subseteq G$ such that $G^+ + G^+ \subseteq G^+$ and $G^+ \cap (-G^+) = \{0\}$. The relation $\leq$ defined by $a \leq b$ if and only if $b - a \in G^+$ is a translation invariant partial order. $\langle G, G^+ \rangle$ is
1) *unperforated* if, for each $n \in \mathbf{N} \setminus \{0\}$, $na \in G^+$ implies $a \in G^+$;
2) *directed* if every pair of elements of G has an upper bound;
3) has the *Riesz interpolation property* if, for each a, b, c, d such that $a, b \leq c, d$, there exists w such that $a, b \leq w \leq c, d$;
4) *lattice ordered* if every pair of elements of G has a least upper bound.

A *dimension group* is a p.o. group satisfying 1), 2), 3). A *lattice-ordered group* (ℓ-group) is a p.o. group satisfying 4). For background the reader is referred to [1],[2],[9],[10],[15].

It is easy to prove the following:

Proposition 1.1. *Let $\langle G, G^+ \rangle$ be a p.o. group which is countable, unperforated and directed. Then $\langle G, G^+ \rangle$ is a dimension group if and only if, for each $a, b \in G$, either $a \vee b$ exists, or for each c greater than a, b there exists a strictly decreasing infinite chain $c > y_1 > y_2 > \cdots > a, b$ such that each d greater than a, b dominates some y_i.*

Given p.o. groups $G = \langle G, G^+ \rangle$ and $H = \langle H, H^+ \rangle$, and a homomorphism $\varphi\colon G \to H$, we say that φ is *positive* if $\varphi(G^+) \subseteq H^+$. If φ is an algebraic isomorphism and $\varphi(G^+) = H^+$, we say that φ is an order isomorphism, and we write $G \cong H$. Given G,H as before, their *direct sum* $G \oplus H$ is the direct product of G and H, ordered componentwise (i.e., $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$). The *lexicographic sum* of G and H, denoted by $G \oplus_{\text{lex}} H$, is their direct product, ordered lexicographically (i.e., $(a, b) \leq (c, d)$ if and only if $a < c$ or ($a = c$ and $b \leq d$)). If G and H are dimension groups, then $G \oplus H$ and $G \oplus_{\text{lex}} H$ are dimension groups. If G and H are ℓ-groups, then $G \oplus H$ is an ℓ-group, whereas $G \oplus_{\text{lex}} H$ is an ℓ-group if and only if G is totally ordered.

Lemma 1.2. *Let G be a totally ordered dense p.o. group, H an unperforated p.o. group which satisfies the Riesz interpolation property. Then $G \oplus_{\text{lex}} H$ is a dimension group.*

Proof. It is easy to see that $G \oplus_{\text{lex}} H$ is unperforated and directed. Let $(a, a'), (b, b') \leq (c, c'), (d, d')$. We may assume $a \leq b \leq c \leq d$. If $b < c$, then there exists $x \in G$ such that $b < x < c$; hence $(x, 0)$ is an interpolant. If $b = c < d$, then (c, c') is an interpolant. If $a < b = c = d$, then (b, b') is an interpolant. If $a = b = c = d$, then there exists $x' \in H$ such that $a', b' \leq x' \leq c', d'$, and (a, x') is an interpolant. ∎

A *convex subgroup* of a dimension group G is a subgroup G' of G such that if $a, b \in G'$, $c \in G$, and $a \leq c \leq b$, then $c \in G'$. An *ideal* is a convex directed subgroup. G is *simple* if $\{0\}$, G are its only ideals.

If G' is a convex subgroup of G, then $\langle G', G' \cap G^+ \rangle$ is an unperforated p.o. group that satisfies the Riesz interpolation property. If I is an ideal of G, we denote by G/I the quotient of G by I, ordered by $0/I \leq a/I$

if and only if there exists $b \in I$ such that $0 \leq a+b$; both I and G/I are dimension groups. A *strong unit* of G is an element $u > 0$ such that, for each $a \in G$, there exists a natural number n such that $a \leq nu$.

2. Cones over $\mathbf{Z}^n$

Let $\mathbf{X}$ be any of $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$. A *cone* over $\mathbf{X}^n$ is a subset $P \subseteq \mathbf{X}^n$ such that $P + P \subseteq P$ and $\mathbf{X}^n \cap \alpha P \subseteq P$ for any real number $\alpha \geq 0$. P is *proper* if $P \cap (-P) = \{0\}$, and is *generating* if $\mathbf{X}^n = P - P$. It is easy to prove that P is the positive cone of an unperforated, directed p.o. group $\langle \mathbf{X}^n, P\rangle$ if and only if it is a proper generating cone over $\mathbf{X}^n$.

We look at $\mathbf{Z}^n$,$\mathbf{Q}^n$ as embedded in $\mathbf{R}^n$, and we refer to points in $\mathbf{Z}^n$, $\mathbf{Q}^n$ as *integral points* and *rational points*, respectively. We identify the points in $\mathbf{X}^n$ with the $n \times 1$ matrices of their components with respect to the standard basis $\{e_1, \ldots, e_n\}$.

By $\mathbf{X}^{n+}$ we mean $\{(k_1, \ldots, k_n)^t \in \mathbf{X}^n : k_1, \ldots, k_n \geq 0\}$, so that $\langle \mathbf{X}^n, \mathbf{X}^{n+}\rangle$ is given componentwise ordering. We denote $\langle \mathbf{Z}, \mathbf{Z}^+\rangle$ simply by $\mathbf{Z}$. A p.o. group of the form $\langle \mathbf{Z}^n, \mathbf{Z}^{n+}\rangle$ is called a *simplicial group*. A *direct sequence* of simplicial groups is a sequence of the form

$$\mathbf{Z}^{n_1} \xrightarrow{\varphi_1} \mathbf{Z}^{n_2} \xrightarrow{\varphi_2} \mathbf{Z}^{n_3} \xrightarrow{\varphi_3} \cdots$$

where each $\mathbf{Z}^{n_i}$ is simplicially ordered and the φ_i's are positive homomorphisms. The direct limit G of the sequence is defined as usual. If all the φ_i's can be chosen to be 1–1, then G is said to be *ultrasimplicial*. If all the $\mathbf{Z}^{n_i}$ are equal to $\mathbf{Z}^n$ for some fixed n, and all the φ_i's are algebraic isomorphisms, then the sequence is called *unimodular*. By the Effros, Handelman and Shen theorem [4, Theorem 2.2], countable dimension groups can be identified with direct limits of direct sequences of simplicial groups.

