Università degli Studi di Udine Anno Accademico 1999/2000

Facoltà di Scienze Matematiche, Fisiche e Naturali
Corso di Laurea in Informatica

Esercizi di Analisi Matematica II

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Esercizio 1. Trovare l'integrale generale delle seguenti equazioni:

(a) $y' + 3y = t + e^{-2t}$,

(b) $y' + y = te^{-t} + 1$,

(c) $y' + \dfrac{2}{t}y = \dfrac{\sin t}{t} \quad (t > 0)$,

(d) $y' + \dfrac{3}{t}y = \dfrac{\operatorname{sen} t}{t^3} \quad (t < 0)$,

(e) $y' + \dfrac{2}{t}y = e^t \quad (t > 0)$.

Esercizio 2. Risolvere i seguenti problemi di Cauchy, specificando il massimo intervallo in cui la soluzione è definita:

(a) $\begin{cases} y' - y = 2te^{2t} \\ y(0) = 1 \end{cases}$

(b) $\begin{cases} y' + \dfrac{2}{t}y = \dfrac{\cos t}{t^2} \\ y(\pi) = 0 \end{cases}$

(c) $\begin{cases} y' + \dfrac{2y}{t} = \dfrac{\sin t}{t} \\ y(\pi/2) = 1 \end{cases}$

(d) $\begin{cases} y' + \dfrac{2y}{t} = t - 1 + \dfrac{1}{t} \\ y(1) = \frac{1}{2} \end{cases}$

(e) $\begin{cases} y' + \dfrac{y}{t} = \dfrac{e^t}{t} \\ y(1) = 1 \end{cases}$

(f) $\begin{cases} y' + \dfrac{\cos t}{\operatorname{sen} t}y = \dfrac{2}{\operatorname{sen} t} \\ y(\pi/2) = 1 \end{cases}$

Esercizio 3. Risolvere i seguenti problemi di Cauchy:

(a) $\begin{cases} y' = y \\ y(0) = 2 \end{cases}$

(b) $\begin{cases} y' = \dfrac{2t}{1 + 2y} \\ y(2) = 0 \end{cases}$

(c) $\begin{cases} y' = -\dfrac{\operatorname{sen} 2t}{\cos 3y} \\ y(\pi/2) = \pi/3 \end{cases}$

(d) $\begin{cases} y' = \dfrac{ty^3}{\sqrt{1 + t^2}} \\ y(0) = 1 \end{cases}$

Esercizio 4. Risolvere i seguenti problemi di Cauchy, specificando il massimo intervallo in cui la soluzione è definita:

(a) $\begin{cases} 1 + y^2 + ty^2 + (t^2 y + y + 2ty)y' = 0 \\ y(1) = -1 \end{cases}$

(b) $\begin{cases} \cos t \cos y - \dfrac{\cos t}{\operatorname{sen} t} - (\operatorname{sen} t \operatorname{sen} y)y' = 0 \\ y(\pi/2) = \pi/2 \end{cases}$

(c) $\begin{cases} \dfrac{t}{(t^2 + y^2)^{3/2}} + \dfrac{y}{(t^2 + y^2)^{3/2}}y' = 0 \\ y(1) = 1 \end{cases}$

(d) $\begin{cases} 3t^2 + 4ty + 2(y + t^2)y' = 0 \\ y(1) = 1 \end{cases}$

Esercizio 5. Mostrare che i seguenti problemi di Cauchy non hanno soluzione unica

(a) $\begin{cases} t^3 - t(1-t^2) + (y-1)y' = 0 \\ y(0) = 1 \end{cases}$ (b) $\begin{cases} t + 2y + (2t+y)y' = 0 \\ y(1) = -2\,. \end{cases}$

Esercizio 6. Trovare per quali valori della costante reale a le seguenti equazioni risultano esatte e risolvere quindi le equazioni usando tali valori di a, ricavando gli integrali generali in forma implicita:

(a) $ty^2 + at^2y + (t+y)t^2y' = 0$ (b) $ye^{2ty} + t + ate^{2ty}y' = 0\,.$

Esercizio 7. Trovare l'integrale generale delle seguenti equazioni:

