The logic of partially ordered abelian groups with strong unit

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ABSTRACT

The Chang-Mundici equivalence between the category of MV-algebras and the category of lattice-ordered abelian groups with strong unit allows us to translate facts/problems about Łukasiewicz many valued logics into facts/problems about partially ordered abelian groups, and conversely. After giving a brief survey of the theory, we study the automorphism groups of the free MV-algebras, i.e., the Lindenbaum algebras of Łukasiewicz logics.

1. Preliminaries

We list here some of the standard definitions and facts about partially ordered abelian groups. By a *group* we always mean an *abelian group*, except when discussing automorphism groups. For background the reader is referred to [2],[8],[9].

A *partially ordered abelian group* (p.o. group) is a pair $\langle G, G^+\rangle$, where G is an abelian group, G^+ is a subsemigroup of G such that $0 \in G^+$, $G^+ \cap (-G^+) = \{0\}$, and G is *positively generated*, i.e., every element of G can be expressed as the difference of two elements of G^+ (not necessarily in a unique way). The elements of G^+ are called *positive*, and G^+ is the *positive cone* of G. The relation $\leq$ between elements of G given by $x \leq y$ iff $y - x \in G^+$ is a translation invariant partial order; in other words, for every $x, y, t \in G$, if $x \leq y$, then $x + t \leq y + t$. Conversely, given a translation invariant partial order on G such that $G^+ = \{x \in G \colon 0 \leq x\}$ generates G, then G^+ is a positive cone.

An element $u \in G^+$ such that for every $x \in G$ there exists an $n \in \mathbf{N}$ for which $x \leq nu$ is said to be a *strong unit*. $\langle G, G^+\rangle$ is *unperforated* if, for every $x \in G$, if for some $n \in \mathbf{N} \setminus \{0\}$ nx is positive, then x itself is positive. $\langle G, G^+\rangle$ is a *Riesz group* if it has the *Riesz interpolation property*: for every $x, y, z, w \in G$ such that $x, y \leq z, w$, there exists $t \in G$ with $x, y \leq t \leq z, w$. The Riesz interpolation property is equivalent to the *Riesz decomposition property*: if $0 \leq x \leq x_1 + \cdots + x_n$, then there exist $x'_1, \ldots, x'_n$ such that $0 \leq x'_i \leq x_i$ and $x = x'_1 + \cdots + x'_n$.

A *dimension group* is an unperforated Riesz group. A *lattice-ordered group* (ℓ-group) is a p.o. group whose underlying partial order is a lattice. A *totally ordered group* is a p.o. group in which the order is total. Every ℓ-group is a dimension group. Actually, the Effros-Handelman-Shen theorem tells us that the dimension groups are exactly the inductive limits of direct systems of ℓ-groups ([9, Theorem 3.21.]). Loosely speaking, the Riesz property is what remains of the lattice property in forming direct limits.

Let P, P' be posets, P' a subposet of P. P' is *convex* in P if $x, y \in P'$, $z \in P$ and $x \leq z \leq y$ imply $z \in P'$. Let $\langle G, G^+\rangle$ be a p.o. group, and let H be a subgroup of G. Define $H^+ = H \cap G^+$, and assume that H is generated by H^+. Then $\langle H, H^+\rangle$ is a *p.o. subgroup* of $\langle G, G^+\rangle$. If, furthermore, H is convex in G, then H is an *ideal* of G (from now on, as soon as no confusion is possible, we will drop references to the positive cones). Note that a convex subgroup of a Riesz group is positively generated iff it is directed; hence an ideal of a Riesz group is a convex directed subgroup. A subsemigroup of G^+, containing 0 and convex in G^+, is called a *face* (of G^+). Let us call Ideals(G) the poset of ideals of G, ordered by inclusion, and Faces(G^+) the poset of faces of G^+, ordered by inclusion. There is an isomorphism between Ideals(G) and Faces(G^+), which is given by $H \mapsto H^+$, the inverse isomorphism being given by $H^+ \mapsto H^+ - H^+$.

The set of faces of G^+ is closed under arbitrary intersections. Let $X \subseteq G^+$; the face generated by X is $\langle X\rangle^+ = \bigcap\{f \in \mathrm{Faces}(G^+) \colon f \supseteq X\}$. It is straightforward to prove that $\langle X\rangle^+ = \{y \in G^+ \colon \exists x_1, \ldots, x_n \in$

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X such that $0 \leq y \leq x_1 + \cdots + x_n\}$. The ideal generated by X is $\langle X\rangle = \langle X\rangle^+ - \langle X\rangle^+$. If $X = \{x\}$, we just write $\langle x\rangle$. We have that $x \in G^+$ is a strong unit iff $\langle x\rangle = G$.

Assume that H is an ideal of G. We order the quotient group G/H by using as positive cone the image of the positive cone of G, i.e., we put $(G/H)^+ = G^+/H$. Then $\langle G/H, (G/H)^+\rangle$ is a p.o. group. Equivalently, define $x/H \leq y/H$ in G/H iff there exists $z \in H$ such that $x \leq y + z$ in G. If H is an ideal of the Riesz group (dimension group, ℓ-group) G, then both H and G/H are Riesz groups (dimension groups, ℓ-groups).

Let G, K be p.o. groups. Their *direct sum* $G \oplus K$ is the ordinary group direct sum, ordered componentwise: $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$. Their *lexicographic sum* $G \oplus_{\mathrm{lex}} K$ has the same support group as $G \oplus K$, and is ordered lexicographically: $(x, y) \leq (x', y')$ iff $(x < x')$ or $(x = x'$ and $y \leq y')$.

A *homomorphism* $\varphi: G \to K$ is a group homomorphism such that $\varphi(G^+) \subseteq K^+$. If $\varphi(G^+) = K^+$, then φ is an *epimorphism*, and an injective epimorphism is an *isomorphism*. Of course, an *automorphism* of G is an isomorphism of G onto itself. An *ℓ-homomorphism* of lattice-ordered groups is a homomorphism which preserves infima and suprema. If u is a distinguished strong unit of G, and v is a distinguished strong unit of K, a homomorphism $\varphi: (G, u) \to (K, v)$ is *unital* if $\varphi(u) = v$.

Here are some examples:
1) Let $C[0,1]$ be the additive group of continuous real valued functions over the unit interval $[0,1]$, with pointwise order: $x \leq y$ iff $\forall t \in [0,1]\ x(t) \leq y(t)$. Then $C[0,1]$ is an ℓ-group, having the constant function $\mathbf{1}$ as a strong unit.
2) Let $C_1[0,1]$ be the subgroup of $C[0,1]$ whose elements are the differentiable functions, with the inherited order. Then $C_1[0,1]$ is a p.o. subgroup of $C[0,1]$ which is not convex. $C_1[0,1]$ is a dimension group, but not an ℓ-group. Any positive function which is never 0 is a strong unit.
3) Same construction as in 2), but taking polynomial functions, instead of differentiable ones.
4) Let $\mathbf{X}$ be any of $\mathbf{Z}, \mathbf{Q}, \mathbf{R}$, and take G to be the additive group $\mathbf{X}^2$. Let $G^+ = \{(0,0)\} \cup \{(x, y) \in \mathbf{X}^2 : 0 < x, 0 \leq y\}$. Then $\{(x, 0) : x \in \mathbf{X}\}$ is an ideal of G, $\{(x, x) : x \in \mathbf{X}\}$ is a p.o. subgroup of G which is not convex, $\{(0, y) : y \in \mathbf{X}\}$ is a subgroup of G which is not positively generated. If $\mathbf{X}$ is $\mathbf{Q}$ or $\mathbf{R}$, then G is a dimension group, whereas if $\mathbf{X} = \mathbf{Z}$, it is not: e.g., there is no interpolant to $(1,0), (0,1) \leq (2,1), (2,2)$.
5) Let Γ be a poset, and for each $\gamma \in \Gamma$, let G_γ be a totally ordered group. For every $x \in \prod_{\gamma\in\Gamma} G_\gamma$, let the *support* of x be $\{\gamma \in \Gamma : x(\gamma) \neq 0\}$, and let G be the set of elements of $\prod_{\gamma\in\Gamma} G_\gamma$ whose support satisfies the ascending chain condition. Define a nonzero element of G to be positive iff it is positive at each maximal element of its support. Groups obtained by using this construction are called Hahn-type, and they are fairly general. In fact:
- every totally ordered group can be embedded in a Hahn-type group in which Γ is a chain and each G_γ is a subgroup of $\mathbf{R}$ (Hahn, 1907);
- every ℓ-group can be embedded in a Hahn-type group in which Γ is a *root system* (i.e., $\forall\gamma \in \Gamma$, the elements greater than γ form a chain) (Conrad, Harvey, Holland, 1963);
- if Γ is a poset with a finite numer of maximal chains, then G, as defined above, is a dimension group (Teller, [14]).