Definition 2.1. Let $f \in \mathbf{R}^{n*}$, the dual space of $\mathbf{R}^n$. We identify f with the $1 \times n$ matrix of its components with respect to the dual standard basis $\{\epsilon_1, \ldots, \epsilon_n\}$. We say that $f = (\alpha_1, \ldots, \alpha_n)$ is *of type m* if the dimension of the $\mathbf{Q}$-vector space $\mathbf{Q}\alpha_1 + \cdots + \mathbf{Q}\alpha_n$ is m.

If $\alpha_1, \ldots, \alpha_n$ are independent over $\mathbf{Q}$ (for short, $\mathbf{Q}$-independent), we denote the free $\mathbf{Z}$-module $\mathbf{Z}\alpha_1 + \cdots + \mathbf{Z}\alpha_n$ by $\mathbf{Z}[\alpha_1, \ldots, \alpha_n]$, equipped with the standard linear order. If $n > 1$, then $\mathbf{Z}[\alpha_1, \ldots, \alpha_n]$ is densely ordered. For $f \in \mathbf{R}^{n*}$, let π_f be $\{x \in \mathbf{Z}^n : f(x) = 0\}$. It is clear that f is of type m if and only if π_f, as a direct factor of $\mathbf{Z}^n$, has rank $n - m$.

Definition 2.2. Let $\mathbf{X}$ be any of $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and let $f_1, \ldots, f_m, g_1, \ldots, g_k \in \mathbf{R}^{n*}$ be $\mathbf{R}$-independent. We denote by $\langle \mathbf{X}^n, f_1, \ldots, f_m, \overline{g_1}, \ldots, \overline{g_k}\rangle$ the p.o. group over $\mathbf{X}^n$ whose positive cone is $\{x \in \mathbf{X}^n : f_1(x), \ldots, f_m(x) > 0 \text{ and } g_1(x), \ldots, g_k(x) \geq 0\} \cup \{0\}$, provided that the latter is proper.

Let $\alpha_1, \ldots, \alpha_n$ be $\mathbf{Q}$-independent. It is immediate that $\langle \mathbf{Z}^n, (\alpha_1, \ldots, \alpha_n)\rangle$ and $\mathbf{Z}[\alpha_1, \ldots, \alpha_n]$ are order isomorphic, the isomorphism being given by $e_i \mapsto \alpha_i$. The following is a generalization of [15, Lemma 4.7].

Lemma 2.3. *$\langle \mathbf{Z}^n, f_1, \ldots, f_m\rangle$ is order isomorphic to $\langle \mathbf{Z}^n, f'_1, \ldots, f'_m\rangle$ if and only if there exist a unimodular matrix $C \in \mathrm{Mat}_n\mathbf{Z}$ and $r_1, \ldots, r_m \in \mathbf{R}^+ \setminus \{0\}$ such that, for each $i \in \{1, \ldots, m\}$, it is $f'_i = r_i f_i C$ (a matrix being unimodular if its determinant is $+1$ or -1).*

Proof. ($\Leftarrow$) For $i \in \{1, \ldots, m\}$, let w_i be defined by $w_i = f_i C$. Then $w_1, \ldots, w_m$ are $\mathbf{R}$-independent and $\langle \mathbf{Z}^n, w_1, \ldots, w_m\rangle = \langle \mathbf{Z}^n, f'_1, \ldots, f'_m\rangle$. Define $\varphi : \mathbf{Z}^n \to \mathbf{Z}^n$ by $\varphi(x) = C^{-1}x$. As C is unimodular, φ is an automorphism of $\mathbf{Z}^n$. As $f_i(x) = f_i x = f_i C C^{-1} x = w_i(\varphi(x))$, it follows that φ is an order isomorphism of $\langle \mathbf{Z}^n, f_1, \ldots, f_m\rangle$ onto $\langle \mathbf{Z}^n, w_1, \ldots, w_m\rangle = \langle \mathbf{Z}^n, f'_1, \ldots, f'_m\rangle$.
($\Rightarrow$) Let $\varphi : \langle \mathbf{Z}^n, f'_1, \ldots, f'_m\rangle \to \langle \mathbf{Z}^n, f_1, \ldots, f_m\rangle$ be an order isomorphism. Then in particular φ is an automorphism of $\mathbf{Z}^n$ and there exists a unimodular matrix $C \in \mathrm{Mat}_n\mathbf{Z}$ such that $\varphi(x) = Cx$. Again, for $i \in \{1, \ldots, m\}$, define $w_i = f_i C$. Then, by the first part of the proof, $\psi : \langle \mathbf{Z}^n, f_1, \ldots, f_m\rangle \to \langle \mathbf{Z}^n, w_1, \ldots, w_m\rangle$ defined by $\psi(x) = C^{-1}x$ is an order isomorphism. Hence $\psi\varphi : \langle \mathbf{Z}^n, f'_1, \ldots, f'_m\rangle \to \langle \mathbf{Z}^n, w_1, \ldots, w_m\rangle$ is the identity order isomorphism. This implies that f'_i is a positive multiple of $w_i = f_i C$. ■

Let P be a cone in $\mathbf{R}^n$. We say that P is *cosimplicial* if there exist $\mathbf{R}$-independent $f_1, \ldots, f_m \in \mathbf{R}^{n*}$ such that $P = \{x \in \mathbf{R}^n : f_1(x), \ldots, f_m(x) \geq 0\}$.