(a) $y'' + y' = \operatorname{sen} t$ (b) $y'' - 3y' + 2y = 2t^2 + 1$

(c) $y'' - 3y' + 2y = 2t^3 - 9t^2 + 6t$ (d) $y'' + 4y = 5e^t - 4t^2$

(e) $y'' - 3y' - 4y = 5e^{4t}$ (f) $y'' - y' - 2y = 1 - 2t - 9e^{-t}$

(g) $y'' + 4y = 4\operatorname{sen}^2 t$

Esercizio 8. Determinare, per ciascuno dei seguenti problemi di Cauchy, i valori dei parametri a, b per i quali la soluzione $y(t)$ è limitata sulla semiretta $t \in [0, +\infty[$:

(a) $\begin{cases} y'' - 4y' + 3y = 20\cos t \\ y(0) = a,\ y'(0) = b \end{cases}$ (b) $\begin{cases} y'' - 3y' + 2y = 2t^2 + 1 \\ y(0) = a,\ y'(0) = b \end{cases}$

(c) $\begin{cases} y'' - 4y' + 3y = 2\cos t + 4\operatorname{sen} t \\ y(0) = a,\ y'(0) = b \end{cases}$ (d) $\begin{cases} y'' + 2y' + y = 7 + 75\operatorname{sen} 2t \\ y(0) = a,\ y'(0) = b \end{cases}$

Esercizio 9. Studiare i ritratti di fase per i seguenti sistemi (trovare i punti di equilibrio, l'equazione differenziale delle orbite; tracciare un disegno delle orbite specificandone il verso):

(a) $\begin{cases} x' = 2y \\ y' = -4x \end{cases}$ (b) $\begin{cases} x' = 2xy - 4y \\ y' = 8x - 4x^2 \end{cases}$ (c) $\begin{cases} x' = \cos x \\ y' = y\operatorname{sen} x \end{cases}$

(d) $\begin{cases} x' = 3x^2y^2 \\ y' = -2xy^3 \end{cases}$ (e) $\begin{cases} x' = \dfrac{x}{xy+1} \\ y' = -\dfrac{y}{xy+1} \end{cases}$ (f) $\begin{cases} x' = \dfrac{x^2 - xy}{xy+1} \\ y' = \dfrac{y^2 - xy}{xy+1} \end{cases}$

Esercizio 10. Discutere il moto dei seguenti sistemi newtoniani studiando, nel piano delle fasi, le corrispondenti linee di livello dell'energia totale:

(a) $x'' = x^2 - x$ (b) $x'' = -\dfrac{1}{1+x^2}$ (c) $x'' = (1-x^2)(x^2-4)$

(d) $x'' = -\sin x$ (e) $x'' = \dfrac{1 + 2x - x^2}{(x^2+1)^2}$

Esercizio 11. Determinare la natura del punto di equilibrio $(0,0)$ dei seguenti sistemi lineari e tracciare un grafico qualitativo delle orbite:

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ (b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 3/2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ (d) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(e) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 9 \\ 1/4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ (f) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 \\ -1 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(g) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ (h) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Esercizio 12. Rappresentare i seguenti insiemi (verificando che sono misurabili secondo Peano-Jordan) e calcolarne l'area e le coordinate baricentriche:

$$\begin{aligned}
D_1 &:= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; 0 \le y \le a^2 - x^2\right\} \qquad \text{con } a > 0 \text{ costante,} \\
D_2 &:= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; xy \ge 1,\ (x-4)(y-4) \ge 1,\ (x-2)^2 \le 4\right\}, \\
D_3 &:= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; y^2 + 1 \le x,\ 3 - x \le y\right\}, \\
D_4 &:= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; y + 2|x| \le 11/2,\ |y - x^2| \le 5/2\right\}, \\
D_5 &:= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; (1+|x|)(1+|y|) \le 4\right\}.
\end{aligned}$$

Esercizio 13. Calcolare l'area del sottoinsieme di $\mathbb{R}^2$ (misurabile secondo Peano-Jordan)

$$D = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \sqrt{x^2+y^2} + \frac{x}{\sqrt{x^2+y^2}} \le 1,\ x \ge 0\right\}.$$

(Suggerimento: usare le coordinate polari.)

Esercizio 14. Calcolare il volume dei seguenti solidi:

(a) cono di altezza h e base un cerchio di raggio R;
(b) sfera di raggio R;
(c) solido dato dalla rotazione di 2π attorno all'asse z del dominio del piano xz

$$D := \left\{(x,z) \;\middle|\; x \ge 1,\ 4 - 2x \ge |z|\right\};$$

(d) solido dato dalla rotazione di 2π attorno all'asse z del dominio del piano xz

$$D = \left\{(x,z) \;\middle|\; 0 \le x \le z^2,\ |z| \le 2\right\}.$$

RISPOSTE

Esercizio 1. (a) $y = Ce^{-3t} + t/3 - 1/9 + e^{-2t}$, (b) $y = Ce^{-t} + 1 + t^2e^{-t}/2$, (c) $y = C/t^2 - (\cos t)/t + (\operatorname{sen} t)/t^2$, (d) $y = (C - \cos t)/t^3$, (e) $y = (C + (t^2 - 2t + 2)e^t)/t^2$.