From now on we concentrate on Riesz groups. Our present goal is to develop the spectral theory for Riesz groups; in the next section we will analyze how the structure of the interval $[0, u]$ in a Riesz group with strong unit u is related to the structure of the whole group. Some words are in order to explain why we are working at this level of generality. As a matter of fact, the categorical equivalence between algebras of Łukasiewicz many valued logics and p.o. groups with strong unit works only at the level of ℓ-groups. Nevertheless, we feel that the class of Riesz groups is appropriate to introduce most of the theory. In fact:
1) the arguments involved do not require the lattice property, but just the interpolation and decomposition properties;
2) Riesz groups and dimension groups arise in functional analysis as an important tool for classifying algebras of operators. It is then possible to use them as a bridge between operator theory and many valued logics ([11]);
3) there are many open problems on the borderline between Riesz groups, dimension groups and ℓ-groups. For example: which are the minimal conditions for a Riesz group to be an ℓ-group ([7])? how much of the spectral theory for ℓ-groups can be extended to Riesz groups?
4) it would be very interesting to extend the duality between MV-algebras and ℓ-groups to the larger classes of

dimension groups and Riesz groups. The logic which seems to arise would have partially defined connectives, and a "limiting process" to approximate undefined formulas.

Let then G be a fixed Riesz group. Ideals(G) is a complete distributive Brouwerian algebraic lattice ([1],[8]). Finite meets are given by intersections, whereas if $\{I_t : t \in T\}$ is an infinite family of ideals, then $\bigwedge_{t\in T} I_t = (\bigcap_{t\in T} I_t^+) - (\bigcap_{t\in T} I_t^+)$. Joins, either finite or infinite, are given by direct sums: $\bigvee_{t\in T} I_t = \sum_{t\in T} I_t =$ subgroup generated by $\bigcup_{t\in T} I_t$. Observe also that meets distribute over infinite suprema (but not conversely). Ideals generated by singletons are compact, and every ideal is clearly the supremum of such ones.

A proper ideal p of G is said to be *prime* if it is meet irreducible in Ideals(G), i.e., $p = I \cap J$ implies $p = I$ or $p = J$. As in any distributive lattice, this is equivalent to saying that $p \supseteq I \cap J$ implies $p \supseteq I$ or $p \supseteq J$.

Proposition 1.1. *Let G be a Riesz group, $I \in$ Ideals(G). Then $I = \bigwedge\{p \in \mathrm{Spec}(G) : p \supseteq I\}$.*

Proof. It suffices to prove that $I^+ = \bigcap\{p^+ : p \in \mathrm{Spec}(G)$ and $p^+ \supseteq I^+\}$. Assume that $x \in G^+ \setminus I^+$, and observe that the union of a chain of faces is a face. Then, by Zorn lemma, we can find a face f which is maximal with respect to the property of containing I^+ and excluding x, and such an f is clearly meet irreducible. Hence the prime ideal $p = f - f$ satisfies the conditions $p^+ = f \supseteq I^+$ and $x \notin p^+$. ■

The set of prime ideals of G, equipped with the Jacobson-Zariski topology, is a topological space called the *spectrum* of G, and denoted by Spec(G). The topology can be conveniently described by describing the closure operator: given $P \subseteq \mathrm{Spec}(G)$, consider the ideal $\bigwedge P$ (the *kernel* of P), and then define $\{p \in \mathrm{Spec}(G) : p \supseteq \bigwedge P\}$ (the *hull* of $\bigwedge P$) to be the closure of P.

The membership relation between elements of G^+ and elements of Spec(G) induces, as always, a Galois connection. Let $X \subseteq G^+$, $P \subseteq \mathrm{Spec}(G)$. Let $X' = F_X = \{p \in \mathrm{Spec}(G) : \forall x \in X \quad x \in p\} = \{p \in \mathrm{Spec}(G) : p \supseteq X\}$, and let $P' = \{x \in G^+ : \forall p \in P \quad x \in p\} = \bigcap\{p^+ : p \in P\} \in \mathrm{Faces}(G^+)$. Then $X \mapsto X''$ and $P \mapsto P''$ are both Moore closure operators. It is clear that X'' is exactly the face generated by X, whereas P'' is the hull-kernel closure of P. It also follows:

- the complete lattice of faces of G^+ (or, equivalently, of ideals of G) is antiisomorphic to the complete lattice of closed sets of Spec(G), and isomorphic to the complete lattice of open sets of Spec(G). The antiisomorphism is given by $I \mapsto F_I = \{p \in \mathrm{Spec}(G) : p \supseteq I\}$, and the isomorphism by $I \mapsto O_I = \{p \in \mathrm{Spec}(G) : p \not\supseteq I\}$ ($I \in$ Ideals(G)). Hence, every closed (open) set of Spec(G) is of the form F_I (O_I) for some $I \in$ Ideals(G);
- there is a 1-1 correspondence between clopen sets of Spec(G) and direct sum decompositions of G. In fact, assume $P \subseteq \mathrm{Spec}(G)$ is clopen. Then P' and $(\mathrm{Spec}(G) \setminus P)'$ have $\{0\}$ as their meet, and their direct sum (i.e., their supremum) is G^+. Let $I = P' - P'$, $J = (\mathrm{Spec}(G) \setminus P)' - (\mathrm{Spec}(G) \setminus P)'$ be the ideals they generate. Then G is isomorphic to $I \oplus J$ as a group and, by [8, Proposition 5.8.], it is also isomorphic to $I \oplus J$ as a p.o. group.

Let $X \subseteq G^+$. As in any Galois connection, $X' = X'''$. Hence $F_X = F_{\langle X\rangle}$ and $O_X = O_{\langle X\rangle}$. In particular, we are interested in the open sets of the form O_x, for $x \in G^+$.

Proposition 1.2. *Let G be a Riesz group. The compact open sets of* Spec(G) *are exactly those of the form O_x, for $x \in G^+$. They form a basis for* Spec(G).

Proof. Choose any open set of Spec(G); it will be of the form O_I for some $I \in$ Ideals(G). Clearly $I = \bigvee_{x\in I^+}\langle x\rangle$. By the above mentioned isomorphism it is $O_I = O_{\bigvee_{x\in I^+}\langle x\rangle} = \bigvee_{x\in I^+} O_{\langle x\rangle} = \bigcup_{x\in I^+} O_x$. Let $O_x \subseteq \bigcup_{t\in T} O_{I_t}$: then $\langle x\rangle \subseteq \bigvee_{t\in T} I_t = \sum_{t\in T} I_t$. Hence there exist $t_1, \ldots, t_n \in T$ and $x_{t_1} \in I_{t_1}, \ldots, x_{t_n} \in I_{t_n}$ such that $x = x_{t_1} + \cdots + x_{t_n}$. Hence $\langle x\rangle \subseteq I_{t_1} + \cdots + I_{t_n}$, and $O_x \subseteq O_{I_{t_1}} \cup \cdots \cup O_{I_{t_n}}$. Conversely, assume that O_I is compact. As $O_I = \bigcup_{x\in I^+} O_x$, there exist $x_1, \ldots, x_n \in I^+$ such that $O_I = O_{x_1} \cup \cdots \cup O_{x_n} = O_{x_1+\cdots+x_n}$ (this last equality follows at once by the straightforward equality $\langle x_1\rangle + \cdots + \langle x_n\rangle = \langle x_1 + \cdots + x_n\rangle$). ■

Corollary 1.3. Spec(G) *is compact if and only if G has a strong unit.*

Proof. If G has a strong unit u, then $\mathrm{Spec}(G) = O_u$, which is compact. If Spec(G) is compact, then it must be $\mathrm{Spec}(G) = O_x$, for some $x \in G^+$. Hence $\forall p \in \mathrm{Spec}(G)$ $x \notin p$. But then x is a strong unit. In fact, if $\langle x\rangle$ were different from G, then by Proposition 1.1. there would exist a prime ideal of G containing x. ■

In a topological space a closed set is called an *irreducible* if it cannot be expressed as a non trivial union of two closed sets. Every closure of a point is clearly an irreducible set. A topological space is *sober* if every irreducible set is the closure of a point.

Proposition 1.4. *Let G be a Riesz group. Then* $\mathrm{Spec}(G)$ *is*
i) T_0
ii) sober
iii) has a basis of compact open sets (and hence is locally compact).

Proof. The third statement is proved in Proposition 1.2.. Let $p, q \in \mathrm{Spec}(G)$, $p \neq q$. By Proposition 1.1. $\{r \in \mathrm{Spec}(G) \colon r \supseteq p\}$ (the closure of p) is different from $\{r \in \mathrm{Spec}(G) \colon r \supseteq q\}$ (the closure of q). Hence $\mathrm{Spec}(G)$ is T_0. Assume that F_I is an irreducible set. This amounts to saying that F_I cannot be expressed in a nontrivial way as the union (i.e., the join in the lattice of closed sets of $\mathrm{Spec}(G)$) of two sets of the form $F_J, F_{J'}$. By the above mentioned antiisomorphism, this means that I cannot be expressed in a nontrivial way as the meet of two ideals J, J'. It follows that I is prime, hence a point of $\mathrm{Spec}(G)$, and F_I is its closure. ■

A topological space which is T_0, sober, and with a countable basis of compact open sets is called a *spectral space*. By [4, Theorem 5] any spectral space is homeomorphic to the spectrum of a dimension group (not necessarily unique). The problem of characterizing, up to homeomorphism, the spectra of ℓ-groups is still open (see [7]).