Lemma 2.4. *Let $\langle \mathbf{Z}^n, P\rangle$ be a dimension group, $\widetilde{P} = \{qx : q \in \mathbf{Q}^+ \text{ and } x \in P\}$, $\overline{P}$ the topological closure of $\widetilde{P}$ in $\mathbf{R}^n$. Then $\overline{P}$ is a cosimplicial cone, $P = \widetilde{P} \cap \mathbf{Z}^n$, and any rational point in the topological interior of $\overline{P}$ belongs to $\widetilde{P}$.*

Proof. By [7, Lemmas 2.1 and 1.1], $\langle \mathbf{Q}^n, \widetilde{P}\rangle$ is a dimension group and $\overline{P}$ is cosimplicial. Assume $x \in \widetilde{P} \cap \mathbf{Z}^n$. Then there exist $y \in P$, $k \in \mathbf{Z}^+$, $k' \in \mathbf{Z}^+ \setminus \{0\}$, such that $x = ky/k'$. Hence $k'x = ky \in P$ and, as $\langle \mathbf{Z}^n, P\rangle$ is unperforated, $x \in P$.
Assume that $x \in \mathbf{Q}^n$ is in the topological interior of $\overline{P}$. As open n-dimensional simplexes form a basis for the standard topology in $\mathbf{R}^n$, we may choose $n+1$ points $x_1, \ldots, x_{n+1} \in \overline{P}$ such that x is in the interior of the simplex spanned by $x_1, \ldots, x_{n+1}$. As $\widetilde{P}$ is dense in $\overline{P}$, we may assume that $x_1, \ldots, x_{n+1} \in \widetilde{P}$. There exist uniquely determined rational numbers $q_1, \ldots, q_{n+1}$ such that $0 < q_i < 1$ and $x = q_1 x_1 + \cdots + q_{n+1} x_{n+1}$. As every $q_i x_i$ belongs to $\widetilde{P}$, it follows that x belongs to $\widetilde{P}$. ∎

It follows that for any dimension group $\langle \mathbf{Z}^n, P\rangle$ there exist $\mathbf{R}$-independent $f_1, \ldots, f_m \in \mathbf{R}^{n*}$ and some $\{0\} \subseteq B \subseteq \pi_{f_1} \cup \cdots \cup \pi_{f_m}$ such that $P = \mathrm{int}(P) \cup B$, where $\mathrm{int}(P) = \{x \in \mathbf{Z}^n : f_1(x), \ldots, f_m(x) > 0\} \neq \emptyset$. It is easy to show that $\mathrm{int}(P)$ is the set of strong units of $\langle \mathbf{Z}^n, P\rangle$. If $m = n$, the dimension of the space, then, by [10, Theorem 3.13] and the observation that there is no infinite descending chain in P, we conclude that $\langle \mathbf{Z}^n, P\rangle$ is simplicially ordered. If $m = 1$, we have the following proposition, which is essentially due to Shen [15, Proposition 1.6].

Proposition 2.5. *Let $G = \langle \mathbf{Z}^n, P\rangle$ be a dimension group such that $P = \{x \in \mathbf{Z}^n : f(x) > 0\} \cup B$ for some $f = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^{n*}$ and $\{0\} \subseteq B \subseteq \pi_f$. We have:*
i) if f is of type n, then $G \cong \mathbf{Z}[\alpha_1, \ldots, \alpha_n]$;
ii) if f is of type $m \in \{2, \ldots, n-1\}$, then $G \cong \mathbf{Z}[\beta_1, \ldots \beta_m] \oplus_{\mathrm{lex}} \langle \mathbf{Z}^{n-m}, P'\rangle$, for some $\beta_1, \ldots, \beta_m \in \mathbf{R}$ and some p.o. group $\langle \mathbf{Z}^{n-m}, P'\rangle$ which is unperforated and satisfies the Riesz interpolation property;
iii) if f is of type 1, then $G \cong \mathbf{Z} \oplus_{\mathrm{lex}} \langle \mathbf{Z}^{n-1}, P'\rangle$, where $\langle \mathbf{Z}^{n-1}, P'\rangle$ is a dimension group.

Proof. Let f be of type m. Identify $\langle \pi_f, B\rangle$ with $\langle \mathbf{Z}^{n-m}, P'\rangle$; it is a convex subgroup of G and hence an unperforated p.o. group that satisfies the Riesz interpolation property. Let $\{x_1, \ldots, x_n\}$ be a basis for $\mathbf{Z}^n$ whose last $n-m$ elements form a basis for π_f, and let $\beta_i = f(x_i)$ for $1 \leq i \leq m$. Then $\beta_1, \ldots, \beta_m$ are $\mathbf{Q}$-independent and $f(\mathbf{Z}^n) = \mathbf{Z}[\beta_1, \ldots, \beta_m]$. Define $\varphi : G \to \mathbf{Z}[\beta_1, \ldots, \beta_m] \oplus_{\mathrm{lex}} \langle \mathbf{Z}^{n-m}, P'\rangle$ by: if $x = k_1 x_1 + \cdots + k_n x_n$, then $\varphi(x) = (f(x), k_{m+1} x_{m+1} + \cdots + k_n x_n)$. It is clear that φ is an algebraic isomorphism and that $x \geq 0$ in G if and only if $\varphi(x) \geq 0$ in $\mathbf{Z}[\beta_1, \ldots, \beta_m] \oplus_{\mathrm{lex}} \langle \mathbf{Z}^{n-m}, P'\rangle$, i.e., φ is an order isomorphism. We also have:
- if f is of type n, then $B = \pi_f = \{0\}$ and $G \cong \mathbf{Z}[\alpha_1, \ldots, \alpha_n]$;
- if f is of type 1, then $G \cong \mathbf{Z}[\beta_1] \oplus_{\mathrm{lex}} \langle \mathbf{Z}^{n-1}, P'\rangle \cong \mathbf{Z} \oplus_{\mathrm{lex}} \langle \mathbf{Z}^{n-1}, P'\rangle$. In this case, $\langle \mathbf{Z}^{n-1}, P'\rangle$ must be directed, and hence a dimension group. In fact, assume that $a, b \in \langle \mathbf{Z}^{n-1}, P'\rangle$ do not have an upper bound. Then $-a, -b$ do not have a lower bound and there is no interpolant for $(0, a), (0, b) \leq (1, -a), (1, -b)$ in $\mathbf{Z} \oplus_{\mathrm{lex}} \langle \mathbf{Z}^{n-1}, P'\rangle$. ∎

