GENERAL THEORY OF DYNAMICAL SYSTEMS

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This is a *personal* geodesic along dynamical systems. Corrections, suggestions, observations, ..., are most welcome. Version of July 24, 2024.

1. Basics

A *topologic dynamical system* is a triple $(X, \mathcal{T}, R)$. A *measurable dynamical system* is a triple $(X, \mathcal{X}, R)$. An *metric dynamical system* is a quadruple $(X, \mathcal{X}, \mu, R)$. All spaces are polish and all measures Borel and σ-finite, thus regular [Rud87, Theorem 2.18]. If μ is finite, then it is usually normalized to be a probability. Usually R is surjective in the appropriate sense. Note that in the last case the maps are actually equivalence classes of maps, as in the L_p spaces.

Exercise 1.1. The obvious composition of equivalence classes is well defined.

Example 1.2.
(1) Left translations L_g in a topological group: rotations of a circle, the odometer $(\mathbb{Z}_p, +1)$.
(2) Continuous endomorphisms of a compact abelian group: $x \mapsto kx \pmod 1$, nonsingular integer matrices acting on d-dimensional tori.
(3) Affine transformations on a compact group.
(4) One-sided and two-sided shifts.
(5) Interval-exchange transformations.
(6) Markov maps of the interval.
(7) Gauss-type maps.

The above can be generalized to the action of a topological monoid or group on a topological, measurable, or measure space; a key issue here is measure rigidity.

Example 1.3.
(1) Flows in smooth differential varieties.
(2) Billiards.
(3) The geodesic and horocycle flow in $\Gamma \backslash \operatorname{PSL}_2 \mathbb{R}$.

In all three cases, the dynamical systems form the objects of a category. The arrows of the category are then the continuous, measurable, or measure-preserving [equivalence classes of] maps that make the appropriate square commute.

Recall that a *semialgebra* in X is a subset $\mathcal{S}$ of $\mathscr{P}(X)$ that contains $\emptyset$, is closed under finite intersections, and is such that the complement of every element is a finite disjoint union of elements.

Example 1.4.
(1) *Cylinders* in $\mathbb{R}$: finite intersections of $(-\infty, a]$'s and their complements.
(2) *Cylinders* in $\mathbb{R}^n$: finite intersections of π_i^{-1}'s of cylinders in $\mathbb{R}$.
(3) *Cylinders* in $n = \{0, \dots, n-1\}$: elements of $\mathscr{P}(n)$.
(4) *Cylinders* in n^I: finite intersections of π_i^{-1}'s of cylinders in n.

(5) *Blocks* in n^ω: all $[a_0, \dots, a_r]$'s.
(6) *Blocks* in $n^{\mathbb{Z}}$: all $[a_{-r}, \dots, a_r]$'s.

Lemma 1.5 (The Monotone Class Theorem)**.** *Let $\mathcal{A}$ be an algebra of sets, and let $\mathcal{M}$ be the smallest monotone* (*i.e., closed under countable increasing and decreasing limits*) *class containing $\mathcal{A}$. Then $\mathcal{F}(\mathcal{A}) = \mathcal{M}$.*

Lemma 1.6. *Let $(X, \mathcal{X}, \mu)$, $(Y, \mathcal{Y}, \nu)$ be measure spaces, and let $\mathcal{S}$ be a semialgebra generating $\mathcal{Y}$ and such that Y is an increasing countable union of ν-finite elements $S_n \in \mathcal{S}$. Let $R : X \to Y$ be a map that is measurable and measure-preserving on $\mathcal{S}$. Then R is measurable and measure-preserving.*

Proof. R is surely measurable; just note that $\{A \subseteq Y : R^{-1}A \in \mathcal{X}\}$ is a σ-algebra. Fix S_n, and let $\mathcal{M} = \{M \in \mathcal{Y} : \mu(R^{-1}(M \cap S_n)) = \nu(M \cap S_n)\}$. We have $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{M}$, and $\mathcal{M} = \mathcal{Y}$ by the Monotone Class Theorem. □

2. Topological groups

We restrict to polish topological groups (more generally, Hausdorff and having a countable basis of open precompact sets).

Theorem 2.1. *For every such group G there exists a unique* (*up to product by constants*) left Haar measure *namely a nontrivial measure λ that is*:

(1) *Radon, i.e.,*
- *Borel and* $[0, +\infty]$*-valued*;
- *regular* (*i.e.,* (*i*) *the measure of every Borel set is the infimum of the measures of the open sets containing it and* (*ii*) *the measure of every open set is the supremum of the measures of compact sets contained in it*);
- *finite on compact sets*;

(2) *strictly positive on nonempty open sets*;
(3) *invariant w.r.t. left translations.*

Proof. [Loo53, §29]. □

Remark 2.2. Every Borel measure on a polish space is regular [Rud87, Theorem 2.18].

Example 2.3. $(\mathbb{R}, +, \lambda)$ and $(\mathbb{R}_{>0}, \cdot, \exp_* \lambda)$ (note that $\mathrm{d}(\exp_* \lambda)(x) = x^{-1}\,\mathrm{d}x$). Discrete finite and countable groups. $\mathbb{T}^n$. $\mathbb{Z}_p$ and $\mathbb{Q}_p$. Z_p^ω. Linear groups: $\mathrm{GL}_n \mathbb{R}$, $\mathrm{SL}_n \mathbb{R}$, $\mathrm{O}_n \mathbb{R}$, ...

Example 2.4. $(\mathbb{Q}, +)$ with the topology induced by $\mathbb{R}$ is not locally compact and does not have a Haar measure. Indeed, any countable group possessing a Haar measure λ must be discrete. Indeed, since λ is notrivial and the group is countable, all singletons have the same strictly positive measure; outer regularity then implies discreteness. There's usually no relation between the Haar measures of a group and that of a subgroup, even if the latter carries the induced topology: look at $\mathbb{R} < \mathbb{R}^2$.

Example 2.5. Let

$$\mathrm{Aff}_+\mathbb{R} = \left\{ \begin{pmatrix} x & y \\ & 1 \end{pmatrix} : x \in \mathbb{R}_{>0}, y \in \mathbb{R} \right\}.$$

Then the left measure is $x^{-2}\,\mathrm{d}x\,\mathrm{d}y$, while the right one is $x^{-1}\,\mathrm{d}x\,\mathrm{d}y$.

Lemma 2.6. *Let G be a polish topological group and λ a left Haar measure. Then G is compact iff λ is finite. Also, G is discrete iff G is at most countable iff λ is positive on singletons.*

Proof. Let G be not compact, and fix a compact neighborhood K of 1. By basics of topological groups, there exists an open neighborhood U of 1 such that $U = U^{-1}$ and $UU \subseteq K$. We can choose a sequence $g_0, g_1, \dots$ s.t. $g_n \notin \bigcup_{k<n} g_k K$. We claim that $(g_n U)$ is disjoint. Indeed, if $k < t$ and $h \in g_k U \cap g_t U$, then $g_t = hu^{-1} = (g_k v)u^{-1} \in g_k UU \subseteq g_k K$, which is impossible. Hence λ is infinite.

A discrete group with a countable basis must be countable, and we already established the other implications. □

For every $g \in G$, $(R_g)_* \lambda$ is again a left Haar measure; indeed it is surely Borel regular, and

$$\big[(R_g)_*\lambda\big](xA) = \lambda(xAg^{-1}) = \lambda(Ag^{-1}) = \big[(R_g)_*\lambda\big](A).$$

Therefore, the *modular function* $m : G \to \mathbb{R}_{>0}$ remains defined —independently of the choice of λ— by $(R_g)_*\lambda = m(g)\lambda$.

Theorem 2.7. (1) *m is a continuous homomorphism.*

(2) *$m = \mathbb{1}$ iff every left measure is also a right measure, and conversely. Such a group is said to be* unimodular.

(3) *Abelian groups and compact groups are unimodular.*

Proof. (1) We have $m(zw)\lambda = (R_{zw})_*\lambda = (R_w \circ R_z)_*\lambda = (R_w)_*\big[(R_z)_*\lambda\big] = (R_w)_*\big[m(z)\lambda\big] = m(w)m(z)\lambda$. For the continuity, fix $f \in C_c(G)$ such that $\int f \,\mathrm{d}\lambda \neq 0$. Then the map

$$z \mapsto \int f(-z)\,\mathrm{d}\lambda = \int f\,\mathrm{d}(R_z)_*\lambda = m(z)\int f\,\mathrm{d}\lambda$$

is continuous. Indeed, for metric spaces, continuity amounts to sequential continuity. Let then $z_n \to z$; without loss of generality all z_n belong to a compact neighborhood H of z. Let K be the support of f. Then:

(a) H^{-1} is compact;
(b) $K \times H^{-1}$ is compact (because the induced topology on $K \times H^{-1}$ is the product of the induced topologies on K and on H^{-1}), and therefore KH^{-1} is compact;
(c) for every n, $f(-z_n)$ is supported on KH^{-1}. Indeed, $w \notin KH^{-1}$ implies $w \notin Kz_n^{-1}$ implies $wz_n \notin K$ implies $f(wz_n) = 0$.
(d) the family $f(-z_n)$ is dominated by $\max|f|\mathbb{1}_{KH^{-1}} \in L_1(\lambda)$, and sequential continuity follows from the dominated convergence theorem.

Thus m is continuous.
(2) Assume $m = \mathbb{1}$, let λ be a left measure, ρ a right one, and fix a nonempty open set U with compact closure. Normalize so that $\lambda(U) = \rho(U)$. For every A and every z, $\lambda(Az^{-1}) = \big[(R_z)_*\lambda\big](A) = m(z)\lambda(A) = \lambda(A)$, so that λ is right invariant and thus a multiple of ρ. By our normalization, $\lambda = \rho$.
(3) When G is compact, the range of m must be a compact subgroup of $\mathbb{R}_{>0}$, containing 1. The only such subgroup is $\{1\}$. □

Exercise 2.8. Compute the modular function for the group of Example 2.5.

Lemma 2.9. *Let G be compact, $\varphi : G \to G$ a surjective continuous homomorphism, μ the Haar probability. Then (G, μ, φ) is a metric system.*

Proof. Since φ is continuous, $\varphi_*\mu$ is Borel and (1), (3), (4), (5) in Theorem 2.1 hold. Fix g and let $h \in \varphi^{-1}\{g\}$; it is easy to see that $\varphi^{-1}(gA) = h\varphi^{-1}(A)$. Therefore $(\varphi_*\mu)(gA) = \mu(\varphi^{-1}(gA)) = \mu(h\varphi^{-1}(A)) = \mu(\varphi^{-1}(A)) = (\varphi_*\mu)(A)$, and $\varphi_*\mu$ is a left Haar probability. Hence $\varphi_*\mu = \mu$. $\square$

Compactness of G is essential: look at $x \mapsto 2x$ in $\mathbb{R}$.