2. The unit interval

Let (G, u) be a Riesz group with a distinguished strong unit u. We are interested in the *unit interval* $[0, u] = \{x \in G \colon 0 \le x \le u\}$ of (G, u). The structure of the unit interval is interesting because:
i) it tells us "almost everything" about the structure of G;
ii) it has a logical interpretation.

Let us consider i). First of all, the unit interval generates G as a group; this follows from the Riesz decomposition property and the definition of strong unit. Let $X \subseteq [0, u]$. We call X a *face* of $[0, u]$ if $0 \in X$, X is convex in $[0, u]$ and is conditionally closed under sum (i.e., $x, y \in X$ and $x + y \le u$ imply $x + y \in X$). Let us denote the poset of faces of $[0, u]$ by $\mathrm{Faces}[0, u]$, ordered by inclusion. As arbitrary intersections of faces of $[0, u]$ are faces of $[0, u]$, it follows that $\mathrm{Faces}[0, u]$ is a complete lattice.

Proposition 2.1. $\mathrm{Faces}[0, u]$ *is isomorphic to* $\mathrm{Faces}(G^+)$ *(and hence to* $\mathrm{Ideals}(G)$*).*

Proof. For $X \in \mathrm{Faces}[0, u]$, let $a(X)$ be the face of G^+ generated by X. For $f \in \mathrm{Faces}(G^+)$ let $b(f) = f \cap [0, u]$. We must verify that $b(f) \in \mathrm{Faces}[0, u]$, that both $a \colon \mathrm{Faces}[0, u] \to \mathrm{Faces}(G^+)$ and $b \colon \mathrm{Faces}(G^+) \to \mathrm{Faces}[0, u]$ are order preserving, that ab is the identity on $\mathrm{Faces}(G^+)$, and that ba is the identity on $\mathrm{Faces}[0, u]$. All these verifications are straightforward. ■

By the above Proposition, meet irreducible elements of $\mathrm{Ideals}(G)$ correspond to meet irreducible elements of $\mathrm{Faces}[0, u]$, and the same holds for compact elements. For $X \subseteq [0, u]$, define the face of $[0, u]$ generated by X to be $\langle X \rangle^+ \cap [0, u]$; *principal faces* (either of G^+ or of $[0, u]$) are those generated by singletons.

Proposition 2.2. *The principal faces of G^+ are those generated by the elements of the unit interval. It follows that principal faces of $[0, u]$ correspond to principal faces of G^+ (and hence to principal ideals of G).*

Proof. Let $x \in G^+$. By the Riesz decomposition property there exist $x_1, \ldots, x_n \in [0, u]$ such that $x = x_1 + \cdots + x_n$. By the interpolation property there exists $y \in [0, u]$ such that $x_1, \ldots, x_n \le y \le x, u$, and clearly $\langle x \rangle^+ = \langle y \rangle^+$. ■

A Riesz group (or a poset) is an *antilattice* ([8]) if it has only the infima that it cannot fail to have. More formally, G is an antilattice if, for any $x, y \in G$, the existence of $x \wedge y$ implies that $x \le y$ or $y \le x$.

Proposition 2.3. *Let H be an ideal of the Riesz group G. The following conditions are equivalent:*
i) H is prime;
ii) if $x, y \in G^+$ and $[0, x] \cap [0, y] \subseteq H$, then $x \in H$ or $y \in H$;

iii) G/H is an antilattice;
iv) $G^+ \setminus H^+$ is lower directed.

Proof. The equivalence of ii) and iv) is clear.
i)$\Rightarrow$ii) Let $x, y \in G^+$. We claim that $\langle x\rangle^+ \cap \langle y\rangle^+ = \langle [0,x] \cap [0,y]\rangle^+$. The right to left inclusion being trivial, assume $z \in \langle x\rangle^+ \cap \langle y\rangle^+$. Then $0 \leq z \leq nx$ for some $n \in \mathbf{N}$. By the Riesz decomposition property there exist $x_1, \ldots, x_n \in [0,x]$ such that $z = x_1 + \cdots + x_n$. Analogously $z = y_1 + \cdots + y_k$ for some $y_1, \ldots, y_k \in [0,y]$. By [9, Proposition 2.2.(c)] there exist $z_1, \ldots, z_r \in [0,x] \cap [0,y]$ such that $z = z_1 + \cdots + z_r$. Our claim is proved; it follows that $\langle x\rangle \wedge \langle y\rangle = \langle [0,x] \cap [0,y]\rangle$. Assume that $[0,x] \cap [0,y] \subseteq H$; then $\langle x\rangle \wedge \langle y\rangle = \langle [0,x] \cap [0,y]\rangle \subseteq H$. Since H is prime, it follows that $\langle x\rangle \subseteq H$ or $\langle y\rangle \subseteq H$, and so $x \in H$ or $y \in H$.
ii)$\Rightarrow$iii) Assume $x/H \wedge y/H$ exists. By translation invariance we may assume $x/H \wedge y/H = 0/H$, and by definition of quotient ordering we may also assume $x, y \geq 0$. We must have $[0,x] \cap [0,y] \subseteq H$ for, otherwise, if $z \in (G^+ \setminus H) \cap [0,x] \cap [0,y]$, we would have $0/H < z/H \leq x/H, y/H$. Hence $x \in H$ or $y \in H$, and so $x/H = 0/H$ or $y/H = 0/H$.
iii)$\Rightarrow$i) Assume H is not prime. Then there exist $I, J \in \mathrm{Ideals}(G)$ and $x, y \in G^+$ such that $H = I \cap J$, $x \in I \setminus H$ and $y \in J \setminus H$. Hence $x/H, y/H > 0/H$. We claim that $x/H \wedge y/H = 0/H$. In fact, assume that there exists $z \in G^+$ such that $x/H, y/H \geq z/H > 0/H$. Then $0/I = x/I \geq z/I \geq 0/I$ and $z \in I$. Analogously $z \in J$. Hence $z \in H = I \cap J$. Contradiction. ∎

Note that, in contrast with the ℓ-group case, the fact that the ideals greater than H form a chain is a sufficient but not necessary condition for H to be prime. As a counterexample, consider the prime ideal $\{0\}$ in the antilattice $(\mathbf{Z} \oplus \mathbf{Z}) \oplus_{\mathrm{lex}} \mathbf{Z}$ ([13]), whose lattice of ideals is

We have already seen how the lattice of ideals of a Riesz group with strong unit is faithfully reflected in the lattice of faces of the unit interval. The order relation of the unit interval also determines the order relation of the whole group.

Theorem 2.4. *Let (G, u) be a Riesz group with strong unit u. Then each of the following properties holds in the poset $[0,u]$ if and only if it holds in G:*
i) each bounded countable chain has a supremum;
ii) ω-interpolation (i.e., if $\{x_i\}, \{y_j\}$ are countable families of elements such that, for each i, j, $x_i \leq y_j$, then there exists z such that, for each i, j, $x_i \leq z \leq y_j$);
iii) to be lattice-ordered;
iv) to be antilattice-ordered;
v) to be totally ordered.

Proof. The statements corresponding to properties i) and ii) are proved in [9, Proposition 16.9.] and [9, Proposition 16.3.], respectively. The statement corresponding to property iii) is proved in [12, Lemma 3.1.]. About iv): if G is an antilattice, then $[0,u]$ is clearly an antilattice. Assume that $[0,u]$ is an antilattice and that, for certain $x, y \in G$, $x \wedge y$ exists. By translation invariance we may assume $x \wedge y = 0$, so that x, y are positive. By the Riesz decomposition property there exist $x_1, \ldots, x_n, y_1, \ldots, y_m \in [0,u]$ such that $x = x_1 + \cdots + x_n$ and $y = y_1 + \cdots + y_m$. We claim that, for every i, j, $x_i \wedge y_j$ exists and has value 0. In fact, $0 \leq x_i, y_j$. If $z \leq x_i, y_j$, then $z \leq x, y$, and hence $z \leq 0$. Our claim is settled. We need only to show that one of x, y equals 0. If $x_1 = x_2 = \cdots = x_n = 0$, then $x = 0$. If for some i $x_i > 0$ then, as $[0,u]$ is an antilattice, it must be $y_1 = y_2 = \cdots = y_m = 0$, so that $y = 0$. For v), simply observe that total orders are posets which are lattices and antilattices at the same time. ∎

Now we turn to the logical interpretation of the unit interval: a *Many Valued algebra* (MV-algebra) is an algebra $\langle A, \oplus, \cdot, ^*, 0, 1\rangle$ such that $\langle A, \oplus, 0\rangle$ is an abelian monoid, $x \oplus 1 = 1$, $x^{**} = x$, $0^* = 1$, $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$, $x \cdot y = (x^* \oplus y^*)^*$. Setting $x \vee y = (x^* \oplus y)^* \oplus y$, $x \wedge y = (x^* \cdot y)^* \cdot y$, $x \to y = x^* \oplus y$, we have that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice. MV-algebras are to Łukasiewicz many valued logics what Boolean algebras are to classical (2-valued) logic; see [5],[6] for a completeness theorem and further references.