We note that all the ℓ-groups over a finite product of integers can be constructed by applying the following procedure:
i) the only ℓ-group over $\mathbf{Z}$ is $\langle \mathbf{Z}, \mathbf{Z}^+\rangle$;
ii) suppose we have constructed all the ℓ-groups over $\mathbf{Z}^i$ for $1 \leq i < n$. The ℓ-groups over $\mathbf{Z}^n$ are $\mathbf{Z}[\alpha_1, \ldots, \alpha_n]$, plus all direct sums $G \oplus G'$ (where G is an ℓ-group over $\mathbf{Z}^i$ for some $i \in \{1, \ldots, n-1\}$ and G' is an ℓ-group over $\mathbf{Z}^{n-i}$), plus all lexicographic sums $G \oplus_{\mathrm{lex}} G'$ (where G and G' are as before, with the additional assumption that G is totally ordered).

Parentheses can be simplified observing that $\oplus$ and $\oplus_{\mathrm{lex}}$ are both associative and that $\oplus$ is commutative. Conrad's structure theorems for ℓ-groups having a finite basis [1, Theorems 7.4.6 and 7.4.7] may be used to show that the above construction generates all the ℓ-groups over $\mathbf{Z}^n$. We give an elementary proof, which does not involve the concept of basis.

Proposition 2.6. *Assume that $G = \langle \mathbf{Z}^n, P\rangle$ is an ℓ-group. Then:*
i) if G is simple, then $G \cong \mathbf{Z}[\alpha_1, \ldots, \alpha_n]$ for some $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$;
ii) if G has a unique nonzero maximal ideal, then $G \cong \mathbf{Z}[\beta_1, \ldots, \beta_m] \oplus_{\mathrm{lex}} \langle \mathbf{Z}^{n-m}, P'\rangle$, for some $\beta_1, \ldots, \beta_m \in \mathbf{R}$ and some ℓ-group $\langle \mathbf{Z}^{n-m}, P'\rangle$;

iii) if G has more that one maximal ideal, then G is a direct sum of ℓ-groups over free groups of rank less than n.

Proof. If G is simple, then by Hölder's theorem G is order isomorphic to a subgroup of $\mathbf{R}$. Since G has rank n, it follows that $G \cong \mathbf{Z}[\alpha_1, \ldots, \alpha_n]$ for some $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$.
Suppose that G is not simple and that $M \neq \{0\}$ is the only maximal ideal of G. Every element of $G \setminus M$ is either positive or negative; this follows easily from Clifford's representation theorem [2, Theorem XIII 22]. By the observation following Lemma 2.4, and noting that no proper ideal may contain a strong unit, we see that there exists $f \in \mathbf{R}^{n*}$, say of type m, such that $M \subseteq \pi_f$ and $P = \{x \in \mathbf{Z}^n : f(x) > 0\} \cup B$, where $B = P \cap \pi_f$. Identifying $\langle \pi_f, B \rangle$ with $\langle \mathbf{Z}^{n-m}, P' \rangle$, we have from Proposition 2.5 that $G \cong \mathbf{Z}[\beta_1, \ldots, \beta_m] \oplus_{\text{lex}} \langle \mathbf{Z}^{n-m}, P' \rangle$. It is now easy to show that $\langle \mathbf{Z}^{n-m}, P' \rangle$ is lattice-ordered.
Finally, if G has more than one maximal ideal, then G is a direct sum of lower rank ℓ-groups by [2, Theorem XIII 23]. ∎

3. Dimension groups over $\mathbf{Z}^3$

The following is Shen's classification of dimension groups over $\mathbf{Z}$ and $\mathbf{Z}^2$.

Theorem 3.1. [15, Corollary 2.6 and Section 4] *Up to order isomorphism, there is exactly one dimension group over $\mathbf{Z}$, namely $\langle \mathbf{Z}, \mathbf{Z}^+ \rangle$.*
There are exactly three classes of dimension groups over $\mathbf{Z}^2$:
1) totally ordered groups of the form $\mathbf{Z}[\alpha, \beta]$, which are simple and partition in isomorphism classes according to Lemma 2.3;
2) $\mathbf{Z} \oplus_{\text{lex}} \mathbf{Z}$;
3) $\mathbf{Z} \oplus \mathbf{Z}$.
All of these are ℓ-groups.

Simple dimension groups over $\mathbf{Z}^n$ for arbitrary n have been characterized by Effros.

Theorem 3.2. [3, Theorem 4.8] *A p.o. group $\langle \mathbf{Z}^n, P \rangle$ is a simple dimension group if and only if it is of the form $\langle \mathbf{Z}^n, f_1, \ldots, f_m \rangle$ for some $f_1, \ldots, f_m \in \mathbf{R}^{n*}$ such that $(\mathbf{R}f_1 + \cdots + \mathbf{R}f_m) \cap (\mathbf{Z}\epsilon_1 + \cdots + \mathbf{Z}\epsilon_n) = \{0\}$.*

We are now in a position to classify all dimension groups over $\mathbf{Z}^3$.