Esercizio 2. (a) $y = 3e^t + 2(t-1)e^{2t}$, (b) $y = (\operatorname{sen} t)/t^2$, (c) $y = (\pi^2/4 - 1 - t\cos t + \operatorname{sen} t)/t^2$, (d) $y = (3t^4 - 4t^3 + 6t^2 + 1)/(12t^2)$, definita per $t > 0$, (e) $y = (e^t + 1 - e)/t$, definita per $t > 0$, (f) $y = (2t + 1 - \pi)/\operatorname{sen} t$, definita per $0 < t < \pi$.

Esercizio 3. (a) $y = 2e^t$, (b) $y = (\sqrt{4t^2 - 15} - 1)/2$, (c) $y = (\pi - \operatorname{arcsen}(3\cos^2 t))/3$, (d) $y = 1/\sqrt{3 - 2\sqrt{1 + t^2}}$.

Esercizio 4. (a) $y = -\sqrt{6 - 2t}/(1 + t)$ per $-1 < t < 3$; (b) $y = \arccos((\log \operatorname{sen} t)/\operatorname{sen} t)$ per $\operatorname{arcsen} t_0 < t < \pi - \operatorname{arcsen} t_0$, dove t_0 è l'unica soluzione di $\log t = -t$; (c) $y = \sqrt{2 - t^2}$ per $|t| < \sqrt{2}$; (d) $y = -t^2 + \sqrt{t^4 - t^3 + 4}$ per $t \in \mathbb{R}$.

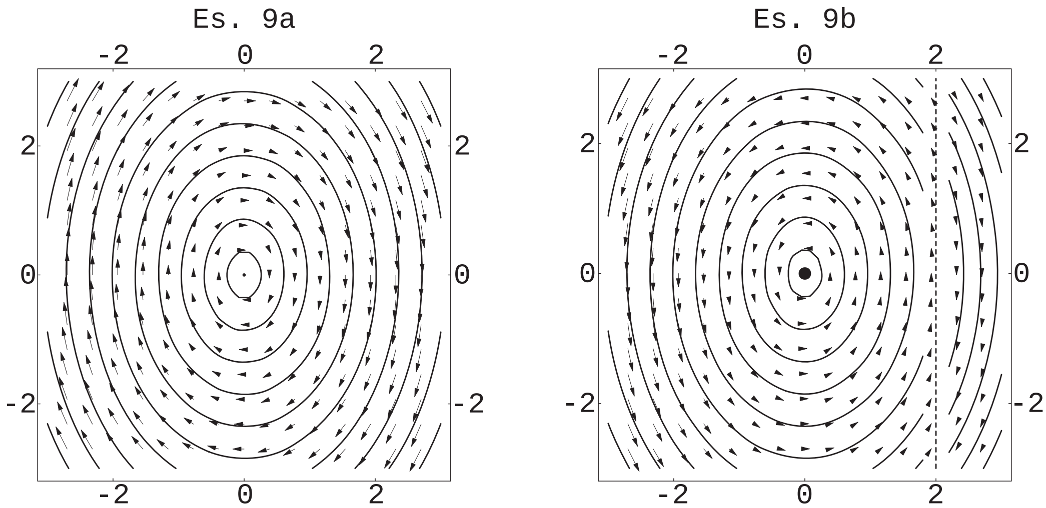
Esercizio 5. (a) ha due soluzioni $y = 1 \pm t\sqrt{1 - t^2}$, definite per $-1 < t < 1$; (b) non ha soluzioni in senso proprio, mentre ci sono due soluzioni $y = -2t \pm \sqrt{3t^2 - 3}$ in senso improprio, non derivabili per $t = 1$.

Esercizio 6. (a) $a = 3$, $2t^3y + t^2y^2 = c$; (b) $a = 1$, $t^2 + e^{2ty} = c$.