The prototypical example of an MV-algebra is the unit real interval $[0, 1]$, equipped with truncated addition $x \oplus y = (x + y) \wedge 1$, and with the other operations defined by $x^* = 1 - x$, $x \cdot y = (x^* \oplus y^*)^*(= (x + y - 1) \vee 0)$. The unit real interval is generic (but not free) in the variety of MV-algebras; equivalently, a propositional sentence is a theorem in Łukasiewicz infinite valued logic iff it receives value 1 under any interpretation in the unit real interval. The construction that gives rise to the unit real interval (as an MV-algebra) from the ℓ-group with strong unit $(\mathbf{R}, \mathbf{1})$ is generic in another sense: *every* MV-algebra is obtained in such a way.

Theorem 2.5. [11] *Let (G, u) be an ℓ-group with strong unit. Equip the unit interval $[0, u]$ with the operations $x \oplus y = (x + y) \wedge u$, $x^* = u - x$, $x \cdot y = (x^* \oplus y^*)^*$, $1 = u$. Then $\Gamma(G, u) = \langle [0, u], \oplus, \cdot, ^*, 0, 1\rangle$ is an MV-algebra, whose lattice order, as induced by the MV operations, coincides with the order inherited by (G, u). If $\varphi\colon (G, u) \to (K, v)$ is a unital ℓ-homomorphism, then the restriction of φ to the unit interval, denoted by $\Gamma(\varphi)$, is an MV-algebra homomorphism $\Gamma(\varphi)\colon \Gamma(G, u) \to \Gamma(K, v)$. Γ is a full, faithful and representative functor (i.e., a categorical equivalence) between the category of ℓ-groups with strong unit and unital ℓ-homomorphisms and the category of MV-algebras and MV-algebra homomorphisms; in particular, for every MV-algebra A there exists a unique ℓ-group (G, u) such that $A \simeq \Gamma(G, u)$.*

Definition 2.6. Let A be an MV-algebra. An ideal I of A is a subset of A containing 0, closed under $\oplus$, and such that $x \leq y$ and $y \in I$ implies $x \in I$.

Proposition 2.7. *Let (G, u) be an ℓ-group with strong unit. The complete lattice of ideals of $\Gamma(G, u)$ is isomorphic to* $\mathrm{Ideals}(G, u)$.

Proof. In the light of Proposition 2.1., it suffices to show that a subset $I \subseteq [0, u]$ is an ideal of $\Gamma(G, u)$ iff it is a face of $[0, u]$. Assume I is an ideal of $\Gamma(G, u)$, $x, y \in I$, $x + y \leq u$. Then $x + y = (x + y) \wedge u = x \oplus y \in I$. Assume I is a face of $[0, u]$, and let $x, y \in I$. Then $0 \leq x \oplus y = (x + y) \wedge u \leq x + y$. By the Riesz decomposition property there exist x', y' such that $0 \leq x' \leq x$, $0 \leq y' \leq y$, $x' + y' = x \oplus y \leq u$. Hence $x', y' \in I$ and $x \oplus y \in I$. The other conditions (containing 0 and closure downwards) are obvious in both directions. ■

If I is an ideal of the MV-algebra A, the canonical epimorphism $A \to A/I$ is defined as usual, by using the congruence relation $\sim_I$ defined by $x \sim_I y$ iff $((x \to y) \cdot (y \to x))^* = (x^* \cdot y) \oplus (x \cdot y^*) \in I$. Let us recall that in any ℓ-group G the *absolute value* of $x \in G$ is defined to be $|x| = x \vee (-x) = (x \vee 0) - (x \wedge 0) \geq 0$ ([2, 1.3.13.]); for any ideal J of G, $x \in J$ iff $|x| \in J$. Assume $A = \Gamma(G, u)$, I is an ideal of A, J is the ideal of (G, u) corresponding to I (i.e., $J = \langle I\rangle$). For $x, y \in A$, a bit of computation shows that $(x^* \cdot y) \oplus (x \cdot y^*) = |x - y| \wedge u$. By [2, 1.3.12.], $|x - y| = (x \vee y) - (x \wedge y)$; hence, $0 \leq |x - y| \leq u$. It follows that $x \sim_I y$ in A iff $|x - y| \in I$ iff $x - y \in J$ iff $x/J = y/J$ in $(G/J, u/J)$.

Proposition 2.8. *Take A, (G, u), I, J as above. Then $A/I \simeq \Gamma(G/J, u/J)$.*

Proof. Define $\varphi\colon A/I \to \Gamma(G/J, u/J)$ by $\varphi(x/I) = x/J$. By the preceding remarks φ is well defined and 1–1. Let $y/J \in \Gamma(G/J, u/J)$. By [8, Proposition 5.7.] there exists $x \in [0, u]$ such that $y/J = x/J$ Hence $y/J = \varphi(x/I)$ and φ is onto. Now we just need to prove that φ preserves $\oplus$ and *, and this is straightforward.■

3. Free objects, maximal ideals, and automorphism groups

The results in the previous section allow us to move freely between the category of MV-algebras and their homomorphisms and the category of ℓ-groups with strong unit and unital ℓ-homomorphisms. In particular, there is a correspondence between ℓ-groups with strong unit and theories in Łukasiewicz many valued logics. Such a correspondence is established by making use of free objects, and the rest of this paper is devoted to an analysis of such objects and their automorphism groups.

The free MV-algebras are, of course, the Lindenbaum algebras of Łukasiewicz logics. By Theorem 2.5. they are in 1–1 correspondence with free objects in the category of ℓ-groups with strong unit (note that the latter is not an equational class, as it is not closed under infinite products; indeed it is not even an elementary class). Free MV-algebras have been characterized by McNaughton ([10]). Let $n \in \mathbf{N} \setminus \{0\}$. A *McNaughton function* over $[0,u]^n$ is a $[0,1]$-valued continuous piecewise linear function with integral coefficients. Here by piecewise linear function with integral coefficients we mean a function x of domain $[0,1]^n$ for which there exist linear affine polynomials $x_1, \ldots, x_n$ (each x_j of the form $c_{1j}h_1 + \cdots + c_{nj}h_n + c_{(n+1)j}$, with $c_{1j}, \ldots, c_{(n+1)j}$ integers) such that for every $h \in [0,1]^n$ there exists j for which $x(h) = x_j(h)$. A McNaughton function over $[0,1]^\omega$ is a function $y\colon [0,1]^\omega \to [0,1]$ for which there exist $0 < m_1 < \cdots < m_n < \omega$ such that $y(h) = x(h_{m_1}, \ldots, h_{m_n})$ for some McNaughton function x over $[0,1]^n$. For $1 \leq \kappa \leq \omega$, McNaughton's representation theorem ([10, Theorem 2]) states that the free MV-algebra over κ generators is the subalgebra A_κ of $[0,1]^{([0,1]^\kappa)}$ whose elements are the McNaughton functions over $[0,1]^\kappa$. The free generators of A_κ are the projections $\{p_{i+1}\}_{i<\kappa}$, i.e., the functions $p_i(h) = h_i$.

As a consequence of Theorem 2.5., the ℓ-group with strong unit $(M_\kappa, \mathbf{1})$ of real-valued continuous piecewise linear functions with integral coefficients defined over $[0,1]^\kappa$ exibits freeness properties (we denote by $\mathbf{1}$ the function that maps every point to 1). In other words, we have $A_\kappa = \Gamma(M_\kappa, \mathbf{1})$. Note that M_κ is not the free ℓ-group over κ generators ([1, Theorem 6.3.]). We define $M_0 = \mathbf{Z}$, $A_0 =$ the two element boolean algebra. $\mathrm{Aut}(M_\kappa)$ is the automorphism group of $(M_\kappa, \mathbf{1})$; we always require that $\mathbf{1}$ stays fixed under any automorphism, whence $\mathrm{Aut}(M_\kappa)$ may be identified with the automorphism group of A_κ.

Let G be a Riesz group. An ideal m of G is *maximal* if it is proper and is not properly contained in any proper ideal. Of course, every maximal ideal is prime. We denote by $\mathrm{Maxspec}(G)$ the space of maximal ideals of G, endowed with the topology induced by $\mathrm{Spec}(G)$.

Proposition 3.1. [11, Proposition 4.17.] *Let $1 \leq \kappa \leq \omega$, let $U \subseteq [0,1]^\kappa$ be open, and let $h \in U$. Then there exists $x \in M_\kappa^+$ such that $x(h) = 0$ and $x = \mathbf{1}$ over $[0,1]^\kappa \setminus U$.*

Proposition 3.2. [11, Proposition 8.1.] *Let $1 \leq \kappa \leq \omega$, $h \in [0,1]^\kappa$, $m_h = \{x \in M_\kappa : x(h) = 0\}$. Then $m_h \in \mathrm{Maxspec}(M_\kappa)$. The mapping $h \mapsto m_h$ is a homeomorphism of $[0,1]^\kappa$ onto $\mathrm{Maxspec}(M_\kappa)$.*

Let $m \in \mathrm{Maxspec}(M_\kappa)$. By Hölder's theorem ([2, Corollaire 2.6.7.]) M_κ/m is isomorphic to a subgroup of $\mathbf{R}$; if we require that $\mathbf{1}/m$ is mapped to 1, then the embedding $\varphi_m\colon M_\kappa/m \to \mathbf{R}$ is uniquely determined ([2, Lemme 13.2.2.]).