Theorem 3.3. *There are exactly eight classes of ℓ-groups over $\mathbf{Z}^3$:*
1) $\mathbf{Z}[\alpha, \beta, \gamma]$;
2) $\mathbf{Z}[\alpha, \beta] \oplus_{\text{lex}} \mathbf{Z}$;
3) $\mathbf{Z} \oplus_{\text{lex}} \mathbf{Z}[\alpha, \beta]$;
4) $\mathbf{Z}[\alpha, \beta] \oplus \mathbf{Z}$;
5) $\mathbf{Z} \oplus_{\text{lex}} \mathbf{Z} \oplus_{\text{lex}} \mathbf{Z}$;
6) $\mathbf{Z} \oplus_{\text{lex}} (\mathbf{Z} \oplus \mathbf{Z})$;
7) $(\mathbf{Z} \oplus_{\text{lex}} \mathbf{Z}) \oplus \mathbf{Z}$;
8) $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$.
There are exactly four classes of dimension groups over $\mathbf{Z}^3$ which are not ℓ-groups:
9) $\langle \mathbf{Z}^3, f_1, f_2 \rangle$, provided that $(\mathbf{R}f_1 + \mathbf{R}f_2) \cap (\mathbf{Z}\epsilon_1 + \mathbf{Z}\epsilon_2 + \mathbf{Z}\epsilon_3) = \{0\}$;
10) $\mathbf{Z}[\alpha, \beta] \oplus_{\text{lex}} \langle \mathbf{Z}, \{0\} \rangle$;
11) $\langle \mathbf{Z}^3, \overline{\epsilon_1}, (\alpha, \beta, 1) \rangle$, provided that $\beta \in \mathbf{R} \setminus \mathbf{Q}$;
12) $(\mathbf{Z} \oplus \mathbf{Z}) \oplus_{\text{lex}} \mathbf{Z}$.
The groups of type 1) and 9) partition in isomorphism classes according to Lemma 2.3. Two groups $\mathbf{Z}[\alpha, \beta] \oplus_{\text{lex}} \mathbf{Z}$ and $\mathbf{Z}[\alpha', \beta'] \oplus_{\text{lex}} \mathbf{Z}$ are order isomorphic if and only if $\mathbf{Z}[\alpha, \beta]$ and $\mathbf{Z}[\alpha', \beta']$ are order isomorphic; analogous statements hold for groups of type 3), 4), 10). Two groups $\langle \mathbf{Z}^3, \overline{\epsilon_1}, (\alpha, \beta, 1) \rangle$ and $\langle \mathbf{Z}^3, \overline{\epsilon_1}, (\alpha', \beta', 1) \rangle$ of type 11) are order isomorphic if and only if $\langle \mathbf{Z}^3, \epsilon_1, (\alpha, \beta, 1) \rangle$ and $\langle \mathbf{Z}^3, \epsilon_1, (\alpha', \beta', 1) \rangle$ are order isomorphic.

Proof. All the groups of type 1)–8) are ℓ-groups. The groups of type 9) are simple dimension groups by Theorem 3.2; we will prove later that the groups of type 11) are dimension groups. As $\mathbf{Z} \oplus \mathbf{Z}$ is not totally ordered, the group 12) is a dimension group which is not an ℓ-group. The groups of type 10) are dimension groups by Lemma 1.2; they are not ℓ-groups as $(0,0) \vee (0,1)$ does not exist. The claims about isomorphism

classes of the groups of type 2), 3), 4), 10), follow by observing that, in all cases, we can recover an isomorphic copy of $\mathbf{Z}[\alpha,\beta]$ from the groups under consideration. Moreover, this process can be unambiguously described in purely p.o. group theoretical terms. So $\mathbf{Z}[\alpha,\beta]$ is isomorphic to:
- the quotient of $\mathbf{Z}[\alpha,\beta] \oplus_{\mathrm{lex}} \mathbf{Z}$ by its only proper nontrivial ideal;
- the only proper nontrivial ideal of $\mathbf{Z} \oplus_{\mathrm{lex}} \mathbf{Z}[\alpha,\beta]$;
- the only densely ordered ideal of $\mathbf{Z}[\alpha,\beta] \oplus \mathbf{Z}$;
- the quotient of $\mathbf{Z}[\alpha,\beta] \oplus_{\mathrm{lex}} \langle \mathbf{Z}, \{0\}\rangle$ by its maximal convex subgroup.

Assume that $\varphi : \langle \mathbf{Z}^3, \overline{\epsilon_1}, (\alpha,\beta,1)\rangle \to \langle \mathbf{Z}^3, \overline{\epsilon_1}, (\alpha',\beta',1)\rangle$ is an order isomorphism. As π_{ϵ_1} is the only maximal ideal of both groups, φ maps π_{ϵ_1} injectively onto itself. Hence $\{x \in \mathbf{Z}^3 : \epsilon_1(x), (\alpha,\beta,1)(x) > 0\}$ is mapped injectively onto $\{x \in \mathbf{Z}^3 : \epsilon_1(x), (\alpha',\beta',1)(x) > 0\}$, i.e., φ is an order isomorphism of $\langle \mathbf{Z}^3, \epsilon_1, (\alpha,\beta,1)\rangle$ onto $\langle \mathbf{Z}^3, \epsilon_1, (\alpha',\beta',1)\rangle$. Conversely, assume that $\varphi : \langle \mathbf{Z}^3, \epsilon_1, (\alpha,\beta,1)\rangle \to \langle \mathbf{Z}^3, \epsilon_1, (\alpha',\beta',1)\rangle$ is an order isomorphism. By topological considerations, we see that $\pi_{\epsilon_1} \cap \{x \in \mathbf{Z}^3 : (\alpha,\beta,1)(x) > 0\}$ is mapped either into $\pi_{\epsilon_1} \cap \{x \in \mathbf{Z}^3 : (\alpha',\beta',1)(x) > 0\}$, or into $\pi_{(\alpha',\beta',1)} \cap \{x \in \mathbf{Z}^3 : \epsilon_1(x) > 0\}$. It cannot be mapped into the latter because, as $\beta' \in \mathbf{R} \setminus \mathbf{Q}$, we have that $\pi_{(\alpha',\beta',1)}$ has rank 0 or 1, whereas π_{ϵ_1} has rank 2. Hence, for each $x \in \mathbf{Z}^3$, it is ($\epsilon_1(x) \geq 0$ and $(\alpha,\beta,1)(x) > 0$) if and only if ($\epsilon_1(\varphi(x)) \geq 0$ and $(\alpha',\beta',1)(\varphi(x)) > 0$), i.e., φ is an order isomorphism between $\langle \mathbf{Z}^3, \overline{\epsilon_1}, (\alpha,\beta,1)\rangle$ and $\langle \mathbf{Z}^3, \overline{\epsilon_1}, (\alpha',\beta',1)\rangle$.