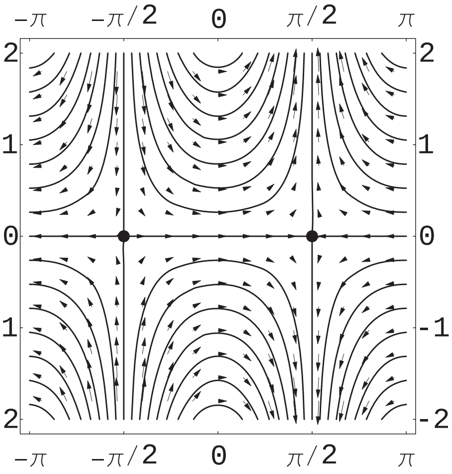
Esercizio 7. (a) $y = a + be^{-t} - (\operatorname{sen} t + \cos t)/2$; (b) $y = ae^t + be^{2t} + t^2 + 3t - 4$; (c) $y = ae^t + be^{2t} + t^3$; (d) $y = a\cos 2t + b\operatorname{sen} 2t + e^t - t^2 + 1/2$; (e) $y = ae^{-t} + be^{4t} + te^{4t}$; (f) $y = ae^{-t} + be^{2t} + t - 1 + 3te^{-t}$; (g) $y = a\operatorname{sen} 2t + b\cos 2t + (1 - t\operatorname{sen} 2t)/2$.

Esercizio 8. (a) $a = 2$, $b = -4$; (b) $a = 4$, $b = 3$; (c) $a = 1$, $b = 0$; (d) $a, b \in \mathbb{R}$.

Esercizio 9.



Es. 9c



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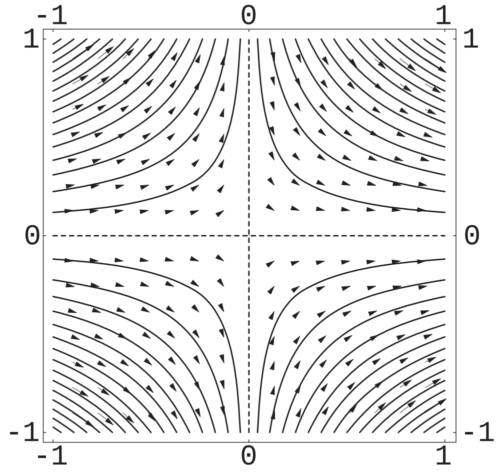

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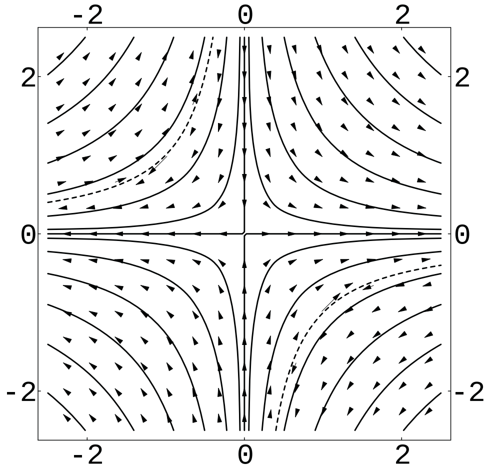

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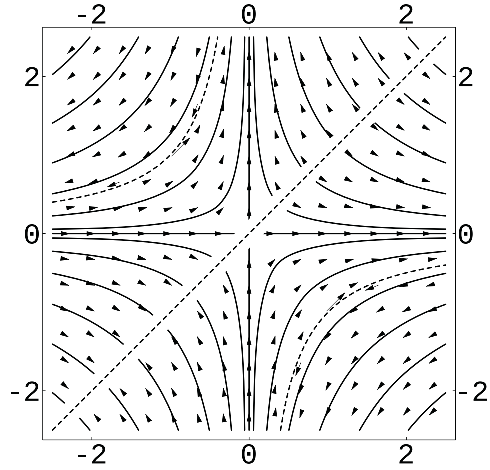
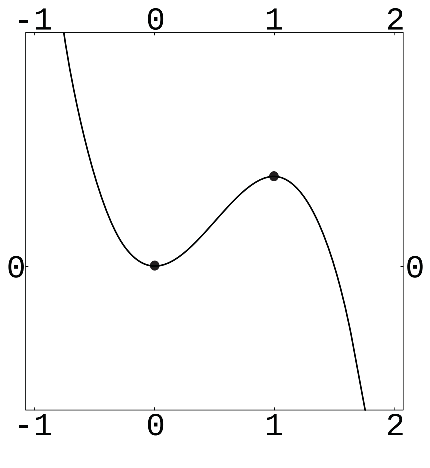
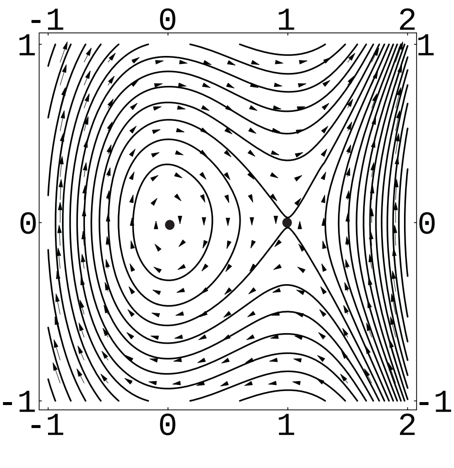


Esercizio 10.