Proposition 3.3. *Adopt the above notation, and fix $x \in M_\kappa$. Then, for any $h \in [0,1]^\kappa$, we have $\varphi_{m_h}(x/m_h) = x(h)$.*

Proof. Define $\sigma\colon [0,1]^\kappa \to \mathbf{R}$ by $\sigma(h) = \varphi_{m_h}(x/m_h)$. The function $x\colon [0,1]^\kappa \to \mathbf{R}$ is continuous by definition of M_κ. It will be sufficient to prove that:
i) σ is continuous;
ii) $x = \sigma$ on a dense subset of $[0,1]^\kappa$.
We first prove ii): let $a/b \in \mathbf{Q}$, $a \in \mathbf{Z}$, $b \in \mathbf{N} \setminus \{0\}$, $h \in [0,1]^\kappa$. We have the following chain of equivalences: $x(h) = a/b$ iff $bx(h) = a = a\mathbf{1}(h)$ iff $(bx - a\mathbf{1})(h) = 0$ iff $bx - a\mathbf{1} \in m_h$ iff $(bx - a\mathbf{1})/m_h = 0/m_h$ iff $\varphi_{m_h}(bx/m_h) = \varphi_{m_h}(a\mathbf{1}/m_h) = a\varphi_{m_h}(\mathbf{1}/m_h) = a$ iff $\sigma(h) = a/b$. Let $n \in \mathbf{N} \setminus \{0\}$ be such that x does not depend on $h_n, h_{n+1} \ldots$. Consider $Q = \{h \in [0,1]^\kappa : h_1, \ldots, h_{n-1} \in \mathbf{Q}\}$ (if $n = 1$, take $Q = [0,1]^\kappa$). Then Q is a dense subset of $[0,1]^\kappa$ and is a subset of $\{h \in [0,1]^\kappa : x(h) \in \mathbf{Q}\}$. Hence by the above chain of equivalences $x = \sigma$ over Q.
About i): we need the following

Claim. *Let $x \in M_\kappa$, $h \in [0,1]^\kappa$. Then $x(h) \geq 0$ iff there exists $y \in m_h$ such that $x + y \geq 0$ in M_κ.*

Proof of Claim. The right to left direction is trivial. If $x(h) = 0$, then $x \in m_h$. As m_h is directed, there is $z \in m_h$ such that $x, 0 \leq z$. Take $y = z - x$. Assume $x(h) > 0$, and let U be the open set

$\{k \in [0,1]^\kappa : x(k) > x(h)/2\}$. Let $M \in \mathbf{N}$ be greater than $\max\{|x(k)| : k \in [0,1]^\kappa \setminus U\}$. By Proposition 3.1. there exists $z \in M_\kappa^+$ such that $z \in m_h$ and has value 1 on $[0,1]^\kappa \setminus U$. Now we can take $y = Mz$. ∎

We prove that σ is continuous by showing that, for any $a \in \mathbf{Z}$, $b \in \mathbf{N}\setminus\{0\}$, it is $\sigma^{-1}([a/b, \infty)) = x^{-1}([a/b, \infty))$ and $\sigma^{-1}((-\infty, a/b]) = x^{-1}((-\infty, a/b])$. In fact we have

$$\begin{aligned} \sigma(h) = \varphi_{m_h}(x/m_h) \geq a/b \qquad & \text{iff} \\ \varphi_{m_h}(bx/m_h) \geq a = \varphi_{m_h}(a\mathbf{1}/m_h) \qquad & \text{iff} \\ bx/m_h \geq a\mathbf{1}/m_h \qquad & \text{iff} \\ \text{there exists } y \in m_h \text{ such that } bx - a\mathbf{1} + y \geq 0 \text{ in } M_\kappa \qquad & \text{iff (by the Claim)} \\ (bx - a\mathbf{1})(h) \geq 0 \qquad & \text{iff} \\ x(h) \geq a/b \qquad & \end{aligned}$$

We argue analogously (modifying the Claim in the obvious way) to prove that $\sigma(h) \leq a/b$ iff $x(h) \leq a/b$. ∎

Proposition 3.4. *The inverse of the homeomorphism $h \mapsto m_h$ is the map $\lambda : \mathrm{Maxspec}(M_\kappa) \to [0,1]^\kappa$ defined by $(\lambda(m))_i = \varphi_m(p_i/m)$ (p_i being the i-th projection, i.e., the i-th free generator).*

Proof. For any i, we have $(\lambda(m_h))_i = \varphi_{m_h}(p_i/m_h) = p_i(h) = h_i$; hence $\lambda(m_h) = h$. On the other hand, fix $m \in \mathrm{Maxspec}(M_\kappa)$. It is then $m = m_h$ for a unique $h \in [0,1]^\kappa$. Hence $m = m_h = m_{\lambda(m_h)} = m_{\lambda(m)}$. ∎

We say that m is *localized* in $\lambda(m)$. In the light of the above results, we will not hesitate to identify an ideal $m \in \mathrm{Maxspec}(M_\kappa)$ with the point $h \in [0,1]^\kappa$ such that $m = m_h$. Moreover, for $x \in M_\kappa$, we will not distinguish between $x(m)$ (the value of x at m) and x/m (the coset of x in M_κ/m).

Let α be an automorphism of M_κ. It is clear that the map $\widetilde{\alpha} : \mathrm{Maxspec}(M_\kappa) \to \mathrm{Maxspec}(M_\kappa)$ defined by $\widetilde{\alpha}(m) = \alpha[m] = \{\alpha(x) : x \in m\}$ is a homeomorphism. We claim that

$$p_i/\widetilde{\alpha}(m) = \alpha^{-1}(p_i)/m$$

In fact, assume $p_i/\widetilde{\alpha}(m) = p_i/\alpha[m] = a/b \in \mathbf{Q}$. Then

$$\begin{gathered} bp_i/\widetilde{\alpha}(m) = a = a\mathbf{1}/\widetilde{\alpha}(m) \\ (bp_i - a\mathbf{1})/\widetilde{\alpha}(m) = 0 \\ bp_i - a\mathbf{1} \in \widetilde{\alpha}(m) = \alpha[m] \\ b\alpha^{-1}(p_i) - a\mathbf{1} \in m \\ b\alpha^{-1}(p_i)/m = a\mathbf{1}/m = a \\ \alpha^{-1}(p_i)/m = a/b \end{gathered}$$

A continuity argument as in Proposition 3.3. shows now that $p_i/\widetilde{\alpha}(m) = \alpha^{-1}(p_i)/m$.

We have then $(\lambda(\widetilde{\alpha}(m)))_i = p_i/\widetilde{\alpha}(m) = \alpha^{-1}(p_i)/m = (\alpha^{-1}(p_i))(m)$. By Proposition 3.2. we consider $\widetilde{\alpha}$ to be a homeomorphism of $[0,1]^\kappa$ onto itself; the last equality tells us that, for any $h \in [0,1]^\kappa$, it is $(\widetilde{\alpha}(h))_i = (\alpha^{-1}(p_i))(h)$; equivalently, $p_i\widetilde{\alpha} = \alpha^{-1}(p_i)$.

Definition 3.5. Let $n \in \mathbf{N} \setminus \{0\}$. A *piece* of $[0,1]^n$ is a finite union of closed convex n-dimensional polyhedra any two of which are either disjoint or intersect in a common face. A *tessellation* of $[0,1]^n$ is a finite set of pieces $\{T_1, \ldots, T_k\}$ such that $[0,1]^n = T_1 \cup \cdots \cup T_k$ and for each i, j, $T_i \cap T_j$ does not contain a ball of affine dimension n.

Definition 3.6. Let $n \in \mathbf{N} \setminus \{0\}$. A *McNaughton homeomorphism* $a : [0,1]^n \to [0,1]^n$ is a homeomorphism such that there exist a tessellation $\{T_1, \ldots, T_k\}$ of $[0,1]^n$ and square matrices $A_1, \ldots, A_k \in \mathrm{Mat}_{n+1}(\mathbf{Z})$ such that:

i) each A_j has its last column of the form

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

ii) each A_j is *unimodular*, i.e., has determinant of absolute value 1. Moreover, $\det(A_1) = \cdots = \det(A_k) = +1$ or $\det(A_1) = \cdots = \det(A_k) = -1$;
iii) for each j, $a = A_j$ over T_j (the map A_j is defined as follows: for $h = (h_1, \ldots, h_n) \in T_j$, let $(h_1, \ldots, h_n, 1)A_j = (k_1, \ldots, k_n, 1)$, and define $A_j(h) = (k_1, \ldots, k_n)$).

Our terminology will be justified after the proof of Theorem 3.9., from which it will follow that a is a McNaughton homeomorphism of $[0,1]^n$ if and only if there exist $2n$ McNaughton functions $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that $a(h) = (x_1(h), \ldots, x_n(h))$ and $a^{-1}(h) = (y_1(h), \ldots, y_n(h))$. Assume that $a, b\colon [0,1]^n \to [0,1]^n$ are McNaughton homeomorphisms. It is easy to see, by refining tessellations, that the homeomorphism ab, defined by $(ab)(h) = a(b(h))$, is a McNaughton homeomorphism. It is also easy to see that the inverse of a McNaughton homeomorphism is a McNaughton homeomorphism. We denote by $\mathrm{Hom}([0,1]^n)$ the (not necessarily abelian) group of McNaughton homeomorphisms over $[0,1]^n$.