Lemma 2.10. *Let $(X, \mathcal{X}, \mu, R)$ be a metric system, let $(Y, \mathcal{Y}, S)$ be a measurable system, and assume that $M : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ makes the square commute. Then $(Y, \mathcal{Y}, M_*\mu, S)$ is a metric system, called a* factor *of (X, μ, R).*

Proof. For every $A \in \mathcal{Y}$ we have $(M_*\mu)(S^{-1}A) = \mu(M^{-1}S^{-1}A) = \mu(R^{-1}M^{-1}A) = \mu(M^{-1}A) = (M_*\mu)(A)$. $\square$

Example 2.11. If X is an interval in $\mathbb{R}$ and the repartition function of μ is a homeomorphism onto $[0, 1]$, then we can take it for M and obtain a conjugacy between (X, μ, R) and $([0, 1], \lambda, MRM^{-1})$. This is the case for the Farey map and the tent map.

Example 2.12. Tent maps and Chebyshev polynomials.

3. Recurrence

Let $(X, \mathcal{X}, R)$ be a measurable system. The set $A \in \mathcal{X}$ is *wandering* if $A, R^{-1}A, R^{-2}A, \ldots$ are pairwise disjoint. A metric system (X, μ, R) is *conservative* is every wandering set has 0 measure (this is surely true if μ is finite).

The following is Halmos's version of the Poincaré recurrence theorem; it is related to Zermelo's gas paradox.

Theorem 3.1. *Let $(X, \mathcal{X}, \mu, R)$ be conservative, $A \in \mathcal{X}$, $NR_A = \{x : x \text{ enters } A \text{ at least once, and finitely many times}\}$. Then $NR_A \in \mathcal{X}$ and has μ-measure 0. In particular*:

- *μ-every x either never enters A, or enters A infinitely many times*;
- *μ-every $x \in A$ returns to A infinitely many times.*

Proof. For every $k \geq 0$, the set

$$N_k = R^{-k}A \cap \bigcap_{h>k} R^{-h}A^c \in \mathcal{X},$$

of all points entering A at time k and leaving it forever, is wandering; indeed, $R^{-q}N_k = N_{k+q}$. Hence it has measure 0, and so does $NR_A = \dot{\bigcup}_{k\geq 0} N_k$. $\square$

The point x in the topological system (X, R) is *recurrent* if it returns infinitely many times to any of its neighbourhoods.

4. Ergodicity

Let $(X, \mathcal{X}, R)$ be a measurable system. The *orbit* of x is $O(x) = \{R^k(x) : k \geq 0\}$; the *grand orbit* of x is $GO(x) = \{y : \text{there exist } h, k \geq 0 \text{ with } R^h(x) = R^k(y)\}$. We say that $A \in \mathcal{X}$ is:

(1) *waterproof from inside* if $R[A] \subseteq A$;
(2) *waterproof from outside* if $R^{-1}A \subseteq A$;
(3) *invariant* if $A = R^{-1}A$, i.e., A is two-sided waterproof, i.e., A is a (disjoint) union of grand orbits;
(4) *fully invariant* if $A = R^{-1}A = R[A]$.

If R is surjective, then invariant sets are fully invariant.

It is worth noting that a (σ)-algebra of sets is indeed an algebra $(\mathcal{X}, \triangle, \cap, 0, 1)$ in the algebraic sense of the word, the base field being $F_2 = \{0, 1\} = \{\emptyset, X\}$. In particular, ($\sigma$)-congruences and ($\sigma$)-ideals are in 1-1 correspondence. Note also that

$$A^c = 1 \triangle A, \quad A \setminus B = A \cap (1 \triangle B) = A \triangle (A \cap B), \quad A \cup B = A \triangle B \triangle (A \cap B).$$

Lemma 4.1. *Let $(X, \mathcal{X}, \mu)$ be a measure space, and define $A \sim B$ by $\mu(A \triangle B) = 0$. Then*:

- $\sim$ *is an equivalence*;
- *actually, $\sim$ is a σ-algebra congruence, corresponding to the σ-ideal of nullsets*;
- *μ is constant on equivalence classes*;
- *in particular, if $\mu(A) = 0$ then $\mu(A \triangle B) = \mu(\emptyset \triangle B) = \mu(B)$.*

Theorem 4.2. *Let $(X, \mathcal{X}, \mu, R)$ be a conservative metric system*; *then the following are equivalent and define an* ergodic *system.*

(1) *if A is invariant, then either $\mu(A) = 0$ or $\mu(A^c) = 0$;*
(2) *if $A \sim R^{-1}A$, then either $\mu(A) = 0$ or $\mu(A^c) = 0$;*
(3) *if $\mu(A) > 0$ and $B = \bigcup_{k \geq 0} R^{-k}A$, then $\mu(B^c) = 0$;*
(4) *if $\mu(A), \mu(B) > 0$, then there exists $k \geq 0$ with $\mu(A \cap R^{-k}B) > 0$.*

Proof. (1)⇒(2) Let A_∞ be the set of all $x \in X$ that enter A infinitely many times; it is clearly invariant. We have $A_\infty \triangle A = (A_\infty \setminus A) \dot\cup (A \setminus A_\infty)$, and $A \setminus A_\infty$ is μ-null by Poincaré. On the other hand

$$A_\infty \setminus A \subseteq (R^{-1}A \setminus A) \cup (R^{-2}A \setminus R^{-1}A) \cup \cdots$$

and each of the right-hand-side sets is μ-null. Hence $A_\infty \sim A$; since A_∞ is null or conull, so is A.

(2)⇒(3) We have $B \setminus R^{-1}B = A \setminus R^{-1}B = \{\text{elements in } A \text{ that never return to } A\}$. This set is null, and equals $B \triangle R^{-1}B$ because $B \supseteq R^{-1}B$. Hence B is null or conull, and the first alternative does not hold since $B \supseteq A$.

(3)⇒(4) By (3), $\mu\big[(\bigcup_k R^{-k}B)^c\big] = 0$. Hence

$$0 < \mu(A) = \mu\big(A \cap (\bigcup_k R^{-k}B)\big) = \mu\big(\bigcup_k (A \cap R^{-k}B)\big),$$

which establishes (4).

(4)⇒(1) Let A be invariant, and assume by contradiction $\mu(A), \mu(A^c) > 0$. Then, for some $k \geq 0$, $\mu(A^c \cap R^{-k}A) > 0$, which is absurd since $R^{-k}A = A$. $\square$

Lemma 4.3. *Let $(X, \mathcal{X}, \mu)$ be a finite measure space, with $\mathcal{X}$ generated by the algebra $\mathcal{A}$. Then, for every $B \in \mathcal{X}$ and every $\varepsilon > 0$, there exists $A \in \mathcal{A}$ such that $\mu(A \triangle B) < \varepsilon$.*

Proof. Let $\mathcal{D}$ be the set of elements of $\mathcal{X}$ satisfying the above property; it is enough to show that $\mathcal{D}$ is a σ-algebra. Since $A \triangle B = A^c \triangle B^c$, we have closure under complements. Let $D = \bigcup_{i \in I} D_i$ be a countable union of elements of $\mathcal{D}$. Fix $\varepsilon > 0$, and find $J \subseteq I$ of finite cardinality n such that $\mu(D \setminus \bigcup_j D_j) < \varepsilon/2$. For every j, choose A_j such that $\mu(D_j \triangle A_j) < (\varepsilon/2)/n$. Then $A = \bigcup_j A_j$ satisfies the requirement. Indeed, $D \triangle A = (D \triangle \bigcup_j D_j) \triangle (\bigcup_j D_j \triangle A)$, and $(\bigcup_j D_j) \triangle (\bigcup_j A_j) \subseteq \bigcup_j (D_j \triangle A_j)$. □

In particular, $|\mu(A^c) - \mu(B^c)| = |\mu(A) - \mu(B)| = |(\mu(A \setminus B) + \mu(A \cap B)) - (\mu(B \setminus A) - \mu(A \cap B))| = |\mu(A \setminus B) - \mu(B \setminus A)| \leq \mu(A \setminus B) + \mu(B \setminus A) < \varepsilon$.

Theorem 4.4. *The full two-sided Bernoulli shift $(m^{\mathbb{Z}}, (p_0, \dots, p_{m-1}), S)$ is ergodic.*

Proof. Let $B \in \mathcal{B}$ be S-invariant, fix $\varepsilon > 0$, and let A be in the algebra generated by cylinders, and such that $\mu(A \triangle B) < \varepsilon$. Since A is defined by finitely many conditions, for some (indeed, for every) $n > 0$ large enough we have that $S^{-n}A$ is μ-independent from A and from A^c. We claim that $\mu(A)\mu(A^c) < 2\varepsilon$. Indeed,

$$\mu(A)\mu(A^c) = \mu(S^{-n}A)\mu(A^c) = \mu(S^{-n}A \cap A^c),$$

and

$$S^{-n}A \cap A^c \subseteq (A \triangle B) \cup (S^{-n}A \triangle B) = (A \triangle B) \cup S^{-n}(A \triangle B),$$

which establishes our claim (the first containment holds by considering the two cases $\omega \in B$ and $\omega \notin B$; the second equality from $B = S^{-n}B$).

We conclude that $\mu(B)\mu(B^c) \leq (\mu(A) + \varepsilon)(\mu(A^c) + \varepsilon) < 2\varepsilon + \varepsilon + \varepsilon^2$. As ε is arbitrary, we must have $\mu(B)\mu(B^c) = 0$, and thus B is null or conull. □

Lemma 4.5. *Any factor of an ergodic system is ergodic.*

It follows that the one-sided Bernoulli shift is ergodic.