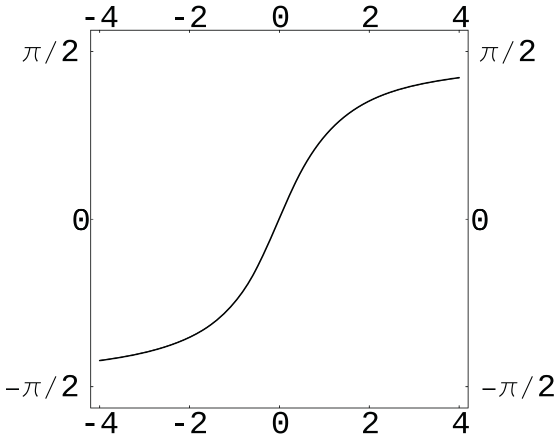
Es. 10a, potenziale



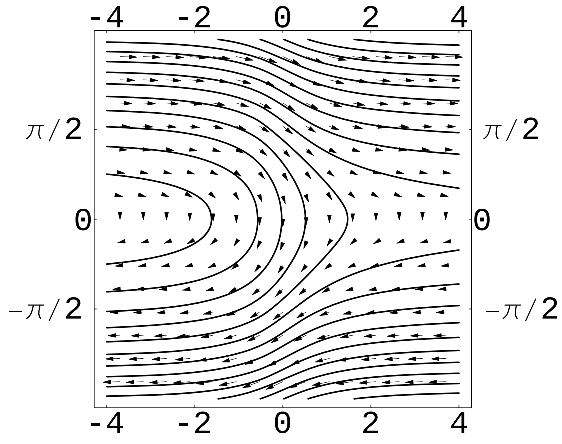
Es. 10a, diagr. di fase



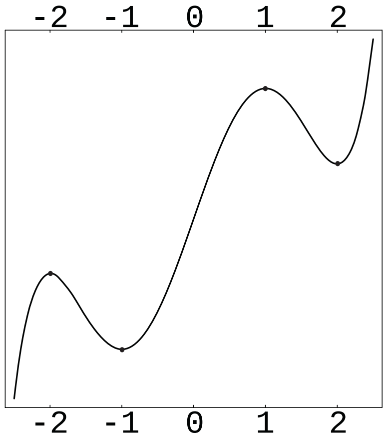
Es. 10b, potenziale



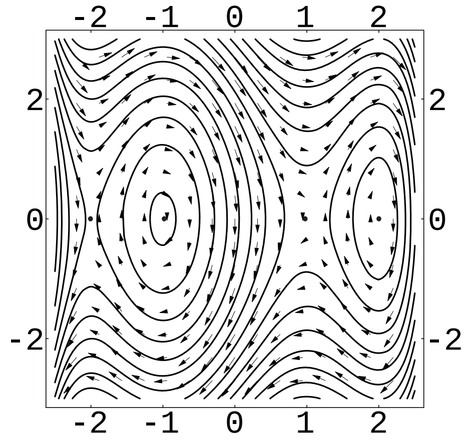
Es. 10b, diagr. di fase



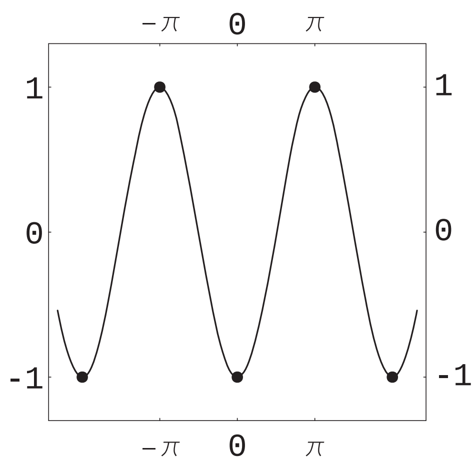
Es. 10c, potenziale



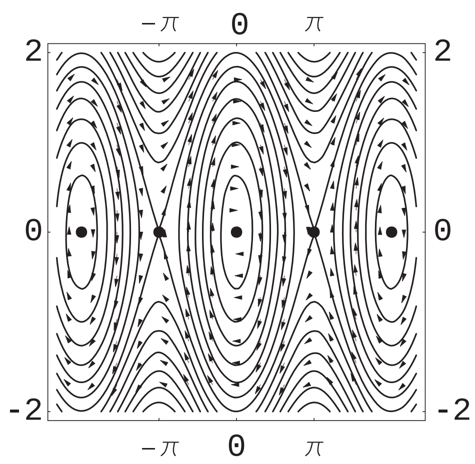
Es. 10c, diagr. di fase

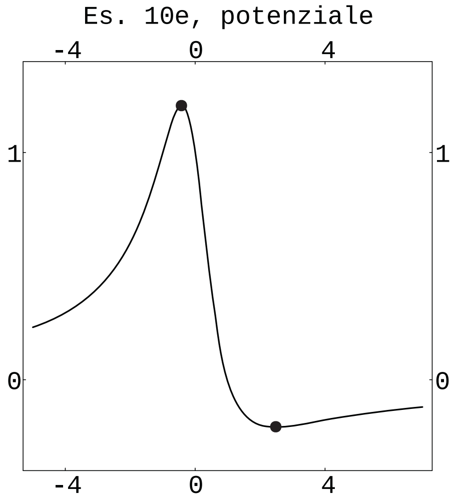