Let $a\colon [0,1]^n \to [0,1]^n$ be a McNaughton homeomorphism. For $i \in \{1, \ldots, n\}$, the function $p_i a^{-1}$ is a McNaughton function with range in $[0,1]$, i.e., an element of the free MV-algebra over n generators A_n.

Definition 3.7. Let $a\colon [0,1]^n \to [0,1]^n$ be a McNaughton homeomorphism. We denote by $\overline{a}$ the MV-algebra homomorphism $\overline{a}\colon A_n \to A_n$ defined by $\overline{a}(p_i) = p_i a^{-1}$. By abuse of notation, we also denote by $\overline{a}$ the associated unital ℓ-homomorphism $\overline{a}\colon M_n \to M_n$, which is defined in the same way.

Lemma 3.8. *Let $a, b \in \mathrm{Hom}([0,1]^n)$. Then $\overline{a}, \overline{b}$ are automorphisms of A_n (and hence automorphisms of M_n), $\overline{1_{[0,1]^n}} = 1_{A_n}$ and $\overline{ab} = \overline{a}\overline{b}$.*

Proof. Is clear that $\overline{1_{[0,1]^n}} = 1_{A_n}$. It will be sufficient to prove that $\overline{ab} = \overline{a}\overline{b}$, because it will then follow immediately that $\overline{a}$,$\overline{b}$ are invertible, the inverses being $\overline{a^{-1}}$,$\overline{b^{-1}}$, respectively. Let $x_i = p_i a^{-1} = \overline{a}(p_i)$ and $y_i = p_i b^{-1} = \overline{b}(p_i)$. By McNaughton's theorem there are polynomials $X_i(p_1, \ldots, p_n)$, $Y_i(p_1, \ldots, p_n)$ in the language of MV-algebras such that $x_i = X_i(p_1, \ldots, p_n)$ and $y_i = Y_i(p_1, \ldots, p_n)$. For any generator p_i we have $(\overline{a}\overline{b})(p_i) = \overline{a}(\overline{b}(p_i)) = \overline{a}(Y_i(p_1, \ldots, p_n)) = Y_i(\overline{a}(p_1), \ldots, \overline{a}(p_n)) = Y_i(X_1(p_1, \ldots, p_n), \ldots, X_n(p_1, \ldots, p_n)) = y_i a^{-1} = p_i b^{-1} a^{-1} = p_i (ab)^{-1} = (\overline{ab})(p_i)$; hence $\overline{ab} = \overline{a}\overline{b}$. ■

Theorem 3.9. *Let $n \in \mathbf{N} \setminus \{0\}$. Then*

$$\begin{gathered} \widetilde{\ }\colon \mathrm{Aut}(M_n) \to \mathrm{Hom}([0,1]^n) \\ \overline{\ }\colon \mathrm{Hom}([0,1]^n) \to \mathrm{Aut}(M_n) \end{gathered}$$

are group isomorphisms, which are each the inverse of the other.

Proof. Let $\alpha \in \mathrm{Aut}(M_n)$. We already showed that $\widetilde{\alpha}\colon [0,1]^n \to [0,1]^n$ defined by $(\widetilde{\alpha}(h))_i = (\alpha^{-1}(p_i))(h)$ is a homeomorphism ($h = (h_1, \ldots, h_n) \in [0,1]^n$, $i \in \{1, \ldots, n\}$). We want to prove that it is a McNaughton homeomorphism. Consider $\alpha^{-1}(p_1), \ldots, \alpha^{-1}(p_n) \in M_n$. We can find a tessellation $\{T_1, \ldots, T_k\}$ of $[0,1]^n$ such that $\forall i \in \{1, \ldots, n\}$, $\forall j \in \{1, \ldots, k\}$ we have $(\alpha^{-1}(p_i))(h) = c_{1ij}h_1 + \cdots + c_{nij}h_n + c_{(n+1)ij}$ over T_j, for some integers $c_{1ij}, \ldots, c_{(n+1)ij}$. Let $A_1, \ldots, A_k \in \mathrm{Mat}_{n+1}(\mathbf{Z})$ be defined by $(A_j)_{rs} = c_{rsj}$ for $1 \le r \le n+1$ and $1 \le s \le n$, $(A_j)_{r(n+1)} = 0$ for $1 \le r \le n$, $(A_j)_{(n+1)(n+1)} = 1$. In order to show that $\widetilde{\alpha}$ fulfils the requirements of Definition 3.6., we need only to show that the A_j's have the same determinant, either $+1$ or -1.

We first prove that the A_j's are unimodular. Choose $j \in \{1, \ldots, k\}$, and choose $h = (h_1, \ldots, h_n) \in T_j$ such that $h_1, \ldots, h_n, 1$ are independent over $\mathbf{Q}$. Then M_n/h is isomorphic to the totally ordered group $\mathbf{Z}[h_1, \ldots, h_n, 1] = \{d_1h_1 + \cdots + d_nh_n + d_{n+1}\colon d_1, \ldots, d_{n+1} \in \mathbf{Z}\}$. Let $(k_1, \ldots, k_n, 1) = (h_1, \ldots, h_n, 1)A_j$. Then we have $\mathbf{Z}[h_1, \ldots, h_n, 1] \simeq M_n/h \simeq \alpha[M_n]/\alpha[h] = M_n/\widetilde{\alpha}(h) \simeq \mathbf{Z}[p_1/\widetilde{\alpha}(h), \ldots, p_n/\widetilde{\alpha}(h), \mathbf{1}/\widetilde{\alpha}(h)] = \mathbf{Z}[\alpha^{-1}(p_1)/h, \ldots, \alpha^{-1}(p_n)/h, \mathbf{1}/h] = \mathbf{Z}[k_1, \ldots, k_n, 1]$. By [13, Lemma 3.6.] A_j is unimodular.

Let us make a definition: we say that $T, T' \in \{T_1, \ldots, T_k\}$ are *mates* if there exists $F \subseteq T \cap T'$ such that the affine dimension of F is $n-1$. We now prove that all the A_j's have determinant of the same sign. This will follow immediately from the following two Claims.

Claim 1. *Let $T, T' \in \{T_1, \ldots, T_k\}$ be mates. Let $A, A' \in \mathrm{Mat}_{n+1}(\mathbf{Z})$ be the matrices associated to T, T', respectively. Then $\det(A)$ and $\det(A')$ have the same sign.*

Claim 2. *For each $T, T' \in \{T_1, \ldots, T_k\}$ there is a chain $T = T_{j_1}, \ldots, T_{j_r} = T'$ such that, for $s \in \{1, \ldots, r-1\}$, T_{j_s} and $T_{j_{s+1}}$ are mates.*

Proof of Claim 1. If $T = T'$, we are through. If not, let $F \subseteq T \cap T'$ be as above. Let $B, B' \in \mathrm{Mat}_n(\mathbf{Z})$ be obtained by A, A' by removing the last line and the last column. Let π_F be the affine hyperplane such that $F \subseteq \pi_F$, and let H_F, H'_F be the halfspaces determined by π_F. Let $h = (h_1, \dots, h_n) \in F$, and let $\pi = \pi_F - h$, $H = H_F - h$, $H' = H'_F - h$ be the translates of π_F, H_F, H'_F through the origin. For any $k = (k_1, \dots, k_n) \in \pi$, we have $kB = kB'$. In fact $A(h) = A'(h)$ and $A(h+k) = A'(h+k)$. Let $i \in \{1, \dots, n\}$. Then

$$(\sum_{j=1}^{n} a_{ji} h_j) + a_{(n+1)i} = (\sum_{j=1}^{n} a'_{ji} h_j) + a'_{(n+1)i}$$

and

$$(\sum_{j=1}^{n} a_{ji} (h_j + k_j)) + a_{(n+1)i} = (\sum_{j=1}^{n} a'_{ji} (h_j + k_j)) + a'_{(n+1)i}$$

(a, a' are generic names for the entries of A, A'). Hence $\sum_{j=1}^{n} a_{ji} k_j = \sum_{j=1}^{n} a'_{ji} k_j$ and, as i is arbitrary, $kB = kB'$.
We also have that $HB = \{(k-h)B : k \in H_F\} = \{kB : k \in H_F\} - hB$ is a translate of $A[H_F] = \{kB : k \in H_F\} + (a_{(n+1)1}, \dots, a_{(n+1)n})$, and analogously $H'B'$ is a translate of $A'[H'_F]$. Hence HB and $H'B'$ are the halfspaces determined by $\pi B = \pi B'$. Choose a basis $\{k^{(1)}, \dots, k^{(n)}\}$ of $\mathbf{R}^n$ such that $k^{(1)}, \dots, k^{(n-1)} \in \pi$ and $k^{(n)} \in H$. Then $-k^{(n)} \in H'$. For $i \in \{1, \dots, n-1\}$, let $w^{(i)} = k^{(i)}B = k^{(i)}B'$. Then both $-(k^{(n)}B)$ and $(-k^{(n)})B'$ lie in $H'B'$. It follows that there exists $C \in \mathrm{Mat}_n(\mathbf{R})$ such that $\det(C)$ is positive nonzero, $w^{(i)}C = w^{(i)}$ for $i \in \{1, \dots, n-1\}$, and $(-(k^{(n)}B))C = (-k^{(n)})B'$. Hence the maps induced by BC and B' coincide over the basis $\{k^{(1)}, \dots, k^{(n-1)}, -k^{(n)}\}$, and so $BC = B'$ as matrices. It follows that $\det(B)$ ($= \det(A)$) and $\det(B')$ ($= \det(A')$) have the same sign. ■