5. Banach spaces

From here on X is compact Hausdorff, with a countable basis of open precompact sets (thus X is metrizable), and μ is a Borel probability on it. We then have the (complex or real) Banach spaces

$$L_1(\mu) \supseteq L_2(\mu) \supseteq \cdots \supseteq L_\infty(\mu) \supseteq C(X).$$

Let $V = V(X)$ be any of the above Banach spaces. An *algebra of operators* on V is any ($\mathbb{C}$ or $\mathbb{R}$)-algebra of continuous linear endomorphisms of V, with composition as multiplication. The *dual* V^* of V is the vector space of all ($\mathbb{C}$ or $\mathbb{R}$)-valued continuous —equivalently, bounded— linear functionals on V. Each $V(X)$ is a function space over X so, for appropriate maps $R : X \to Y$, we have natural maps

$$\begin{aligned} V(X) &\leftarrow V(Y) : R^* \\ f \circ R &\leftarrow f \\ R_* : V(X)^* &\to V(Y)^* \\ \Phi &\mapsto \Phi \circ R^* \end{aligned}$$

For example in the case of $C(X)^*$, seen as the space of complex measures on X, the 2nd map reads

$$\int_Y f \, \mathrm{d}R_* \mu = \int_Y R^* f \, \mathrm{d}\mu = \int_Y f \circ R \, \mathrm{d}\mu, \qquad \text{for every } f \in C(X).$$

The weakest topology on V^* that makes all evaluation functions $\Phi \mapsto \Phi f$ continuous is the *weak-$*$ topology*; it is usually weaker than the norm topology.

Theorem 5.1 (Banach-Alaoglu)**.** *Let V be a normed space. Then the unit sphere $\{\Phi \in V^* : \|\Phi\| = 1\}$ in V^*, as well as the unit ball, are compact in the weak-$*$ topology.*

In the infinite-dimensional case the unit sphere is never compact in the norm topology. Consider, for example, the unit sphere in $(C[0,1])^*$: every countable set of Direc measures has a weak accumulation point. However, two different Dirac measures are always at distance 2 in the norm metric, and by the triangle inequality there is no accumulation point.

Theorem 5.2 (Riesz representation theorem)**.** *There exists a natural bijection between the positive Borel probabilities on X and the positive normalized functionals in $C(X)^*$.*

Lemma 5.3. *Let (X, μ, R) be a metric system.*

(1) *If $f \in L_1$, then*
$$\int R^* f \, \mathrm{d}\mu = \int f \, \mathrm{d}R_* \mu = \int f \, \mathrm{d}\mu.$$
(2) *For every $1 \leq p \leq \infty$, $R^* : L_p \to L_p$ is an isometric vector space immersion* (*hence of norm* 1)*, having all its eigenvalues in S^1.*
(3) *If the system is invertible, then R^* is a unitary operator on the Hilbert space L_2* (*it is then denoted by U, and named the* Koopman operator).

Proof. (2) Assume $p < \infty$. Then $|f|^p \in L_1$ and $\|f\|_p^p = \int |f|^p \, \mathrm{d}\mu = \int |f|^p \circ R \, \mathrm{d}\mu = \int |f \circ R|^p \, \mathrm{d}\mu = \|R^* f\|_p^p$.
Assume $p = \infty$. Then, for every $M > 0$, we have $\{x : |(R^* f)(x)| < M\} = R^{-1}\{x : |f(x)| < M\}$, and thus $\mu(|R^* f| < M) = \mu(|f| < M)$. This implies $\|f\|_\infty = \|R^* f\|_\infty$.
If $R^* f = \alpha f$, then $\|f\|_p = \|R^* f\|_p = |\alpha| \|f\|_p$, so that, for $f \neq 0$, we have $|\alpha| = 1$.
(3) Let $U = R^*$; we have to prove $UU^* = U^* U = I$. For every $f, g \in L_2$ we have

$$\langle Uf, Ug \rangle = \int \overline{(f \circ R)} (g \circ R) \, \mathrm{d}\mu = \int (\overline{f} g) \circ R \, \mathrm{d}\mu = \int \overline{f} g \, \mathrm{d}\mu = \langle f, g \rangle,$$

which implies $U^* U = I$. Since R is invertible, so is U, with inverse $U^{-1} = (R^{-1})^*$. Applying the above to U^{-1}, we obtain $(U^{-1})^* U^{-1} = I$, and thus $UU^* = I$. $\square$

The following theorem says that ergodicity is a spectral property.

Theorem 5.4. *Let (X, μ, R) be a metric system, $1 \leq p \leq \infty$. The following are equivalent*:

(1) *the system is ergodic*;
(2) *the* 1*-autospace of R^* in L_p boils down to the constants $\mathbb{C} \cdot \mathbb{1}$.*

Proof. (2) implies (1). Let $\mu(A \triangle R^{-1} A) = 0$. Of course $\mathbb{1}_A \in L_p$, and $R^* \mathbb{1}_A = \mathbb{1}_{R^{-1} A}$; thus the assumption implies $\mathbb{1}_A = R^* \mathbb{1}_A$ in L_p. Therefore $\mathbb{1}_A$ is constantly 0 or 1.
(1) implies (2). $\square$

If (X, μ, R) is a continuous metric system, the fact that the only continuous invariant functions are the constants is not sufficient for ergodicity.

6. Characters for topological abelian groups

Let $\mathbf{LCA}$ be the category of locally compact Hausdorff abelian groups. Every $G \in \mathbf{LCA}$ has a *dual group* $\hat{G}$, whose elements are the *characters* of G, namely the continuous homomorphisms $\chi : G \to S^1$. We endow $\hat{G}$ with the compact-open topology (note that the defining subbasis is not necessarily a basis).

Theorem 6.1 (Pontryagin)**.** (1) *$\hat{G} \in \mathbf{LCA}$ and $\hat{}$: $\mathbf{LCA} \to \mathbf{LCA}$ is a contravariant functor.*
(2) *G is naturally isomorphic to $\hat{\hat{G}}$, via $g \mapsto (\chi \mapsto \chi(g))$.*
(3) *G is compact iff $\hat{G}$ is discrete.*
(4) *Let H be a closed subgroup of G. Then the restriction map $\chi \mapsto \chi \restriction H$ is an epimorphism $\hat{G} \to \hat{H} \simeq \hat{G}/H'$, where $H' = \{\chi \in \hat{G} : \chi \text{ is trivial on } H\}$. Moreover, the map $H \mapsto H'$ is a Galois correspondence between the lattice of closed subgroups of G and that of $\hat{G}$.*
(5) *$\hat{}$ is a categorical equivalence; in particular, $\widehat{G \times H} \simeq \hat{G} \times \hat{H}$.*
(6) *G is 2nd countable iff so is $\hat{G}$.*

Proof. (3) Assume G compact, and let O be a small neighborhoud of $1 \in S^1$. Then $W(G, O)$ is open, and equals $\{\mathbb{1}\}$ because the only subgroup of S^1 contained in O is the trivial one. Conversely —and using (2)— assume G discrete. Since the only compact sets in a discrete space are the finite ones, the compact-open topology of $\hat{G}$ equals the point-open topology, namely the product topology of $(S^1)^G$. Thus $\hat{G}$ identifies both algebraically and topologically with a subgroup of $(S^1)^G$. This subgroup is closed (because, for every $g, h \in G$, the function $\pi_{gh} \cdot (\pi_h)^{-1} \cdot (\pi_g)^{-1} : (S^1)^G \to S^1$ is continuous) and hence compact.
The two maps

$$\hat{G} \times \hat{H} \ni (\chi, \eta) \mapsto ((g, h) \mapsto \chi(g)\eta(h)) \in \widehat{G \times H},$$

$$\hat{G} \times \hat{H} \ni (\psi \circ \iota_1, \psi \circ \iota_2) \leftarrow \psi \in \widehat{G \times H},$$

are continuous homomorphisms, each the inverse of the other. □

Example 6.2.
- The only closed subgroups of S^1 are S^1 and the finite cyclic ones.
- Each finite cyclic group is selfdual; therefore so is every finite abelian group.
- $(S^1)^d$ and $\mathbb{Z}^d$ are dual of each other.
- $\mathbb{Z}_p$ and the Prüfer group

$$Z_{p^\infty} = \{\alpha \in S^1 : \alpha \text{ has order a power of } p\}$$

are dual of each other. This can be seen as follows: as $\chi \in \hat{\mathbb{Z}}_p$ is determined by $\chi(1)$, the group $\hat{\mathbb{Z}}_p$ is surely a subgroup of S^1 (with the discrete topology, not the induced one). Any α of order p^n surely determines the character $1 \mapsto \alpha$, i.e.,

$$\mathbb{Z}_p \to Z_{p^n} \to \langle \alpha \rangle.$$

Conversely, let χ be any character and let $O \subset S^1$ be a small neighborood of 1. By continuity of χ, there exists n so large that $\chi[p^n\mathbb{Z}_p] \subseteq O$; since $\chi[p^n\mathbb{Z}_p]$ is a subgroup of S^1, it must be trivial. Thus $\chi[\mathbb{Z}_p]$ is a subgroup of the group of p^n-roots of 1 in S^1, and thus $\chi(1)$ has order a power of p.

Theorem 6.3. *Let $G \in \mathbf{LCA}$ be compact 2nd countable. Then $\hat{G}$ is an at most countable discrete group, whose elements form an orthonormal basis for $L_2(G)$. Moreover, the map*

$$L_2(G) \to L_2(\hat{G}) = \ell_2$$
$$f \mapsto (\alpha_\chi)_{\chi \in \hat{G}}$$

where $\alpha_\chi = \langle \chi, f \rangle$ is the χ-th Fourier coefficient of f, is an isometric isomorphism of Hilbert spaces, with inverse $(\alpha_\chi) \mapsto \sum_\chi \alpha_\chi \chi$.

Proof. We only prove orthonormality; clearly $\|\chi\|_2 = 1$ for every χ. If $\chi \neq \psi$, then $\eta = \chi^{-1}\psi \neq \mathbb{1}$ and we claim $\int \eta \, \mathrm{d}\mu = 0$. Let $\eta(h) \neq 1$; since μ is invariant under translation by h we have

$$\int \eta(x) \, \mathrm{d}\mu(x) = \int \eta(hx) \, \mathrm{d}\mu(x) = \eta(h) \int \eta(x) \, \mathrm{d}\mu(x),$$

and our claim follows. □

Theorem 6.4. *Let $R(x) = ax$ be a rotation in the compact group G. T.f.a.e.*:

1. *R is ergodic*;
2. *a is a topological generator for G* (*i.e.*, *$\langle a \rangle^f = G$*);
3. *G is abelian and the only character χ which is trivial on a* (*i.e.*, *$\chi(a) = 1$*) *is $\mathbb{1}$.*

Example 6.5.
- $(a_1, \dots, a_d) \in \mathbb{T}^d$ is a topological generator iff $1, a_1, \dots, a_d$ are linearly independent over $\mathbb{Q}$.
- The topological generators of $\mathbb{Z}_p$ are precisely the invertible elements of the local ring $\mathbb{Z}_p$, namely those not in the maximum ideal $p\mathbb{Z}_p$.