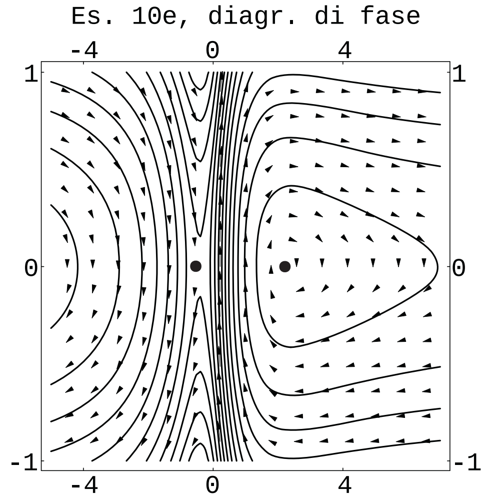
Es. 10d, potenziale



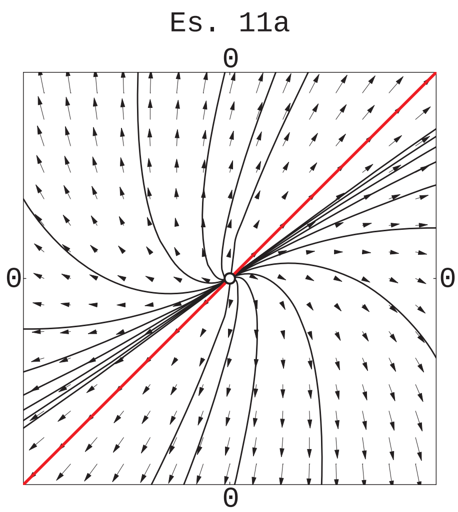
Es. 10d, diagr. di fase

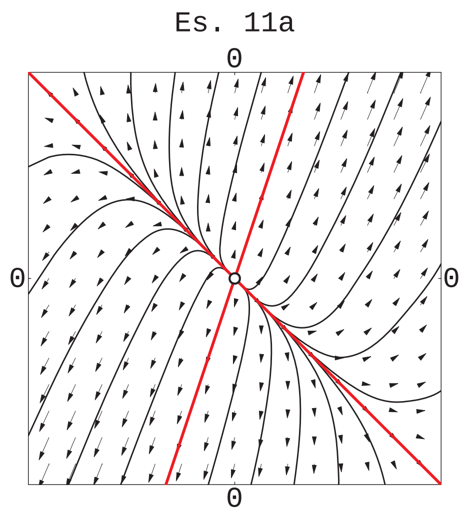


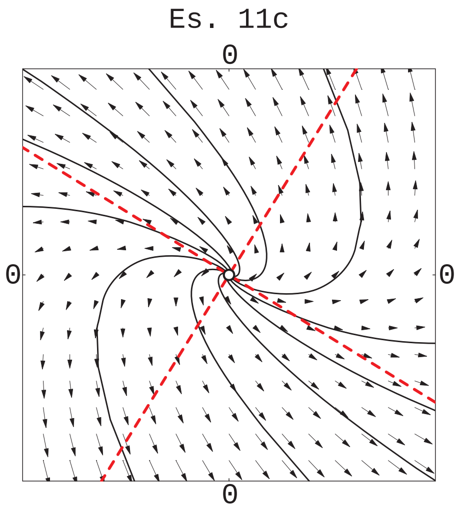


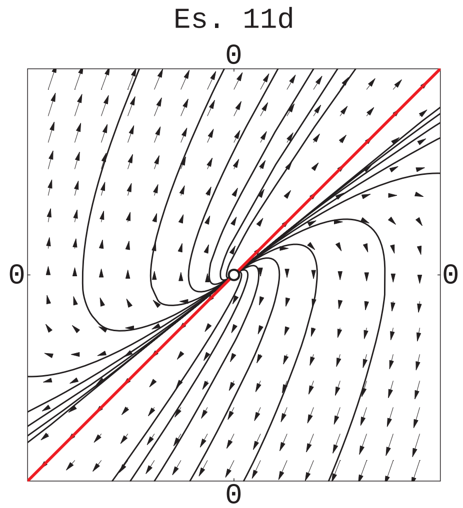


Esercizio 11. (a) nodo improprio instabile; (b) nodo instabile; (c) fuoco instabile; (d) nodo improprio instabile; (e) nodo stabile; (f) centro; (g) nodo improprio stabile; (h) punto di sella .