Proof of Claim 2. Induction over k. If $k = 1$, there is nothing to prove. Assume that the statement is true for $k-1$, and choose $T, T' \in \{T_1, \dots, T_k\}$, $T \neq T'$. There must be a $T'' \in \{T_1, \dots, T_k\}$ such that T, T'' are mates. If $T'' = T'$, we are through. Otherwise, consider the tessellation $\{R_1, \dots, R_{k-1}\} = \{T \cup T''\} \cup \{R \in \{T_1, \dots, T_k\} : R \neq T \text{ and } R \neq T''\}$. By inductive hypothesis, there is a chain $T \cup T'' = R_{j_1}, \dots, R_{j_r} = T'$ such that for $s \in \{1, \dots, r-1\}$, R_{j_s} and $R_{j_{s+1}}$ are mates. We may also assume that $T \cup T'' \notin \{R_{j_2}, \dots, R_{j_r}\}$. It is now sufficient to observe that R_{j_2} must be mate either to T or to T'' (or both). ■

It follows that $\widetilde{\alpha}$ is a McNaughton homeomorphism and range$(\widetilde{\ })\subseteq \mathrm{Hom}([0,1]^n)$. We prove that $\widetilde{\ }$ is an isomorphism: clearly $\widetilde{1_{M_n}} = 1_{[0,1]^n}$. Let $\alpha\beta \in \mathrm{Aut}(M_n)$. Then, for any $h \in [0,1]^n \simeq \mathrm{Maxspec}(M^n)$, $\widetilde{\alpha\beta}(h) = (\alpha\beta)[h] = \alpha[\beta[h]] = \alpha[\widetilde{\beta}(h)] = \widetilde{\alpha}\widetilde{\beta}(h)$; hence $\widetilde{\alpha\beta} = \widetilde{\alpha}\widetilde{\beta}$. Moreover $(\widetilde{\alpha})^{-1} = \widetilde{\alpha^{-1}}$, because $\widetilde{\alpha}\widetilde{\alpha^{-1}} = \widetilde{\alpha\alpha^{-1}} = 1_{[0,1]^n} = \widetilde{\alpha^{-1}\alpha} = \widetilde{\alpha^{-1}}\widetilde{\alpha}$.
By Lemma 3.8. $\overline{\ } : \mathrm{Hom}([0,1]^n) \to \mathrm{Aut}(M_n)$ is an isomorphism. For any generator p_i, it is $\overline{\widetilde{\alpha}}(p_i) = p_i(\widetilde{\alpha})^{-1} = p_i\widetilde{\alpha^{-1}} = \alpha(p_i)$; hence $\overline{\widetilde{\alpha}} = \alpha$.
For any $h \in [0,1]^n$, it is $\widetilde{\overline{a}}(h) = (((\overline{a})^{-1}(p_1))(h), \dots, ((\overline{a})^{-1}(p_n))(h)) = ((\overline{a^{-1}}(p_1))(h), \dots, (\overline{a^{-1}}(p_n))(h)) = ((p_1 a)(h), \dots, (p_n a)(h)) = a(h)$; hence $\widetilde{\overline{a}} = a$. This completes the proof of Theorem 3.9.. ■

4. Some computations

McNaughton homeomorphisms have rather strong properties: first of all, they preserve volumes.

Proposition 4.1. *Let* $a\colon [0,1]^n \to [0,1]^n$ *be a McNaughton homeomorphism, and let* μ *denote Lebesgue measure on* $[0,1]^n$*. Then for any measurable set* $S \subseteq [0,1]^n$, $\mu(S) = \mu(a[S])$.

Proof. Immediate, since unimodular matrices preserve measure, and each $a_{|T_j}$ can be expressed as a product by an $n \times n$ unimodular matrix followed by a translation. ∎

Second, they preserve denominators.

Proposition 4.2. *Let* a *as above, let* $c_1, \ldots, c_n, d \in \mathbf{N}$ *be relatively prime,* $d \neq 0$, $a((c_1/d, \ldots, c_n/d)) = (e_1/d, \ldots, e_n/d)$*. Then* $e_1, \ldots, e_n, d$ *are relatively prime.*

Proof. Assume $h = (c_1/d, \ldots, c_n/d) \in T_j$. We may express $a_{|T_j}$ by using homogeneous coordinates, so that $a(h) = (c_1, \ldots, c_n, d)A_j = (e_1, \ldots, e_n, d)$ (in homogeneous coordinates). Now just observe that A_j (as a mapping $A_j\colon \mathbf{Z}_{n+1} \to \mathbf{Z}_{n+1}$) is an automorphism of $\mathbf{Z}_{n+1}$ (as an abelian group). ∎

In a sense, we may think of McNaughton homeomorphisms as stresses of a crystal: the volume of the unit cell is preserved, and points of the base lattice are mapped to points of the same kind.

Of course, $\mathrm{Hom}([0,1]^n)$ contains at least the symmetry group of the n-cube, let us call it C_n. It corresponds to the group of automorphisms of M_n which are obtained by permuting the generators and flipping some of them (here we mean the following: if $x \in A_n = \Gamma(M_n, \mathbf{1})$, the *flip* of x is $\mathbf{1} - x$). Clearly, C_n has $n!2^n$ elements. The following observation is due to Mundici.

Proposition 4.3. $\mathrm{Aut}(M_1) = C_1 \simeq \mathbf{Z}_2$.

Proof. Let $a \in \mathrm{Hom}([0,1])$. Assume $a(0) = 0$. By Proposition 4.1. we have, for any $x \in [0,1]$, $|x - 0| = |a(x) - a(0)| = |a(x) - 0|$; hence $a(x) = x$ and a is the identity. If $a(0) = 1$ we have $|x - 0| = |a(x) - a(0)| = |a(x) - 1|$; hence $a(x) = 1 - x$ and a is the rotation about $1/2$, i.e., $\overline{a}$ is the automorphism induced by flipping the generator p_1. ∎

On the other hand, the structure of $\mathrm{Aut}(M_2)$ is still an open problem. It is not an abelian group, since C_2 is nonabelian. We will prove that the free abelian group over ω generators is embeddable in $\mathrm{Aut}(M_2)$.

Lemma 4.4. *Let* $h^1, h^2, h^3, k^1, k^2, k^3 \in \mathbf{Z}^3$, $h^i = (h^i_1, h^i_2, h^i_3)$, $k^i = (k^i_1, k^i_2, k^i_3)$, $h^i_3 = k^i_3$ *for* $i \in \{1,2,3\}$,

$$\begin{vmatrix} h^1_1 & h^1_2 & h^1_3 \\ h^2_1 & h^2_2 & h^2_3 \\ h^3_1 & h^3_2 & h^3_3 \end{vmatrix} = \begin{vmatrix} k^1_1 & k^1_2 & k^1_3 \\ k^2_1 & k^2_2 & k^2_3 \\ k^3_1 & k^3_2 & k^3_3 \end{vmatrix} = +1$$

Then there exists $A \in \mathrm{Mat}_3(\mathbf{Z})$ *such that* $\det(A) = +1$, $a_{13} = a_{23} = 0$, $a_{33} = 1$ *and*

$$\begin{pmatrix} k^1_1 & k^1_2 & k^1_3 \\ k^2_1 & k^2_2 & k^2_3 \\ k^3_1 & k^3_2 & k^3_3 \end{pmatrix} = \begin{pmatrix} h^1_1 & h^1_2 & h^1_3 \\ h^2_1 & h^2_2 & h^2_3 \\ h^3_1 & h^3_2 & h^3_3 \end{pmatrix} A \qquad (*)$$

Proof. Both $\{h^1, h^2, h^3\}$ and $\{k^1, k^2, k^3\}$ are bases for the free abelian group $\mathbf{Z}^3$. As automorphisms of $\mathbf{Z}^3$ are in 1–1 correspondence with unimodular, 3×3 matrices with entries in $\mathbf{Z}$, it follows that there exists a unimodular matrix $A \in \mathrm{Mat}_3(\mathbf{Z})$ such that $(*)$ is satisfied. Clearly $\det(A) = +1$. As $\{h^1, h^2, h^3\}$ is a basis, there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbf{Z}$ such that $\sum_{i=1}^3 \lambda_i h^i = (1,0,0)$. Hence the first row of A is $(1,0,0)A = \sum_{i=1}^3 \lambda_i h^i A = \sum_{i=1}^3 \lambda_i k^i$. It follows that $a_{13} = \sum_{i=1}^3 \lambda_i k^i_3 = \sum_{i=1}^3 \lambda_i h^i_3 = 0$. Analogously $a_{23} = 0$ and $a_{33} = 1$. ∎

Let us work in projective coordinates. For $h^1, \ldots, h^m \in [0,1]^2$, $h^i = (h^i_1, h^i_2, h^i_3)$, let $\overline{h^1, \ldots, h^m}$ be the closed convex hull generated by $h^1, \ldots, h^m$, i.e., the set $\{\sum_{i=1}^m r_i h^i$: the r_i's are positive reals, not all zero$\}$.