Lemma 6.6. *Let G be compact abelian, (G, μ, φ) as in Lemma 2.9. Write $\hat{\varphi}$ for $\varphi^* \restriction \hat{G}$. Then $\hat{\varphi} : \hat{G} \to \hat{G}$ is an injective homomorphism.*

Theorem 6.7 (The Rokhlin-Halmos Theorem)**.** *The system (G, μ, φ) is ergodic iff all elements of $\hat{G}$, except $\mathbb{1}$, have infinite $\hat{\varphi}$-orbit.*

Example 6.8.
- No examples for G finite. Indeed, every surjective homomorphism is an automorphism, and $\{0\}$ is invariant.
- No examples for $\mathbb{Z}_p$.
- Epimorphisms of $\mathbb{T}^d$ correspond to nonsingular matrices with integer entries. Such an epimorphism is ergodic iff the corresponding matrix has no eigenvalues which are roots of unity.

7. The von Neumann mean ergodic theorem

Let H be a Hilbert space, $T : H \to H$ an isometry, preserving the scalar product but not necessarily invertible, and let $A_n = n^{-1} \sum_{k=0}^{n-1} T^k$. The operator $I - T$ is the *coboundary operator*; its kernel is the closed subspace $\mathrm{Inv} = \{f \in H : f = Tf\}$, and its image the —not necessarily closed— subspace Cob of *coboundaries*. We have an orthogonal decomposition $H = \mathrm{Cob}^\perp \oplus \mathrm{Cob}^{\perp\perp} = \mathrm{Cob}^\perp \oplus \mathrm{Cob}^f$, with projections $P : H \to \mathrm{Cob}^\perp$ and $Q : H \to \mathrm{Cob}^f$.

Theorem 7.1. *The following facts hold.*

1. $\mathrm{Cob}^\perp = \mathrm{Inv}$.

(2) *If* $g \in \mathrm{Cob}^f$, *then* $A_n g \to 0$.
(3) *For every* f, $A_n f \to Pf$.

Proof. (1) We claim that $Tf = f$ iff $T^*f = f$. For the "only if" implication, $T^*T = I$ for every isometry. Conversely, $\|f - Tf\|^2 = \langle f - Tf, f - Tf\rangle = 2\|f\|^2 - \langle f, Tf\rangle - \langle Tf, f\rangle = 2\|f\|^2 - \langle T^*f, f\rangle - \overline{\langle f, Tf\rangle} = 2\|f\|^2 - \langle T^*f, f\rangle - \overline{\langle T^*f, f\rangle} = 0$.

Now we get $f \in \mathrm{Cob}^\perp$ iff, for every g, $\langle f, g - Tg\rangle = 0$ iff, for every g, $\langle f, g\rangle = \langle f, Tg\rangle$ iff, for every g, $\langle f, g\rangle = \langle T^*f, g\rangle$ iff $f = T^*f$ iff $f = Tf$.
(2) If g is a true coboundary, say of f, then

$$A_n g = n^{-1} \sum_{k=0}^{n-1} T^k(f - Tf) = n^{-1}(f - T^n f),$$

whose norm is bounded, by the triangle inequality, by $n^{-1}2\|f\|$. Hence $A_n g \to 0$ in norm.

Let now $\varepsilon > 0$; we shall prove that for every n sufficiently large $\|A_n g\| < \varepsilon$. Let h be a true coboundary at distance $\leq \varepsilon/2$ from g. Then $\|A_n g\| = \|A_n(g - h) + A_n h\| \leq \|A_n(g - h)\| + \|A_n h\|$. Now,

$$\|A_n(g - h)\| = \|n^{-1} \sum_{k=0}^{n-1} T^k(g - h)\| \leq n^{-1} \sum_{k=0}^{n-1} \|T^k(g - h)\| = \|g - h\| \leq \varepsilon/2.$$

(3) We have $f = Pf + Qf$, and thus $A_n f = A_n Pf + A_n Qf = Pf + A_n Qf$, that converges to Pf. □

Corollary 7.2 (von Neumann, 1930)**.** *Let* (X, μ, R) *be a metric system. Then, for every* $f \in L_2(\mu)$, *the average* $n^{-1} \sum_{k=0}^{n-1} f \circ R^k$ *converges to an invariant function* $f^+ = f^+ \circ R$ *in* $L_2(\mu)$.

If (X, μ, R) is ergodic, then $\mathrm{Cob}^f = \mathrm{Inv}^\perp = \mathbb{1}^\perp = \{f : \int f \,\mathrm{d}\mu = 0\}$. Livsic-type theorems say that, under appropriate additional conditions, every f such that $\int f \,\mathrm{d}\mu = 0$ *is* a coboundary.

8. The Birkhoff pointwise ergodic theorem

Theorem 8.1 (The Maximal Inequality)**.** *Let* $Q : L_1(\mu, \mathbb{R}) \to L_1(\mu, \mathbb{R})$ *be a positive linear operator of norm* ≤ 1. *For* $f \in L_1(\mu, \mathbb{R})$ *and* $n, N \geq 1$, *let*

$$S_n f = f + Qf + Q^2 f + \cdots + Q^{n-1} f,$$
$$F_N = 0 \vee f \vee S_2 f \vee \cdots \vee S_N f.$$

We then have

$$(f > 0) = (F_1 > 0) \subseteq (F_2 > 0) \subseteq (F_3 > 0) \subseteq \cdots,$$

and thus

$$\int_{(F_1 > 0)} f \,\mathrm{d}\mu \geq \int_{(F_2 > 0)} f \,\mathrm{d}\mu \geq \cdots.$$

However, for every N, *we have*

$$\int_{(F_N > 0)} f \,\mathrm{d}\mu \geq 0,$$

whence

$$\int_{\bigcup_N (F_N>0)} f \, \mathrm{d}\mu \geq 0.$$

In our case, this means that f has nonnegative integral over the set of all $x \in X$ such that f has positive average at least once along the forward orbit of x.

Theorem 8.2 (The Wiener Maximal Ergodic Theorem, 1939)**.** *Let (X, μ, R) be a metric system, $g \in L_1(\mu, \mathbb{R})$, $a \in \mathbb{R}$. Let*

$$B_a = (a < \sup_{n \geq 1} A_n g),$$

and let Y be R-invariant. Then

$$a \cdot \mu(B_a \cap Y) \leq \int_{B_a \cap Y} g \, \mathrm{d}\mu.$$

As an example, take $B \subseteq Y = X$, $g = \mathbb{1}_B$, $a = \mu(B)^\epsilon > \mu(B)$ for some $0 < \epsilon < 1$. Then

$$B_a = (\mu(B)^\epsilon < \sup A_n \mathbb{1}_B)$$

is the set of points that, sometimes in the future, will have spent in B more time than due (in particular, $B_a \supseteq B$). The theorem then says that

$$\mu(B)^\varepsilon \cdot \mu(B_a) \leq \mu(B),$$

i.e.,

$$\mu(B_a) \leq \mu(B)^{1-\varepsilon}.$$

As ε decreases, B_a shrinks and $\mu(B)^{1-\varepsilon}$ gives an upper bound for the shrinking.

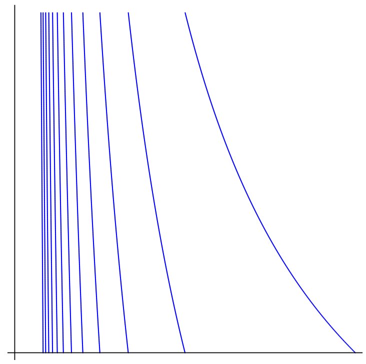
Theorem 8.3 (G. D. Birkhoff, 1930/31)**.** *Let (X, μ, R) be a metric system, $f \in L_1(\mu)$, $A_n f = n^{-1} \sum_{k<n} f \circ R^k$.*

1. *For μ-every x, $\lim_n (A_n f)(x)$ exists in $\mathbb{C}$.*
2. *Let f^+ be the limit function. Then $f^+ \in L_1(\mu)$, f^+ is R-invariant, $\int f^+ \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$; in particular, if the system is ergodic then f^+ is constant.*
3. *If the system is invertible and f^- is defined as above using R^{-1}, then $f^+ = f^-$.*

9. Normal numbers

Theorem 9.1 (E. Borel, 1909, with an erroneous proof)**.** *Lebesgue-all numbers $\alpha \in [0, 1]$ are normal to every base $b \geq 2$ (i.e., the expansion of α in base b contains every block of digits with the correct frequency).*

Proof. It follows either from the strong law of large numbers, or from the pointwise ergodic theorem (these tools have been rigorously established independently and more-or-less simultaneously). □



10. The Gauss map

The ordinary continued fraction expansion provides a homeomorphism between the *Baire space* $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ and $[0,1] \setminus \mathbb{Q}$. It conjugates the *Gauss map* $G(x) = \{1/x\}$ with the one-sided shift on $\mathcal{N}$. Note that $\mathcal{N}$ can be characterized as the only complete metric space which is separable, 0-dimensional, and not locally compact at any point (i.e., no nonempty open set has compact closure).

In a letter to Legendre dated January 30, 1812 [AS17, Appendix], Gauss wrote "12 years ago ... I found by very simple reasoning that"

$$\lim_{n\to\infty} \lambda(\{\alpha = [0, *_1, \dots, *_n, \alpha_{n+1}] : 1/\alpha_{n+1} \leq x\}) = \lim_{n\to\infty} \lambda(\{\alpha : G^n\alpha \leq x\}) = \log_2(1+x),$$

and asks for an extimation of the error term for large n.

We look at $G^0, G^1, G^2, \dots : ([0,1], \mathcal{B}, \lambda) \to ([0,1], \mathcal{B})$ as a stochastic process. Then $\lambda(G^n \leq x) = (G^n_*\lambda)[0,x] = M_n x$ is the repartition function of $G^n_*\lambda$. Gauss's statement means then that $G^n_*\lambda$ converges weakly to the probability γ whose repartition function is $\log_2(1+x)$. This probability γ has a density, namely

$$\mathrm{d}\gamma = \frac{1}{\log 2}\frac{1}{1+x}\,\mathrm{d}x,$$

and is named the *Gauss measure*. Gauss's problem took more than a century to be solved.