What remains to be proved is that every dimension group over $\mathbf{Z}^3$ has one of the previous forms. As the proof is a bit cumbersome, we will make use of the geometric language freely. Moreover, we will sometimes identify $\mathbf{R}^{3^*}$ and $\mathbf{R}^3$ without specific notice.

Let us then assume that we are given a dimension group $G = \langle \mathbf{Z}^3, P\rangle$. By the observation following Lemma 2.4, P has one of the following forms:
i) $\{x \in \mathbf{Z}^3 : f(x) > 0\} \cup B$ for some $f \in \mathbf{R}^{3^*}$ and $\{0\} \subseteq B \subseteq \pi_f$;
ii) $\{x \in \mathbf{Z}^3 : f_1(x), f_2(x) > 0\} \cup B$ for some $\mathbf{R}$-independent $f_1, f_2 \in \mathbf{R}^{3^*}$ and $\{0\} \subseteq B \subseteq \pi_{f_1} \cup \pi_{f_2}$;
iii) $\{x \in \mathbf{Z}^3 : f_1(x), f_2(x), f_3(x) > 0\} \cup B$ for some $\mathbf{R}$-independent $f_1, f_2, f_3 \in \mathbf{R}^{3^*}$ and $\{0\} \subseteq B \subseteq \pi_{f_1} \cup \pi_{f_2} \cup \pi_{f_3}$.

We already observed that there is exactly one dimension group over $\mathbf{Z}^3$ whose positive cone has form iii), namely $\langle \mathbf{Z}^3, \mathbf{Z}^{3^+}\rangle \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ (Case 8)). Assume P has form i) and apply Proposition 2.5.
- If f is of type 3, then $G \cong \mathbf{Z}[\alpha,\beta,\gamma]$ for certain α,β,γ, and we are in Case 1).
- If f is of type 2, then either $G \cong \mathbf{Z}[\alpha,\beta] \oplus_{\mathrm{lex}} \mathbf{Z}$ (Case 2)), or $G \cong \mathbf{Z}[\alpha,\beta] \oplus_{\mathrm{lex}} \langle \mathbf{Z}, \{0\}\rangle$ (Case 10)). In fact, $\langle \mathbf{Z}, \{0\}\rangle$ is trivially unperforated and satisfies the Riesz interpolation property. On the other hand, assume that $\langle \mathbf{Z}, P\rangle$ is unperforated and $0 \neq k \in P$. Without loss of generality, $k > 0$. By unperforatedness $1 \in P$, and hence $P = \mathbf{Z}^+$.
- If f is of type 1, we apply Shen's classification and conclude that G is either $\mathbf{Z} \oplus_{\mathrm{lex}} \mathbf{Z}[\alpha,\beta]$ (Case 3)), or $\mathbf{Z} \oplus_{\mathrm{lex}} \mathbf{Z} \oplus_{\mathrm{lex}} \mathbf{Z}$ (Case 5)), or $\mathbf{Z} \oplus_{\mathrm{lex}} (\mathbf{Z} \oplus \mathbf{Z})$ (Case 6)).

Assume now that P has form ii). If G is simple, then we are in Case 9) by Theorem 3.2. If G is not simple, then $B \neq \{0\}$. Let us fix some notation: we have two $\mathbf{R}$-independent functionals f_1, f_2, two free subgroups $\pi_i = \{x \in \mathbf{Z}^3 : f_i(x) = 0\}$ of $\mathbf{Z}^3$, and two planes $\phi_i = \{x \in \mathbf{R}^3 : f_i(x) = 0\}$. Let g be a nonzero vector in $\phi_1 \cap \phi_2$, and let $\phi = \{x \in \mathbf{R}^3 : g(x) = 0\}$ be the plane through the origin and perpendicular to both ϕ_1 and ϕ_2. Let $\psi : \mathbf{Z}^3 \to \phi$ be the projection map.

Claim 1. i) If g is of type 1, then $\psi(\mathbf{Z}^3) = \mathbf{Z}v_1 + \mathbf{Z}v_2$ for certain $\mathbf{R}$-independent vectors v_1, v_2 in ϕ;
ii) If g is of type 2, then $\psi(\mathbf{Z}^3) = \mathbf{Z}v_1 + \mathbf{Z}v_2 + \mathbf{Z}v_3$, where v_1, v_2, v_3 are $\mathbf{Q}$-independent, v_1, v_2 are $\mathbf{R}$-dependent, and both v_1, v_3 and v_2, v_3 are $\mathbf{R}$-independent;
iii) if g is of type 3, then $\psi(\mathbf{Z}^3)$ is dense in ϕ.

Proof. i) If g is of type 1, then $\mathbf{R}g \cap \mathbf{Z}^3 = \mathbf{Z}w_1$ for some $w_1 \in \mathbf{Z}^3$. Extend w_1 to a basis $\{w_1, w_2, w_3\}$ of $\mathbf{Z}^3$, and define $v_1 = \psi(w_2)$, $v_2 = \psi(w_3)$. Clearly, v_1 and v_2 are $\mathbf{R}$-independent.
ii) If g is of type 2, then $\phi \cap \mathbf{Z}^3 = \mathbf{Z}w$ for some $w \in \mathbf{Z}^3$. Let $\{w_1, w_2\}$ be a basis for π_w, and extend it to a basis $\{w_1, w_2, w_3\}$ for $\mathbf{Z}^3$. Then $v_i = \psi(w_i)$, for $i = 1, 2, 3$, have the required properties.
iii) This is a consequence of [4, Theorem 4.1]. ∎