For $n \in \mathbf{N}$, consider the tessellations $\{T_1, \ldots, T_{10}\}$ and $\{T_1', \ldots, T_{10}'\}$ given by

$$\begin{aligned}
T_1 &= \text{closure of } ([0,1]^2 \setminus \overline{(n,n,2n+1),(n+1,n,2n+1)} \\
&\qquad\qquad\qquad \overline{,(n+1,n+1,2n+1),(n,n+1,2n+1)}) \\
T_2 &= \overline{(n,n,2n+1),(n+1,n+1,2n+3),(n,n+1,2n+1)} \\
T_3 &= \overline{(n,n,2n+1),(n+2,n+1,2n+3),(n+1,n+1,2n+3)} \\
T_4 &= \overline{(n,n,2n+1),(n+1,n,2n+1),(n+2,n+1,2n+3)} \\
T_5 &= \overline{(n+1,n,2n+1),(n+2,n+2,2n+3),(n+2,n+1,2n+3)} \\
T_6 &= \overline{(n+1,n,2n+1),(n+1,n+1,2n+1),(n+2,n+2,2n+3)} \\
T_7 &= \overline{(n+1,n+1,2n+1),(n+1,n+2,2n+3),(n+2,n+2,2n+3)} \\
T_8 &= \overline{(n+1,n+1,2n+1),(n,n+1,2n+1),(n+1,n+2,2n+3)} \\
T_9 &= \overline{(n,n+1,2n+1),(n+1,n+1,2n+3),(n+1,n+2,2n+3)} \\
T_{10} &= \overline{(n+1,n+1,2n+3),(n+2,n+1,2n+3),(n+2,n+2,2n+3)} \\
&\qquad\qquad\qquad \overline{,(n+1,n+2,2n+3)} \\
T_1' &= T_1 \\
T_2' &= \overline{(n,n,2n+1),(n+1,n+2,2n+3),(n,n+1,2n+1)} \\
T_3' &= \overline{(n,n,2n+1),(n+1,n+1,2n+3),(n+1,n+2,2n+3)} \\
T_4' &= \overline{(n,n,2n+1),(n+1,n,2n+1),(n+1,n+1,2n+3)} \\
T_5' &= \overline{(n+1,n,2n+1),(n+2,n+1,2n+3),(n+1,n+1,2n+3)} \\
T_6' &= \overline{(n+1,n,2n+1),(n+1,n+1,2n+1),(n+2,n+1,2n+3)} \\
T_7' &= \overline{(n+1,n+1,2n+1),(n+2,n+2,2n+3),(n+2,n+1,2n+3)} \\
T_8' &= \overline{(n+1,n+1,2n+1),(n,n+1,2n+1),(n+2,n+2,2n+3)} \\
T_9' &= \overline{(n,n+1,2n+1),(n+1,n+2,2n+3),(n+2,n+2,2n+3)} \\
T_{10}' &= T_{10}
\end{aligned}$$

To be precise, the T_j's and the T_j''s, as well as the A_j's to be defined in a moment, should bear an index to express dependence on n, but we omit it in order not to overburden the notation. This should cause no trouble. The following is a scheme of the T_j's (to the left) and the T_j''s (to the right).

We are planning to describe a family $\{a_n : n \in \mathbf{N}\}$ of McNaughton homeomorphisms $a_n : [0,1]^2 \to [0,1]^2$. We will try to reduce our formalism to the minimum compatible with full rigour.

Two concentric squares are inscribed in $[0,1]^2$: an *outer square* of vertices $(n, n, 2n+1), (n+1, n, 2n+1), (n+1, n+1, 2n+1), (n, n+1, 2n+1)$, and an *inner square*, of vertices $(n+1, n+1, 2n+3), (n+2, n+1, 2n+3), (n+2, n+2, 2n+3), (n+1, n+2, 2n+3)$. $T_1 = T_1'$ is the set of points outside the outer square; they stay fixed under a_n, i.e., we define $A_1 =$ the 3×3 identity matrix. $T_{10} = T_{10}'$ is the set of points inside the inner square; they rotate clockwise by an angle of $\pi/2$ about $(1, 1, 2)$. The matrix A_{10} that accomplishes the rotation is

$$A_{10} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

The points between the outer and the inner square are tessellated into the triangles $T_2, \ldots, T_9$. These triangles are mapped to $T_2', \ldots, T_9'$ by matrices $A_2, \ldots, A_9$. Lemma 4.4. guarantees that $A_2, \ldots, A_9$ exist and are appropriate, i.e., have integral entries, determinant $+1$ and last column of the form

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

A bit of computation shows in fact that the hypotheses of Lemma 4.4. are always satisfied; for example

$$\begin{vmatrix} n & n & 2n+1 \\ n+1 & n+1 & 2n+3 \\ n & n+1 & 2n+1 \end{vmatrix} = \begin{vmatrix} n & n & 2n+1 \\ n+1 & n+2 & 2n+3 \\ n & n+1 & 2n+1 \end{vmatrix} = +1$$

whence the existence of an appropriate A_2 follows. Let $a_n : [0,1]^2 \to [0,1]^2$ be the mapping so obtained, according to Definition 3.6.. It is clear that a_n is a homeomorphism, hence a McNaughton homeomorphism. Note also that the inner square of a_n is the outer square of a_{n+1}.

As an example, take $n = 0$. T_1 collapses then to the borders of $[0,1]^2$, and we just disregard it. $\overline{a_0}$ maps the free generator p_1 to $x = \overline{a_0}(p_1) = p_1 a_0^{-1}$ and p_2 to $y = \overline{a_0}(p_2) = p_2 a_0^{-1}$. We may get an explicit expression for x as follows: a_0^{-1} is determined by $A_2^{-1}, \ldots, A_{10}^{-1}$. Each A_j^{-1} maps T_j' to T_j. For example, it must be

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix} A_2^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

whence

$$A_2^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If the point $(\xi, \nu, 1)$ (in projective coordinates) belongs to T_2', then it gets mapped by a_0^{-1} to the point $(\xi, -\xi + \nu, 1) \in T_2$. Hence $x_{|T_2'}$ is the linear affine polynomial ξ and $y_{|T_2'}$ is the linear affine polynomial $-\xi + \nu$. The graph of x is and the graph of y is

Theorem 4.5. *The free abelian group over ω generators is embeddable in* $\mathrm{Aut}(M_2)$.

Proof. Let F be the subgroup of $\mathrm{Hom}([0,1]^2)$ generated by $\{a_0, a_1, \ldots\}$. We claim that F is abelian and that $\{a_0, a_1, \ldots\}$ are free generators for F. We must prove:
i) each a_n has infinite period;
ii) for each n, m, it is $a_n a_m = a_m a_n$ and $a_n a_m^{-1} = a_m^{-1} a_n$;
iii) if for $k_1, \ldots, k_r \in \mathbf{Z}$ it is $a_{n_1}^{k_1} a_{n_2}^{k_2} \ldots a_{n_r}^{k_r} = 1_{[0,1]^2}$, then $k_1 = k_2 = \cdots = k_r = 0$.
About i): let $k \in \mathbf{N} \setminus \{0\}$. If k is not a multiple of 4, then no point in the inner square of a_n is mapped to itself by a_n^k (except $(1,1,2)$), whence $a_n^k \neq 1_{[0,1]^2}$. If $k = 4l$, then look at the line segment $\overline{(n,n,2n+1),(n+1,n+1,2n+3)}$. The point $(n,n,2n+1)$ stays fixed under a^{4l}, whereas $(n+1,n+1,2n+3)$ rotates l times round the inner square. It is clear that a^{4l} cannot be the identity over $\overline{(n,n,2n+1),(n+1,n+1,2n+3)}$.
About ii): if $n = m$, we are through. Assume $n < m$. The function a_m acts identically on all points outside the outer square of a_m, whence $a_n a_m = a_m a_n$ for those points. The points inside the outer square of a_m are also inside the inner square of a_n. Then a_n acts on them by a clockwise rotation of $\pi/2$, and it is clear that such a rotation commutes with a_m. We argue analogously to prove that a_n and a_m^{-1} commute.

About iii): we may assume $0 \le n_1 < n_2 < \cdots < n_r$. The argument in i) shows that, for any n, if $k \in \mathbf{Z} \setminus \{0\}$, we can find an h inside the outer square and outside the inner square of a_n such that $a_n^k(h) \neq h$, i.e., h witnesses $a_n^k \neq 1_{[0,1]^2}$. Now, the points inside the outer square and outside the inner square of a_{n_1} are not moved by $a_{n_2}, \dots, a_{n_r}$. Hence, as $a_{n_1}^{k_1} a_{n_2}^{k_2} \dots a_{n_r}^{k_r} = 1_{[0,1]^2}$, it follows that $k_1 = 0$. By induction, $k_2 = k_3 = \cdots = k_r = 0$. ∎

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