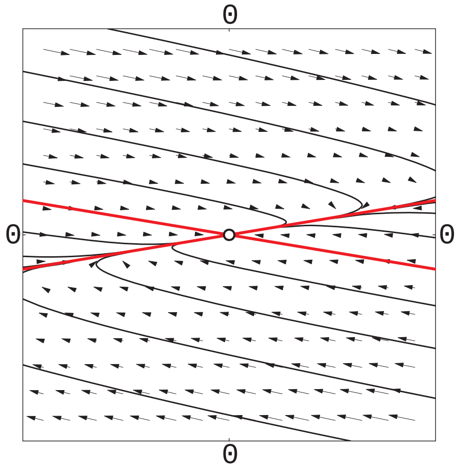




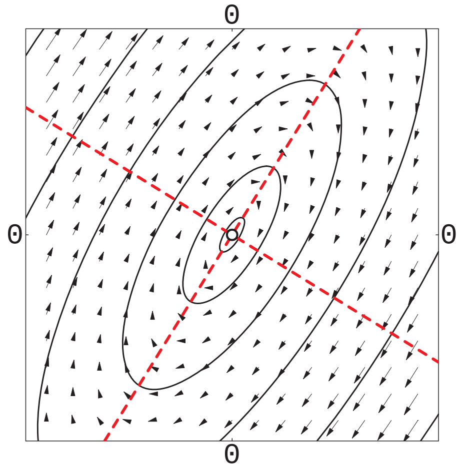




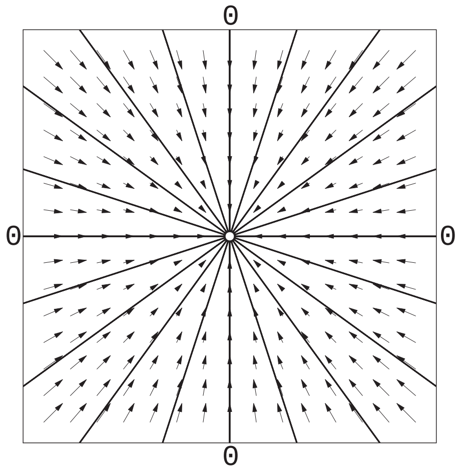
Es. 11e



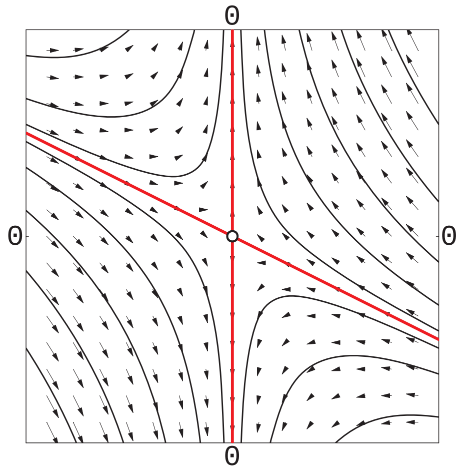
Es. 11f



Es. 11g

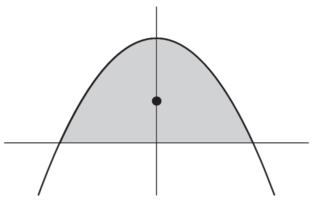


Es. 11h

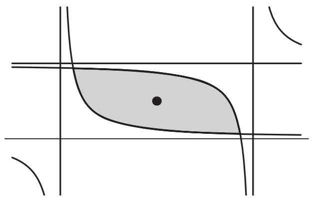


Esercizio 12. area$(D_1) = 4a^3/3$, baric$(D_1) = (0, 2a^2/5)$; area$(D_2) = 8\sqrt{3} + 2\log(7 - 4\sqrt{3})$, baric$(D_2) = (2, 2)$; area$(D_3) = 9/2$, baric$(D_3) = (13/5, -1/2)$; area$(D_4) = 46/3$, baric$(D_4) = (0, 137/230)$; area$(D_5) = 4(-3 + 4\log 4)$, baric$(D_5) = (0, 0)$.

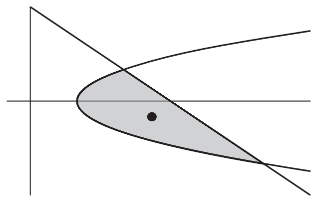