Theorem 10.1 (Kuzmin 1928)**.** *There exists $0 < c < 1$ such that, for every $x \in [0,1]$,*

$$\left|\lambda(G^n \leq x) - \log_2(1+x)\right| = O(c^{\sqrt{n}}).$$

The following year 1929 Lèvy improved the convergence speed to $O(0.7^n)$; a footnote to Lèvy's paper provides a good glimpse of pre-email and pre-ArXiv mathematics. *J'ai appris depuis, par une lettre de M. G. Pòlya, ... qu'au sixième Congres international des Mathèmaticiens (Bologne, septembre 1928), M. Kuzmin a indiqué la démonstration de cette formule. Cette démonstration n'ayant pas encore été publiée, je ne puis encore la comparer á celle donnée dans le présent travail.*

Kuzmin's results immediately implies —but it is stronger than— the following lemma.

Lemma 10.2. *G preserves γ.*

Proof. □

We thus have a stationary (hence i.d.) stochastic process $a_1, a_2, \ldots : ([0,1], \mathcal{B}, \gamma) \to \mathbb{N}$ with discrete-density function

$$m(a) = \gamma(I_a) = \log_2\left(1 + \frac{1}{a}\right) - \log_2\left(1 + \frac{1}{a+1}\right) = \log_2\left(1 + \frac{1}{a(a+2)}\right).$$

(Un)fortunately this is not an independent process. Indeed,

$$\begin{aligned} I_{a_1,\ldots,a_n} &= \{[0, a_1, \ldots, a_n, \alpha_{n+1}] : 1 < \alpha_{n+1} < +\infty\} \\ &= \left\{ \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} * \alpha_{n-1} : 1 < \alpha_{n+1} < +\infty \right\} = \left[\frac{p_n}{q_n}, \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right]. \end{aligned}$$

Therefore,

$$\gamma(I_{1,1}) = \gamma([1/2, 2/3]) = \log_2 \frac{1 + 2/3}{1 + 1/2} \neq (\gamma(I_1))^2.$$

For every n, we have $\gamma(a_n \geq a) = \log_2(1 + 1/a)$ and $\lambda(a_n \geq a) = 1/a$; there exists a constant $C > 1$ such that, for every a, these two numbers are within a C, C^{-1}-multiple of each other. Let $\psi : \mathbb{N} \to \mathbb{N}$ be a function, and consider the family of events $(a_n \geq \psi(n))$. By Borel-Cantelli we have:

(1) If $\sum \log_2(1 + 1/\psi(n)) < \infty$ then

$$\gamma\{x : a_n(x) \geq \psi(n) \text{ for infinitely many } n\} = 0$$

or, equivalently, if $\sum 1/\psi(n) < \infty$ then $\lambda\{x : \ldots\} = 0$.

If the a_n were γ-independent, we would analogously have

(2) If $\sum 1/\psi(n) = \infty$ then $\lambda\{x : \ldots\} = 1$.

The Borel-Bernstein theorem says that (2) holds anyway.

Theorem 10.3. $([0,1], \gamma, G)$ *is ergodic.*

Theorem 10.4. *For γ-every (i.e., λ-every) $\alpha = [0, a_1, a_2, \ldots]$ the following facts are true.*

(1) $b_1 \ldots b_n$ *of digits appears with frequency*

$$\left| \log_2 \frac{1 + p_n/q_n}{1 + (p_n + p_{n-1})/(q_n + q_{n-1})} \right|.$$

(2)

$$\lim_n (a_1 \cdots a_n)^{1/n} = \prod_{1 \leq a} \left(1 + \frac{1}{a(a+2)}\right)^{\log_2 a} = \textit{Khinchin's constant } K_0 = 2.685\cdots$$

(*analogous results hold for the harmonic mean and other means*).

(3)

$$\lim_n \frac{a_1 + \cdots + a_n}{n} = \infty.$$

(4) (*Khinchin-Levy,* 1935-37)

$$\lim_n \frac{\log q_n}{n} = \frac{\zeta(2)}{2 \log 2} = 1.18656\cdots.$$

(5)

$$\chi(\alpha) = \lim_{n\to\infty} \frac{\log\left|(G^n)'(\alpha)\right|}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \log\left|G'(G^k\alpha)\right|$$
$$= \frac{-2}{\log(2)}\int_0^1 \frac{\log(x)}{1+x}\,\mathrm{d}x = \frac{\zeta(2)}{\log(2)} = 2.37313\cdots.$$

(6)

$$h_\gamma(G) = \lim_{n\to\infty} \frac{-\log(\gamma(I_{a_1,\dots,a_n}))}{n} = \lim_{n\to\infty} \frac{-\log(\lambda(I_{a_1,\dots,a_n}))}{n}$$
$$= \lim_{n\to\infty} \frac{-\log|\alpha - p_n/q_n|}{n} = \frac{\zeta(2)}{\log(2)}.$$

11. The natural extension

The natural extension for the 1-sided shift, the tent map, and the Gauss map (the Nakada-Ito-Tanaka construction, 1977).

Lemma 11.1. (1) *A factor of an ergodic map is ergodic.*
(2) *The natural extension of an ergodic map is ergodic.*

12. Mixing

Let (a_n) be a sequence and b a point in some topological vector space. The three following conditions are in increasing order of strength.

(1) *Cesàro convergence*:
$$\lim_n \frac{1}{n}\sum_{k<n} a_k = b;$$
(2) *nameless convergence* (for normed spaces): $\|a_n - b\|$ Cesàro converges to 0;
(3) *ordinary convergence*.

Definition of mixing and weak mixing.

Theorem 12.1. *Let (X,μ,R) be a metric system. T.f.a.e.*:
(1) *the system is wmixing*;
(2) (X^2,μ^2,R^2) *is ergodic*;
(3) (X^2,μ^2,R^2) *is wmixing*;
(4) *for every ergodic (Y,ν,S), the system $(X\times Y,\mu\times\nu,R\times S)$ is ergodic*;
(5) *for every A, B, there exists $J\subset\omega$ of density 0 such that*
$$\lim_{J\not\ni n\to\infty} |\mu(A\cap R^{-n}B) - \mu(A)\mu(B)| = 0.$$

If R is invertible, then all of the above is equivalent to
(6) $U = R^*$ *has continuous spectrum on $L_2(\mu)$ (i.e., no eigenvalues and eigenfunctions, except for 1 and $\mathbb{C}\mathbb{1}$).*

An ergodic rotation R_a in a compact abelian group is never wmixing. Indeed, every character χ is an eigenfunction for $U = R_a^*$, with eigenvalue $\chi(a)$.

Theorem 12.2. *Every ergodic endomorphism of a compact abelian group is strong mixing.*

13. Stochastic processes

Let $(X_k) : (\Omega, P) \to (C, \mathcal{C})$ be a stochastic process, indexed by either ω or $\mathbb{Z}$. If for every finite index set J and every $\{C_j\}_{j \in J}$ we have

$$P\big(\bigcap\{X_j^{-1}C_j : j \in J\}\big) = P\big(\bigcap\{X_{j+1}^{-1}C_j : j \in J\}\big),$$

we say that the process is *stationary*. We have i.i.d. implies stationary implies i.d. Note that the reverse implications does not hold: take X_0 nontrivial simmetrically distributed, and consider $X_0, X_0, -X_0, -X_0, X_0, X_0, -X_0, -X_0, \ldots$.

For simplicity's sake we take $C = n$, a finite alphabeth; let $\mathcal{A} = X_0^{-1}\mathscr{P}(n)$.

Example 13.1. Let $(X, \mathcal{X}, \mu, R)$ be a metric system, with μ a probability. Let $\varphi_0 : X \to n$ be a measurable map, with associated partition $A_0, \ldots, A_{n-1}$. Then $\varphi_k = \varphi_0 \circ R^k$ is a stationary process.

The process $X_k : ([0,1], \lambda) \to \{0, \ldots, 9\}$ given by $X_k(\omega) = k$th decimal digit of ω is of this type.

Every stationary processes arises in this way. Indeed, any stochastic process amounts to a random variable $X : (\Omega, P) \to (\prod_k C, \prod_k \mathcal{C})$, and the stationarity condition corresponds to the shift-invariance of $X_* P$. Applying the construction in the example to $(\prod_k C, X_* P$, shift, projection to the 0th component) gives us the process we started with.

The best setting is when the "metric system with random variable" $(X, \mathcal{X}, \mu, R, \varphi_0)$ is independent because, in that case, we have an i.i.d. process and lots of tools available: the Kolmogorov 0-1 law, the weak and strong law of large numbers, the central limit theorem. Independence means that, for every r, the σ-algebras $\mathcal{A}$,$R^{-1}\mathcal{A}$,...,$R^{-r}\mathcal{A}$ are μ-independent. More explicitely, for every r and every tuple $(i_0, \ldots, i_r) \in n^{r+1}$, we must have

$$\mu(A_{i_0} \cap R^{-1}A_{i_1} \cap \cdots \cap R^{-r}A_{i_r}) = \mu(A_{i_0})\mu(A_{i_1}) \cdots \mu(A_{i_r}).$$

If, in addition, $\mathcal{X} = \bigvee_k R^{-k}\mathcal{A}$, then we say that $\mathcal{A}$ is an *independent generator*. The existence of an independent generator is the maximum degree of stochasticity: it means that the system is conjugate to a full Bernoulli shift.

14. Kryloff-Bogoliuboff and extreme points

Let (X, R) be a topological system, with X compact Hausdorff 2nd countable (hence metrizable).

Theorem 14.1. *$C(X)^*$ is a topological vector space under the weak-$*$ topology. Its subset $\mathcal{P}(X)$ of positive norm-1 functionals is closed, compact and convex. R_* is continuous and affine, and hence $\mathcal{P}(X, R) = \{\mu \in \mathcal{P}(X) : R_*\mu = \mu\}$ is closed, compact and convex, too.*

Theorem 14.2. *Let $\mathcal{E}(X, R)$ be the set of extreme points of $\mathcal{P}(X, R)$. Then*:

1. *if $\mu \in \mathcal{P}(X, R)$ is ergodic and $\nu \ll \mu$, then $R_*^n\nu$ Cesàro weakly converges to μ*;
2. *the points of $\mathcal{E}(X, R)$ are precisely the R-invariant ergodic measures*;
3. *if $\mu, \nu \in \mathcal{E}(X, R)$ are distinct, then $\mu \perp \nu$.*

Actually, Theorem 14.2(1) is an iff, and a similar characterization holds for wmixing and mixing [Wal82, Theorem 6.12].