Assume that g is of type 3. Then ψ is 1–1 and we may identify G with $\langle \psi(\mathbf{Z}^3), \psi(P)\rangle$. Neither π_1 nor π_2 may have rank 2 (otherwise $\psi(\mathbf{Z}^3)$ would not be dense). As $B \neq \{0\}$ and we have unperforatedness, we may assume $P \cap \phi_1 = \mathbf{Z}^+x \subseteq B$ for some $0 \neq x \in \pi_1$. Hence we have on ϕ the situation depicted by the following diagram:

$\psi(\phi_1)$

$\psi(P)$

$\psi(x)$

$\psi(\phi_2)$

$\psi(y)$

0

As $\psi(\mathbf{Z}^3)$ is dense, we can choose $\psi(y)$ in the dotted open half strip in figure. It is clear that $0 \vee \psi(y)$ does not exist and that for no point $\psi(z)$ the inequality $0, \psi(y) < \psi(z) < \psi(x)$ holds. This contradicts Proposition 1.1, and hence g cannot be of type 3.

Assume that g is of type 2. Again ψ is 1–1 and we identify G with $\langle \psi(\mathbf{Z}^3), \psi(P) \rangle$. We claim that for one of ϕ_1,ϕ_2, say ϕ_1, it is $\psi(\phi_1) = \mathbf{R}v_1 = \mathbf{R}v_2$ (v_1, v_2 as in Claim 1). Suppose this is not the case: then again we may assume $P \cap \phi_1 = \mathbf{Z}^+ x \subseteq B$ for some $0 \neq x \in \pi_1$. Hence we have the following situation on ϕ:

$\psi(\phi_1)$

$\psi(P)$

$\psi(\phi_2)$

$\psi(x)$

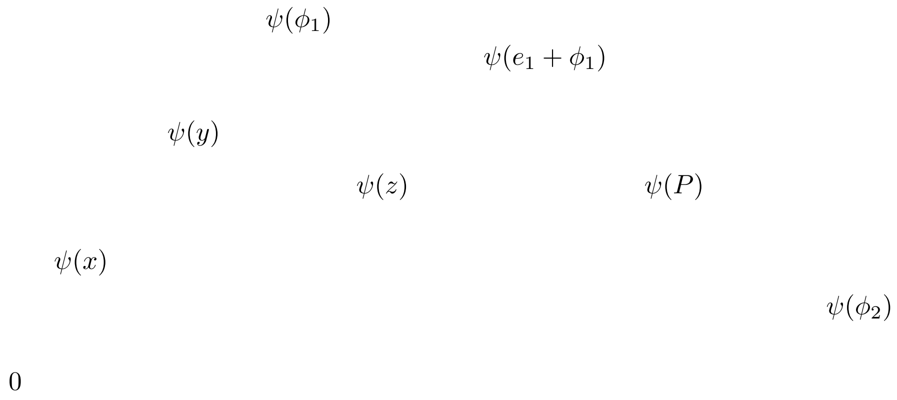
I

$\mathbf{Z}v_1 + \mathbf{Z}v_2$

$\psi(y)$ 0

Choose an open interval I in the segment $\{\alpha\psi(x) : 0 \leq \alpha \leq 1\} \subseteq \phi$. There exists $\psi(y)$ in the open half strip $(I + \psi(\phi_2)) \setminus \psi(P)$. Again, $0 \vee \psi(y)$ does not exists and for no $\psi(z)$ the inequality $0, \psi(y) < \psi(z) < \psi(x)$ holds. This is a contradiction and our claim is settled. We may assume $\psi(\phi_1) = \mathbf{R}v_1 = \mathbf{R}v_2$; this implies also that f_1 is of type 1. We may assume $f_1 = \epsilon_1$. Consider the rank 2 convex subgroup $G' = \langle \pi_1, \pi_1 \cap P \rangle$ (π_1 is now $\mathbf{Z}e_2 + \mathbf{Z}e_3$), which is unperforated and has the Riesz interpolation property. We want to prove that it is a dimension group. Assume it is not directed. Then we can find $x, y \in \pi_1$ such that x, y do not have an upper bound in π_1. We may assume $f_2(x) \leq f_2(y)$. Choose $z \in e_1 + \pi_1$ such that $f_2(y) < f_2(z)$. Then $x, y < z, z - x + y$; in fact $0 = \epsilon_1(x) = \epsilon_1(y) < \epsilon_1(z) = \epsilon_1(z - x + y) = 1$ and $f_2(x) \leq f_2(y) < f_2(z) \leq f_2(z - x + y)$. On the other hand, there is no lower bound for $z, z - x + y$ in $e_1 + \pi_1$ (if w were a lower bound for $z, z - x + y$ in $e_1 + \pi_1$, then $z + y - w$ would be an upper bound for x, y in π_1). Hence there is no interpolant for $x, y < z, z - x + y$, which is a contradiction. It follows that G' is directed and it is a dimension group over $\mathbf{Z}^2$. By Theorem 3.1, we see that either $\{x \in \pi_1 : f_2(x) > 0\} \subseteq B$, or $G' \cong \mathbf{Z} \oplus \mathbf{Z}$.

We claim that the latter can never be the case. For otherwise we could find $x \in \pi_1$ such that $f_2(x) < f_2(y)$, where y is the least upper bound of $0, x$ in G'. Projecting on ϕ we would have the following situation:



Take $\psi(z)$ as in figure: it is $0, x < y, z$, but clearly there is no interpolant. So $\{x \in \pi_1 : f_2(x) > 0\} \subseteq B$. Now consider $f_2 = (\alpha, \beta, \gamma)$. One of β, γ must be different from 0; we may assume $\gamma \neq 0$ and $f_2 = (\alpha, \beta, 1)$ for some α, β. Then $\beta \in \mathbf{R} \setminus \mathbf{Q}$ is a necessary and sufficient condition for g to be of type 2.
• If $B = \{x \in \pi_1 : f_2(x) \geq 0\}$, then we have $G = \langle \mathbf{Z}^3, \overline{\epsilon_1}, (\alpha, \beta, 1)\rangle$. It is easy to show, using Proposition 1.1, that this is a dimension group (Case 11)).
• If there exists x in $B \setminus \pi_1$, then x must be in $e_1 + \pi_1$ (otherwise we could choose $y \in (e_1 + \pi_1) \setminus P$, so that $0 \vee y$ does not exist and there is no infinite chain $x > y_1 > y_2 > \cdots > 0, y$). Hence $G \cong \mathbf{Z}[\alpha, \beta] \oplus \mathbf{Z}$ for some α, β (Case 4)).