D_1



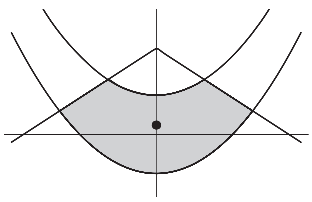
D_2



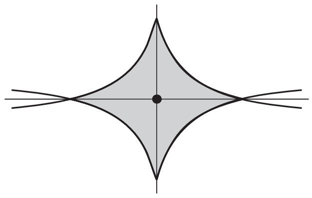
D_3



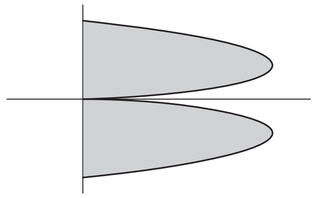
D_4



D_5



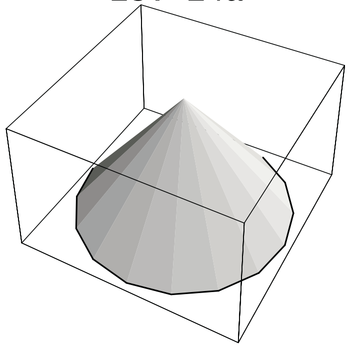
Es. 13



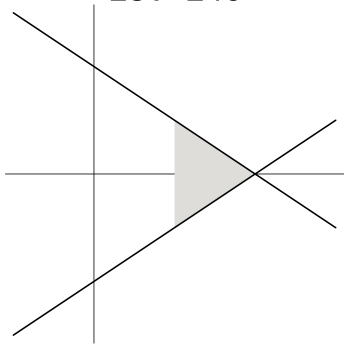
Esercizio 13. $(3\pi - 8)/4$.

Esercizio 14. (a) $\pi R^2 h/3$; (b) $4\pi R^3/3$; (c) $16\pi/3$; (d) $64\pi/5$.

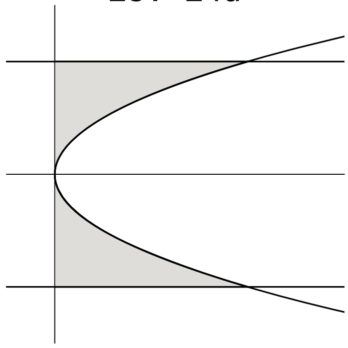
Es. 14a



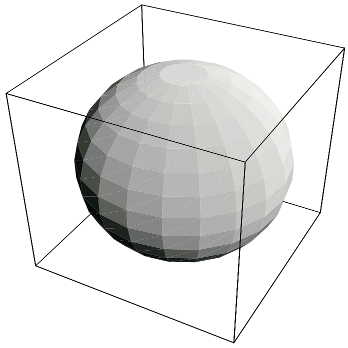
Es. 14c



Es. 14d



Es. 14b



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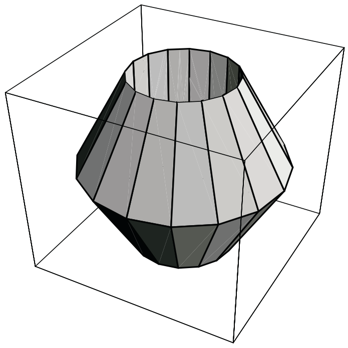

Es. 14d