Theorem 14.3. *Let $(\nu_n)_{n<\omega}$ be any sequence in $\mathcal{P}(X)$. For every n, let*

$$\mu_n = \frac{1}{n} \sum_{k<n} R_*^k \nu_n.$$

Then every accumulation point for (μ_n) is in $\mathcal{P}(X, R)$.

Corollary 14.4 (Kryloff-Bogoliuboff)**.** $\mathcal{P}(X, R) \neq \emptyset$ (*and hence* $\mathcal{E}(X, R) \neq \emptyset$).

Example 14.5. (1) The South-North map $x \mapsto 2x$ on $\mathrm{P}^1\mathbb{R}$.
(2) $x \mapsto x^2$ on $(0, 1)$.
(3) $x \mapsto x^2$ on $(0, 1)$ and $x \mapsto 1/2$ on $\{0, 1\}$.

15. Unique ergodicity

Theorem 15.1 (The Stone-Weierstrass Theorem)**.** *Let X be a compact 2nd countable metrizable space. Let A be*:

(1) *either an $\mathbb{R}$-subalgebra of $C(X, \mathbb{R})$, containing $\mathbb{1}$ and separating points*;
(2) *or a sub-$\mathbb{R}$-vector lattice of $C(X, \mathbb{R})$, containing $\mathbb{1}$ and separating points*;
(3) *or a $\mathbb{C}$-subalgebra of $C(X, \mathbb{C})$, closed under complex conjugation, containing $\mathbb{1}$ and separating points.*

Then, in each case, A is uniformly dense in the ambient space.

Corollary 15.2. *$C(X)$ is separable* (*i.e., contains a countable $\| \ \|_\infty$-dense subset*).

Theorem 15.3. *Let (X, R) be a topological system, with X compact Hausdorff with a countable basis. The following conditions are equivalent, and define the* unique ergodicity *of the system.*

(1) *$\mathcal{P}(X, R)$ is a singleton*;
(2) *$\mathcal{E}(X, R)$ is a singleton*;
(3) *for every $f \in C(X)$ there exists c_f such that $A_n f$ converges pointwise to $c_f \mathbb{1}$*;
(4) *as in* (3)*, with the further requirement that the convergence be uniform*;
(5) *as in* (3)*, with the restriction that the convergence holds for functions in some set whose $\mathbb{C}$-span is uniformly dense in $C(X)$*;
(6) *there exists $\mu \in \mathcal{P}(X)$ such that, for every x, $(R_*^n \delta_x)$ Cesàro converges to μ.*

If the above conditions hold, then the probability μ in (6) *is the unique element in $\mathcal{P}(X, R) = \mathcal{E}(X, R)$, and c_f in* (3) *is $\int f \, \mathrm{d}\mu$.*

Corollary 15.4. *The orbit of any point under any ergodic rotation in a compact abelian group is uniformly distributed w.r.t. the Haar measure.*

Definition 15.5. Let (X, R) be a topological system, $\mu \in \mathcal{P}(X, R)$, $x \in X$. If $(R_*^n \delta_x)$ Cesàro converges to μ, then we say that x is *μ-generic*.

Theorem 15.6. *If $\mu \in \mathcal{E}(X, R)$, then the set of μ-generic points has full μ-measure.*

Let the invertible topological system (X, R) be uniquely ergodic, with $\mathcal{P}(X, R) = \{\mu\}$. Let (G, ν) be a compact group, not necessarily abelian, and let $c : X \to G$ be a continuous function. These data determine the *skew product*

$$\begin{aligned} S : X \times G &\to X \times G \\ (x, g) &\mapsto (Rx, c(x)g). \end{aligned}$$

Theorem 15.7. *$\mu \times \nu \in \mathcal{P}(X \times G, S)$. If $\mu \times \nu \in \mathcal{E}(X \times G, S)$, then S is uniquely ergodic.*

Theorem 15.8. *Let α be irrational. Then*

$$\begin{aligned} S_n : \mathbb{T}^n &\to \mathbb{T}^n \\ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} &\mapsto \begin{pmatrix} \alpha + x_1 \\ x_1 + x_2 \\ \vdots \\ x_{n-1} + x_n \end{pmatrix} \end{aligned}$$

is uniquely ergodic.

Theorem 15.9 (The Weyl Equidistribution Theorem)**.** *Let $g(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$, with at least one of $a_n, \ldots, a_1$ irrational. Then $\big(g(t)\big)_{t<\omega}$ is uniformly distributed modulo 1.*

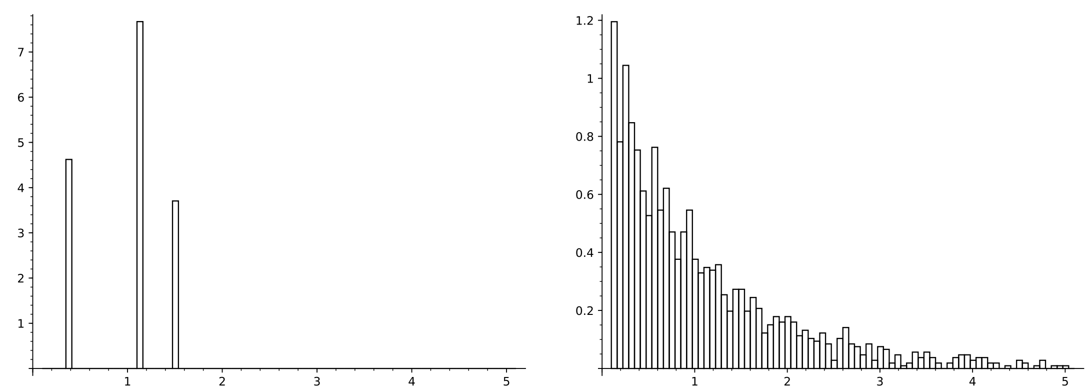


FIGURE 1. Gap distribution of $n/\sqrt{3}$ and of $n^2/\sqrt{3}$

16. Uniform distribution

Theorem 16.1. *Let $(x_n)_{n<\omega}$ be a sequence in $[0, 1]$. T.f.a.e., and define the fact that the sequence is* uniformly distributed (*or* equidistributed).

(1) *(δ_{x_n}) Cesàro converges to λ.*
(2) *For every $f \in C([0, 1], \mathbb{R})$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) = \int_0^1 f \, \mathrm{d}x.$$

(3) *As in* (2)*, with* f *varying in the set of Riemann-integrable functions* (*or in any set whose* $\mathbb{R}$*-span is uniformly dense in* $C([0,1],\mathbb{R})$*, such as the polynomials*).

(4) *For every* $0 \le a \le 1$,
$$\lim_{N\to\infty} \frac{\sharp\{n < N : x_n \le a\}}{N} = a.$$

(5) *As in* (2)*, with* f *varying in* $C(\mathbb{T},\mathbb{C})$.

(6) (*The Weyl Criterion*) *For every* $0 \neq k \in \mathbb{Z}$,
$$\sum_{n=0}^{N-1} \exp(2\pi k i x_n) = o(N).$$

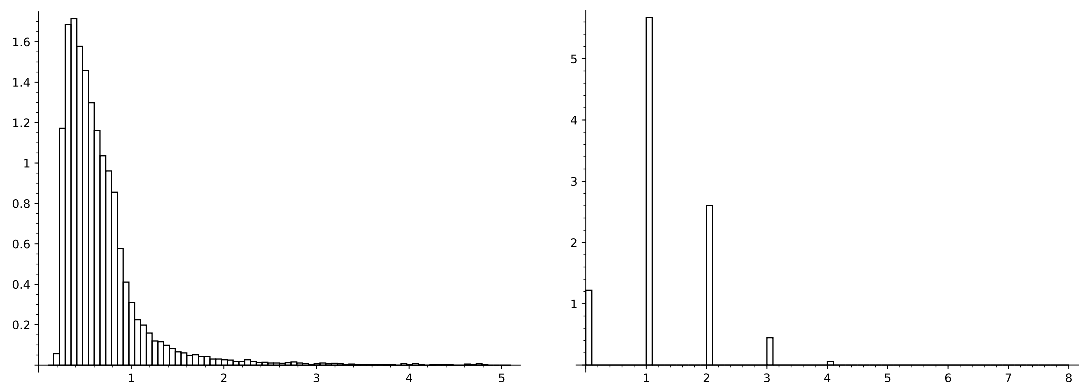


FIGURE 2. Gap and spacing distributions of the Farey sequence

17. The Iwasawa decomposition

Theorem 17.1. *Every* $M \in \mathrm{SL}_2\mathbb{R}$ *decomposes uniquely as* $M = KAN$ (*as well as* LAK, KAL, NAK)*, with* $K \in \mathrm{SO}_2\mathbb{R}$, $A = \begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}$ *for some* $a \in \mathbb{R}_{>0}$, N *upper triangular with* 1 *on the diagonal. Replacing* $\mathrm{SO}_2\mathbb{R}$ *with* $\mathrm{SU}_2\mathbb{C}$ *we obtain an analogous decomposition for* $\mathrm{SL}_2\mathbb{C}$.