Finally we consider the case where g is of type 1. This implies that $\pi_1 \cap \pi_2 = \mathbf{Z}w$ for some $0 \neq w \in \mathbf{Z}^3$.

Claim 2. Either w or $-w$ belongs to P, so that $\mathbf{Z}w$ is a nontrivial ideal.
Proof. Suppose $w, -w \notin P$. Then by unperforatedness $\pi_1 \cap \pi_2 \cap P = \{0\}$. As $B \neq \{0\}$, one of f_1, f_2 must be of type 1, say f_1. As before, $G' = \langle \pi_1, \pi_1 \cap P\rangle$ is a dimension group over $\mathbf{Z}^2$, hence an ℓ-group. Consider $w \vee 0$ ($\vee$ being the supremum in G'); it cannot be $w \vee 0 \in \pi_2$, otherwise we would have $w \vee 0 = 0$, which implies $-w \in P$, contradicting our assumption. Choose $a \in \mathrm{int}(P)$; as $w \vee 0 \notin \pi_2$, there exists $n \in \mathbf{Z}^+$ such that $f_2(a) < nf_2(w \vee 0) = f_2(n(w \vee 0)) = f_2(nw \vee 0)$ (the last equality follows from [1, Proposition 1.3.7]). But then $0, nw < nw \vee 0, a$, and it is clear that there is no interpolant in G, which is a contradiction. ■

As the quotient of a dimension group by an ideal is a dimension group, we see that $G/\mathbf{Z}w \cong \mathbf{Z} \oplus \mathbf{Z}$. This means that we may assume $f_1 = \epsilon_1, f_2 = \epsilon_2$, and that both π_1 and π_2 are dimension groups over $\mathbf{Z}^2$.
• If $\pi_1 \cong \mathbf{Z} \oplus \mathbf{Z}$ and $\pi_2 \cong \mathbf{Z} \oplus_{\mathrm{lex}} \mathbf{Z}$, or conversely, we have the ℓ-group $(\mathbf{Z} \oplus_{\mathrm{lex}} \mathbf{Z}) \oplus \mathbf{Z}$ (Case 7)).
• If $\pi_1 \cong \pi_2 \cong \mathbf{Z} \oplus_{\mathrm{lex}} \mathbf{Z}$, we have the dimension group $(\mathbf{Z} \oplus \mathbf{Z}) \oplus_{\mathrm{lex}} \mathbf{Z}$ (Case 12)).
• If $\pi_1 \cong \pi_2 \cong \mathbf{Z} \oplus \mathbf{Z}$, we may assume that $B = (\mathbf{Z}^+e_1 + \mathbf{Z}^+e_3) \cup (\mathbf{Z}^+e_2 + \mathbf{Z}^+e_3)$ and we do not have a dimension group; in fact $(0,0,0), (0,1,-1) < (0,1,0), (1,1,-1)$ is a counterexample to the Riesz property.

This concludes the analysis of the possible structures for G, and hence the proof of Theorem 3.3. ■

We conclude our paper with the following remarks: by [12, Theorem 2.1], every abelian ℓ-group is ultrasimplicial. The group 12) in Theorem 3.3 is clearly ultrasimplicial, and so are the groups of type 10) by the main result of [13]. We claim that the groups of type 11) are ultrasimplicial. In fact, by [11, Proposition 1], a dimension group $\langle G, G^+\rangle$ is ultrasimplicial if and only if for every $x_1, \ldots, x_n \in P$ there exist $\mathbf{Z}$-independent $y_1, \ldots, y_m \in P$ such that $\{x_1, \ldots, x_n\} \subseteq \mathbf{Z}^+y_1 + \cdots + \mathbf{Z}^+y_m$. Let $x_1, \ldots, x_n$ be positive elements of a group of type 11). Choose a positive $y_1 \in e_1 + \pi_{\epsilon_1}$, such that $0 < (\alpha, \beta, 1)(y_1) < \min\{(\alpha, \beta, 1)(x_i)/\epsilon_1(x_i) : i \in \{1, \ldots, n\}$ and $\epsilon_1(x_i) > 0\}$. For each $i \in \{1, \ldots, n\}$, let $m_{i1} = \epsilon_1(x_i)$, and define $z_i = x_i - m_{i1}y_1$. Then it is $\epsilon_1(z_i) = 0$ and $(\alpha, \beta, 1)(z_i) > 0$, so that $\{z_1, \ldots, z_n\} \subseteq P \cap \pi_{\epsilon_1}$. As $\langle \pi_{\epsilon_1}, \pi_{\epsilon_1} \cap P\rangle \cong \mathbf{Z}[\beta, 1]$, which is an ultrasimplicial group of rank 2, it follows that there exist $\mathbf{Z}$-independent $y_2, y_3 \in \pi_{\epsilon_1} \cap P$ such that, for each i, there exist $m_{i2}, m_{i3} \in \mathbf{Z}^+$ with $z_i = m_{i2}y_2 + m_{i3}y_3$. Hence, for each i, it is $x_i = m_{i1}y_1 + m_{i2}y_2 + m_{i3}y_3$, and clearly y_1, y_2, y_3 are $\mathbf{Z}$-independent.

On the other hand the groups of type 9) are not, in general, ultrasimplicial. In fact, dimension groups of that form are among Riedel's counterexamples [14] to the conjecture that every finitely generated dimension group is the limit of a unimodular sequence, and it is easy to prove that a dimension group of the form $\langle \mathbf{Z}^n, P\rangle$ is ultrasimplicial if and only if it is the limit of a unimodular sequence.

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