18. The hyperbolic plane

Let H be a 2×2 hermitian-symmetric matrix of determinant < 0; it has the form
$$H = \begin{pmatrix} a & \beta \\ \bar\beta & d \end{pmatrix}.$$

Theorem 18.1. *The hermitian-symmetric form*
$$\begin{pmatrix} z \\ 1 \end{pmatrix}^* H \begin{pmatrix} z \\ 1 \end{pmatrix} = a|z|^2 + 2\,\mathrm{re}(\bar\beta z) + d$$
determines a circle $\mathcal{C}_H$ *and its "exterior" and "interior" disks* $\mathcal{D}_H$ *in* $\mathrm{P}^1\,\mathbb{C}$*, according to the value* 0, > 0, < 0 *of the form. Every generalized circle in* $\mathbb{R}^2 \cup \{\infty\}$ *has that form. Two matrices determine the same disk iff they differ multiplicatively by a real* > 0*. We have*:

(1) $a = 0$ *iff* $\mathcal{C}$ *goes through* ∞ *iff* $\mathcal{C}$ *is a straight line in* $\mathbb{C}$;
(2) $d = 0$ *iff* $\mathcal{C}$ *goes through* 0;
(3) $\mathrm{P}^1\,\mathbb{R}$ *and its interior* $\mathcal{H} = \{z : \mathrm{im}(z) > 0\}$ *correspond to* $H = \left(\begin{smallmatrix} & -i \\ i & \end{smallmatrix}\right)$;
(4) S^1 *and its interior* $\mathcal{D}$ *correspond to* $H = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$;
(5) *the action of* $\mathrm{SL}_2\,\mathbb{C}$ *on* $\mathrm{P}^1\,\mathbb{C}$ *preserves orientation and angles, and is transitive on disks* (*for this, the subgroup generated by* $\mathfrak{K} = \mathrm{SU}_2\,\mathbb{C}$ *and the diagonal subgroup* $\mathfrak{A} = \left\{\left(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right) : a \in \mathbb{R}_{>0}\right\}$ *suffices*);
(6) $B[\mathcal{D}_H] = \mathcal{D}_{(B^{-1})^* H B^{-1}}$;
(7) *the* Cayley matrix

$$C = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

maps $\mathcal{H}$ *to* $\mathcal{D}$;
(8) *the* (*setwise*) *stabilizer of* $\mathcal{D}$ *in* $\mathrm{PSL}_2\,\mathbb{C} = \mathrm{PGL}_2\,\mathbb{C}$ *is* (*by definition*) $\mathrm{PSU}_{1,1}\,\mathbb{C}$, *and*

$$\mathrm{PSU}_{1,1}\,\mathbb{C} = \left\{\begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} : |\alpha|^2 - |\beta|^2 = 1\right\};$$

(9) *the stabilizer of* $\mathcal{H}$ *is* $C^{-1}(\mathrm{PSU}_{1,1}\,\mathbb{C})C = \mathrm{PSL}_2\,\mathbb{R}$.

We write points of the tangent bundle $T\mathcal{H}$ as pairs (α, τ), with $\alpha \in \mathcal{H}$ and $\tau = x + iy \in \mathbb{C}^*$. The family of scalar products

$$\langle(\alpha, \tau), (\alpha, \tau')\rangle_\alpha = \frac{xx' + yy'}{(\mathrm{im}\,\alpha)^2} = \frac{\mathrm{re}\langle\tau, \tau'\rangle}{(\mathrm{im}\,\alpha)^2}$$

gives $\mathcal{H}$ the structure of a riemannian manifold, the *hyperbolic plane*. Its *unit tangent bundle* is then $T^1\mathcal{H} = \{(\alpha, \tau) \in T\mathcal{H} : |\tau| = \mathrm{im}\,\alpha\}$.

Lemma 18.2. (1) *For every* $G = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PSL}_2\,\mathbb{R}$ *and* $\alpha \in \mathcal{H}$, *we have*

$$\mathrm{im}(G * \alpha) = \frac{\mathrm{im}\,\alpha}{|c\alpha + d|^2}.$$

In particular, the left action of $\mathrm{PSL}_2\,\mathbb{R}$ *preserves the riemannian structure,* $T^1\mathcal{H}$, *and the area and volume forms*

$$\mathrm{d}A = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}, \quad \mathrm{d}V = \frac{\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\theta}{y^2}.$$

(2) *The action of* $\mathrm{PSL}_2\,\mathbb{R}$ *on* $T^1\mathcal{H}$ *is transitive with trivial stabilizers, thus inducing a bijection* $G \leftrightarrow G * (i, i)$; *under this bijection the above volume form corresponds to the left Haar measure.*

19. Geodesics

Lemma 19.1. *Let* $t_0 < t_1$. *Then the parametric path* $\gamma : [t_0, t_1] \to \mathcal{H}$ *given by* $\gamma(t) = i\exp(t)$ *has constant speed* 1 *and length* $t_1 - t_0$. *Any other path connecting* $\alpha = i\exp(t_0)$ *with* $\beta = i\exp(t_1)$

- *either has strictly greater length*;
- *or has the same length, and is a reparametrization of* γ.

The distance $d(\alpha,\beta)$ between points of $\mathcal{H}$ is, by definition, the infimum of all lengths of all paths connecting α with β.

We use
$$(\alpha_1,\alpha_2;\alpha_3,\alpha_4)=\frac{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)}$$
as our cross-ratio definition; note that $(0,\infty;z,1)=z$.

Lemma 19.2. *Assume the α_i are all distinct. Then their cross-ratio is in $\mathbb{R}^*$ iff they lie on the same generalized circle.*

Theorem 19.3. *Let (α,β) be an ordered pair of points in $\mathcal{H}$, at distance $d>0$.*

(1) *Then there exists a unique $G\in\mathrm{PSL}_2\mathbb{R}$ that maps i to α and $i\exp(d)$ to β; in particular, $\mathrm{PSL}_2\mathbb{R}$ acts simply transitively on the set of ordered pairs of points at a fixed distance.*
(2) *There exists a unique geodesics through α and β, namely the generalized half-circle $\mathcal{C}$ through them that is perpendicular to $\mathrm{P}^1\mathbb{R}$. The unique unit-speed geodesic path from α to β is $[0,d]\ni t\mapsto G*i\exp(t)$.*
(3) *Let ξ^-,ξ^+ be the points in which $\mathcal{C}$ intersects $\mathrm{P}^1\mathbb{R}$, with the agreement that traveling from ξ^- to ξ^+ along $\mathcal{C}$ we meet α first. Then $d=\log(\xi^-,\xi^+;\beta,\alpha)$.*

All of this determines a *right* action of $\mathbb{R}$ on $T^1\mathcal{H}$: given $(\alpha,\tau)=G*(i,i)$, we set
$$(\alpha,\tau)t=G*(i\exp(t),i\exp(t))=GA_t*(i,i),$$
where $A_t=\left[\begin{smallmatrix}\exp(t/2)&\\&\exp(-t/2)\end{smallmatrix}\right]$. This is the *geodesic flow* on $T^1\mathcal{H}$. Under the identification $\mathrm{PSL}_2\mathbb{R}\ni G\mapsto G*(i,i)$, it corresponds to the action of the diagonal subgroup $\mathfrak{A}$ on $\mathrm{PSL}_2\mathbb{R}$ by right translations.

20. Lie groups

Definition 20.1. A *Lie group* is a group $\mathfrak{G}$ which is also a C^∞ differential variety of dimension d, such that both the product and the inverse are C^∞ maps. A *homomorphism* of Lie groups is a C^∞ group homomorphism $\Phi:\mathfrak{H}\to\mathfrak{G}$. If Φ is injective and an embedding of varieties (i.e., the tangent map is injective), then it is an *embedding* of Lie groups. If, moreover, $\Phi[\mathfrak{H}]$ is closed in $\mathfrak{G}$, then we say that Φ is a *closed embedding*.

Theorem 20.2 (Cartan)**.** *If F is a closed subgroup of $\mathfrak{G}$, then there exists a unique Lie group structure on F that makes the identity map an embedding* (*automatically closed*) *of Lie groups.*

Definition 20.3. A *closed linear group* is a closed subgroup of some $\mathrm{GL}_n\mathbb{R}$, with the Lie group structure given by Cartan's theorem.

Example 20.4. $\mathrm{PSL}_2\mathbb{R}$ acts by conjugation on the space of trace 0 matrices in $\mathrm{Mat}_2\mathbb{R}$. Using this, one shows that $\mathrm{PSL}_2\mathbb{R}$ is isomorphic to the connected component of the identity in $\mathrm{SO}_{2,1}\mathbb{R}$, and hence is a closed linear group.

If q is any quadratic form on $\mathbb{R}^n$, with associated symmetric matrix Q, then $\mathrm{O}_Q\mathbb{R}=\{G\in\mathrm{GL}_n\mathbb{R}:G^\top QG=Q\}$. If Q is nonsingular then, up to conjugation, we only have $\mathrm{O}_n\mathbb{R}$, $\mathrm{O}_{n-1,1}\mathbb{R}$, …, $\mathrm{O}_{0,n}\mathbb{R}$; the first and last are compact, the others are not.

21. The exponential map

The exponential map $\exp : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is defined by the absolutely convergent series

$$\exp(U) = \sum_{k\geq 0} \frac{1}{k!} U^k.$$

Theorem 21.1. (1) $\exp$ *distributes under conjugation by elements of* $\mathrm{GL}_n \mathbb{R}$.
(2) $\det(\exp U) = \exp(\operatorname{tr} U)$; *in particular,* $\exp$ *has range in* $\mathrm{GL}_n \mathbb{R}$.
(3) *if U and V commute, then* $\exp(U+V) = \exp(U)\exp(V)$.
(4) *for every U, $t \mapsto \exp(Ut)$ is a Lie group homomorphism from $\mathbb{R}$ to* $\mathrm{GL}_n \mathbb{R}$.
(5)
$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_0} \big(G\exp(Ut)H\big) = GU\exp(Ut_0)H = G\exp(Ut_0)UH.$$
(6) *there are neighborhoods O_1 of 0 and O_2 of I over which* $\exp$ *is a diffeomorphism, with inverse*
$$\log(G) = (G-I) - \frac{1}{2}(G-I)^2 + \frac{1}{3}(G-I)^3 - \cdots$$
and jacobian matrix at 0 the identity.

Theorem 21.2. *Let $\mathfrak{G} \leq \mathrm{GL}_n \mathbb{R}$ be a d-dimensional closed linear group. Then there exists an open* (*in the topology of* $\mathfrak{G}$) *neighborhood O_2 of I such that $\log[O_2] = O_1$ is a neighborhood of 0 inside some d-dimensional subspace $\mathfrak{U} = \mathbb{R}O_1$ of $\mathbb{R}^{n\times n}$. Moreover*:
(1) $\exp[\mathfrak{U}] \subseteq \mathfrak{G}$*, and $\mathfrak{U}$ can be characterized as the maximum subspace of $\mathbb{R}^{n\times n}$ having that property*;
(2) *the map $U \mapsto u = \big[\exp(Ut)\big]'(0)$ identifies $\mathfrak{U}$ with $\mathfrak{g} = T_I\mathfrak{G}$*;
(3) *more generally, for every $G \in \mathfrak{G}$ the map $GU \mapsto \big[G\exp(Ut)\big]'(0)$ identifies $G[\mathfrak{U}]$ with $T_G\mathfrak{G}$*;
(4) *thus $\mathfrak{G}\times\mathfrak{U}$, $T\mathfrak{G}$, and $\dot{\bigcup}\big\{G[\mathfrak{U}] : G \in \mathfrak{G}\big\}$ are naturally identified.*

Example 21.3. Let $\mathfrak{G} = \mathrm{O}_q \mathbb{R}$. Then $\mathfrak{U} = T_I\mathfrak{G} = \{U \in \mathbb{R}^{n\times n} : I + \varepsilon U \in \mathfrak{G} + o(\varepsilon)\}$. We have $(I+\varepsilon U)^\top Q(I+\varepsilon U) = \varepsilon(U^\top Q + QU) + \varepsilon^2 U^\top QU$, and therefore $\mathfrak{U} = \{U : U^\top Q + QU = 0\}$. The condition amounts to QU being antisymmetric, and the space on antisymmetric $n\times n$ matrices has dimension $(n-1)n/2$; thus, if Q is nonsingular, $\mathrm{O}_Q \mathbb{R}$ has dimension $(n-1)n/2$.

22. The Lie algebra

Definition 22.1. A *Lie algebra* is a vector space $\mathfrak{a}$ over a field (usually $\mathbb{R}$ or $\mathbb{C}$), endowed with a bilinear map $[-,-] : \mathfrak{a}\times\mathfrak{a} \to \mathfrak{a}$ satisfying $[u,u] = 0$ and the Jacobi indentity

$$[[u,v],w] + [[v,w],u] + [[w,u],v] = 0.$$

Any associative $\mathbb{R}$-algebra (for example $\mathbb{R}^{n\times n}$) can be given a Lie algebra structure via $[U,V] = UV - VU$.

Under the above identification of $\mathfrak{G}\times\mathfrak{U}$ with $T\mathfrak{G}$, the group $\mathfrak{G}$ acts to the left on $\mathfrak{G}\times\mathfrak{U}$ in two different ways, via L_H and $R_{H^{-1}}$:

$$L_H : (G,U) \mapsto (HG, U), \tag{22.1}$$
$$R_{H^{-1}} : (G,U) \mapsto (GH^{-1}, HUH^{-1}). \tag{22.2}$$

In particular:

(1) $H - H^{-1}$ gives a left action of $\mathfrak{G}$ on $\mathfrak{U}$, i.e., a Lie group homomorphism

$$\mathrm{Ad} : \mathfrak{G} \to \mathrm{GL}(\mathfrak{U}) = \mathrm{GL}_d \mathbb{R},$$

which is called the *adjoint representation* of $\mathfrak{G}$;

(2) equivalently, each $H \in \mathfrak{G}$ acts on $\mathfrak{G}$ by left conjugation $H - H^{-1}$, and $\mathrm{Ad}\, H$ is the tangent map at the identity;

(3) the tangent map to Ad, namely

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{U}) = \mathbb{R}^{d\times d},$$

induces a parenthesis on $\mathfrak{g}$ and $\mathfrak{U}$:

$$[u, v] = w \text{ iff } (\mathrm{ad}\, u)(V) = W \text{ iff } [U, V] = W.$$

Lemma 22.2. *We have $[U, V] = UV - VU$, and therefore $\mathfrak{g} \simeq \mathfrak{U}$ is a Lie subalgebra of $\mathbb{R}^{n\times n}$.*

One can endow a closed linear group $\mathfrak{G}$ with various metrics, all of them inducing the euclidean topology (for example using any matrix norm), but usually these will not be translation-invariant. We construct a left-invariant riemannian structure on [the connected component of the identity of] $\mathfrak{G}$ by choosing arbitrarily a scalar product $\langle -, - \rangle$ on $\mathfrak{U}$ and exporting it to the various tangent spaces $T_G\mathfrak{G}$ via $\langle (G, U), (G, V) \rangle_G = \langle U, V \rangle$. By formula (22.1) this determines a left-invariant riemannian structure on $\mathfrak{G}$, and the standard procedure provides a left-invariant metric.

Lemma 22.3. *For every elements G of the closed linear group $\mathfrak{G}$ there exists a neighborhood of G over which the above metric is Lipschitz-equivalent to the metric induced by any given vector norm.*

Proof. See [EW11, §9.12]. □

Theorem 22.4. *Fix $G \in \mathfrak{G} = \mathrm{PSL}_2 \mathbb{R}$. Then the* stable manifold *of G under the geodesic flow is*

$$\mathcal{W}^s(G) = \{H : \lim_{t\to\infty} d(HA_t, GA_t) = 0\} = G\mathfrak{N},$$

namely the orbit of G under the stable horocyclic flow

$$\begin{aligned} &\mathfrak{G} \times \mathbb{R} \to \mathfrak{G}, \\ &(G, x) \mapsto G\left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right]. \end{aligned}$$

23. Fuchsian groups

Definition 23.1. Let G be a topological group which acts continuously on a topological space X. If for every x and every compact subset K of X the set $\{g : gx \in K\}$ is finite, then we say that the action is *properly discontinuous*.

Lemma 23.2. *Every discrete subgroup of a Hausdorff group is closed.*

Theorem 23.3. *Let Γ be a subgroup of $\mathrm{PSL}_2 \mathbb{R}$; the following conditions are equivalent and define a* fuchsian group.

(1) *the action of Γ on $\mathcal{H}$ is properly discontinuous*;
(2) *every Γ-orbit is discrete, and the stabilizer of every point is finite*;
(3) Γ *is discrete.*

Definition 23.4. Let Γ be a group that acts faithfully and in a measure-preserving way on the measure space (X, μ). A *fundamental domain* for the action is a measurable D such that $X = \bigcup_\gamma \gamma[D]$ and $\mu(D \cap \gamma[D]) = 0$ for every $\gamma \neq 1$. If $D \cap \gamma[D] = \emptyset$ for every $\gamma \neq 1$, then D is a fundamental domain *in the strict sense*.

Theorem 23.5. *For fuchsian groups Γ acting on $(\mathcal{H}, \mu)$ (μ being the measure induced by the riemannian area form), one can always construct fundamental domains D which satisfy*:

- *D is a convex regular closed set (i.e., $D = D^{of}$)*;
- *$D^o \cap G[D^o] = \emptyset$ for every $G \neq I$*;
- *$\mu(\partial D) = 0$.*

Proof. One such construction is due to Dirichlet: fix first z_0 with trivial stabilizer in Γ. Then set

$$D = \{z : \text{ for every } G \in \Gamma \setminus \{I\} \text{ we have } d(z, z_0) \leq d(z, G * z_0)\};$$

it is closed convex because it is an intersection of closed halfplanes. $\square$

We let matrices in $\mathrm{PSL}_2^{\pm} \mathbb{R}$ with determinant -1 act on $\mathcal{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * z = \frac{a\bar{z} + b}{c\bar{z} + d}.$$

Lemma 23.6. *The above is an isometric, orientation-reversing action. There is a bijection between matrices in $\mathrm{PSL}_2^{\pm} \mathbb{R}$ of determinant -1 and trace 0 and reflections in $\mathcal{H}$ of mirror a geodesic.*

Example 23.7. Triangle groups.

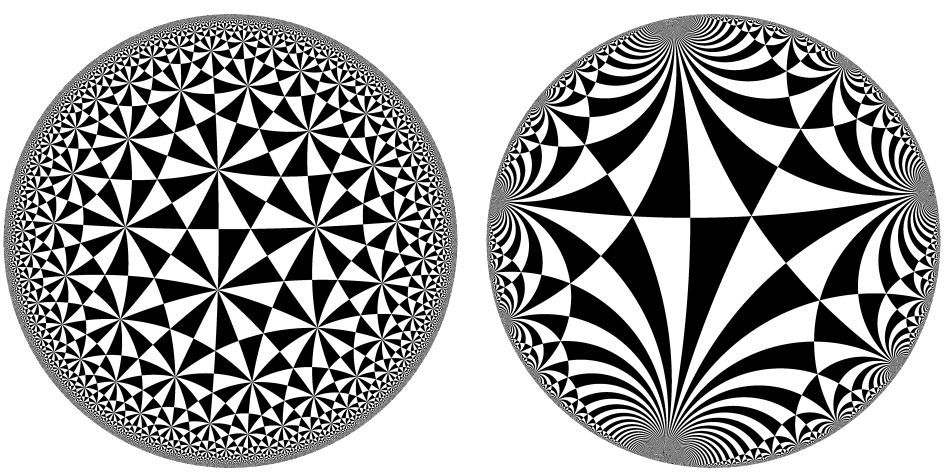


FIGURE 3. The $(2, 3, 7)$ and $(2, 3, \infty)$ extended triangle groups. By Tamfang - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=12806647

24. Dynamics on $\Gamma\backslash\mathcal{H}$

Theorem 24.1. *Let G be a locally compact Hausdorff group with left Haar measure λ. Let Γ be a countable discrete subgroup, with $\pi : G \to \Gamma\backslash G = X$ the canonical projection. Let M, M' be measurable transversals (i.e., fundamental domains in the strict sense for the left-translation action of Γ on G).*

(1) $\lambda(M) = \lambda(M')$.
(2) *If $\lambda(M)$ is finite then Γ* has finite comeasure *or* is a lattice *in G; this is surely the case if M can be taken compact, in which case we say that Γ* is cocompact, *or* is a uniform lattice. *If Γ is a lattice then*:
  (a) *G is unimodular*;
  (b) *the finite measure μ induced on X by $\mu(A) = \lambda(\pi^{-1}A \cap M)$ does not depend on the choice of M, and is right G-invariant.*

Since the projection $T^1\mathcal{H} \to \mathcal{H}$ has compact fibers, a fuchsian group is a lattice iff its action on $\mathcal{H}$ has a fundamental domain of finite area. As fuchsian lattices abound, $\mathrm{PSL}_2\,\mathbb{R}$ is unimodular.

Theorem 24.2 (The Margulis Lemma). *Let $\mathfrak{G}$ be a closed linear group, $\mathfrak{B} \ni B_t$ a 1-parameter subgroup, and $\Gamma < \mathfrak{G}$ a lattice. Let d be a left-invariant metric on $\mathfrak{G}$, and μ the probability on $\Gamma\backslash\mathfrak{G}$ constructed above. Let $f \in L_2(\mu)$ be $\mathfrak{B}$-invariant, and let G satisfy the following*:

- *for every $\varepsilon > 0$ there exists $C \in \mathfrak{G}$ and $B, B' \in \mathfrak{B}$ such that $d(C,G), d(BCB', I) < \varepsilon$.*

Then f is G-invariant.

Theorem 24.3. *Let Γ be a fuchsian lattice, and let $X = \Gamma\backslash\,\mathrm{PSL}_2\,\mathbb{R}$ and μ be as above. Then the geodesic flow – A_t on (X, μ) is ergodic.*

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