A computational approach for the stability of a mechanical system

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1 The problem

We study the stability of the equilibrium $(0, 0)$ for the family of systems

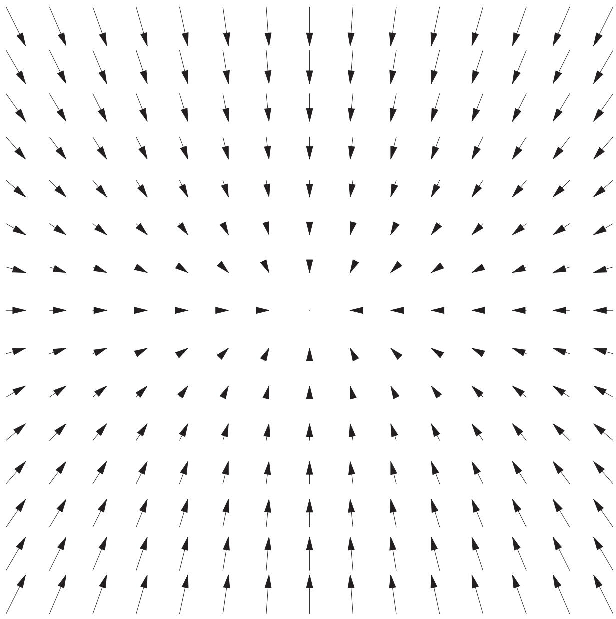
$$\begin{cases} \ddot{x} = -f(x)x \\ \ddot{y} = -g(x)y \end{cases}$$

when f, g are smooth functions defined in a neighbourhood of $0 \in \mathbb{R}$ and $f(0) > 0, g(0) > 0$.

These are the equation of motion of a point in the force field $(-f(x)x, -g(x)y)$. The origin $(0, 0)$ is an equilibrium.

2 A simple example

Take $f(x) \equiv 1$, $g(x) \equiv 2$. The force field looks like this:



The force is not exactly central, but the arrows point generally not very far from the origin. The brazilian authors call this class of forces *"força que aponta"*.

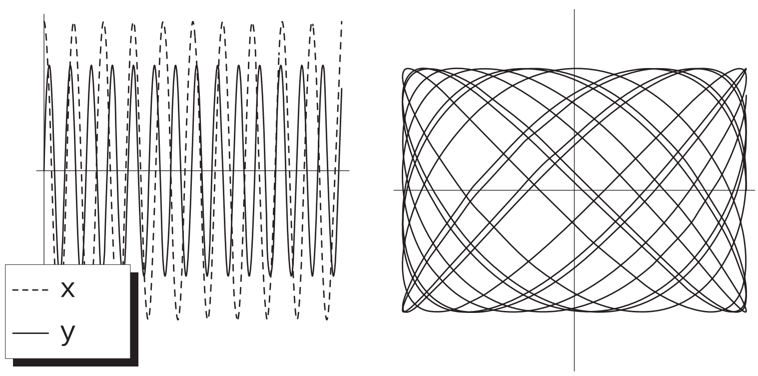
Since the force tends to draw the point towards the origin, we expect that the origin is a *stable equilibrium* for our mechanical system.

With $f(x) \equiv 1$, $g(x) \equiv 2$ this is verified easily because we can solve the system explicitly:

$$x(t, 0, x_0, \dot{x}_0) = x_0 \cos t + \dot{x}_0 \sin t \,,$$

$$y(t, 0, y_0, \dot{y}_0) = y_0 \cos t\sqrt{2} + \frac{\dot{y}_0}{\sqrt{2}} \sin t\sqrt{2} \,,$$

With the initial data $(x_0, \dot{x}_0) = (1, 0)$, $(y_0, \dot{y}_0) = (0, 1)$ the solutions look like this:



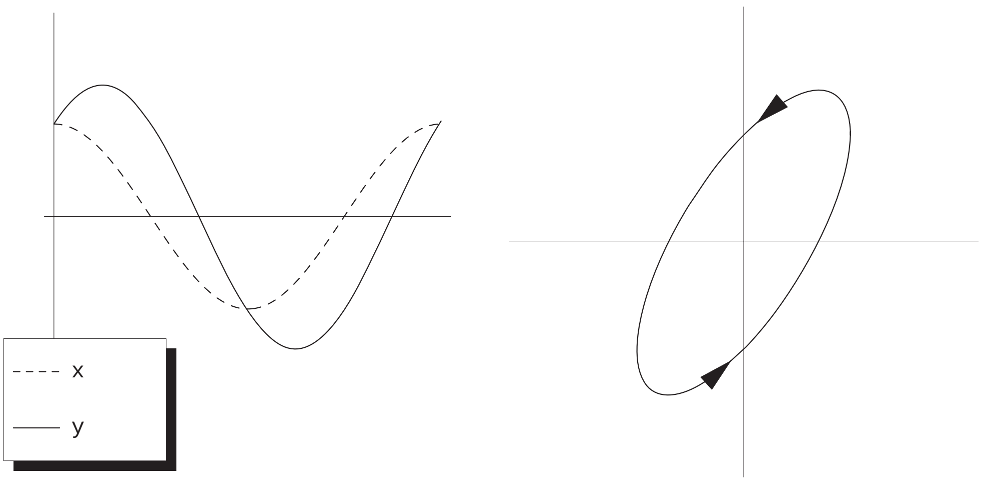
3 Instability in the central force case

When $f \equiv g$ the force field is *central and attractive*. Still, Barone, Cesar and Gaetano Zampieri found already in the 1980s that these systems are *generically unstable*.

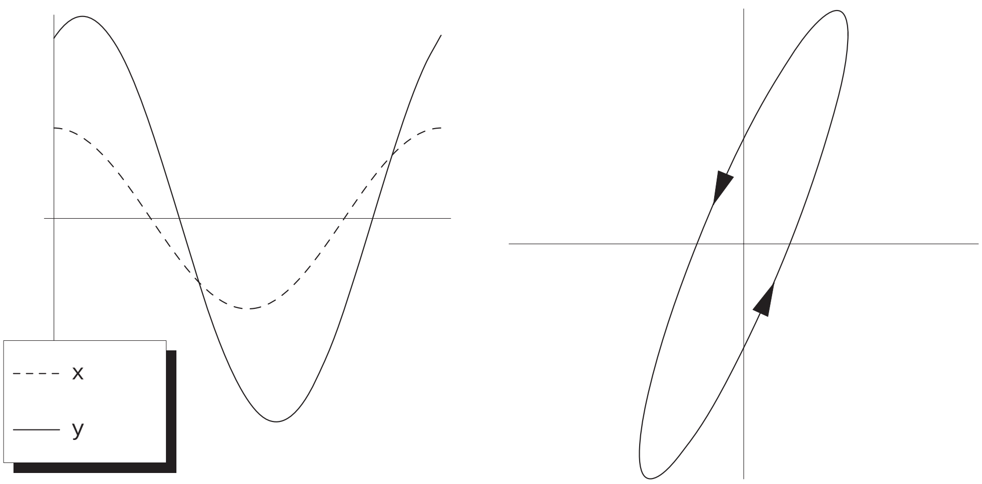
Among other things, they found that a sufficient condition for instability is

$$4f'(0)^2 - 3f(0)f''(0) \neq 0\,.$$

For example, with $f(x) := g(x) := 1 - 2x^2$ and $(x_0, \dot{x}_0) = (1/10, 0)$, $(y_0, \dot{y}_0) = (1/10, 1/10)$ the trajectory looks like this initially



but after 31 cycles they are stretched vertically:



and the vertical stretching goes on in an approximately linear way.

We are seeing a *parametric resonance*.

4 The strategy

The first equation of our system

$$\begin{cases} \ddot{x} = -f(x)x \\ \ddot{y} = -g(x)y \end{cases}$$

is similar to the pendulum equation, and it is well-known: in particular, all solutions are periodic.

Let $x(t)$ be the solution of $x = -f(x)x$ with initial data $x(0) = x_0$, $\dot{x}(0) = 0$, periodic of period $\tau(x_0)$. Plug $x(t)$ into $y = -g(x)y$: we get a linear equation in y with periodic coefficient (Hill's equation):

$$\ddot{y} = -g\big(x(t)\big)y\,.$$

As a first-order system this becomes

$$\frac{d}{dt}\begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -g\big(x(t)\big) & 0 \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \end{pmatrix}.$$

Let $\Phi(t, x_0)$ be the matrix solution of the problem

$$\dot{\Phi}(t, x_0) = \begin{pmatrix} 0 & 1 \\ -g\big(x(t)\big) & 0 \end{pmatrix} \Phi(t, x_0)\,,$$

$$\Phi(0, x_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,.$$

Define the *Floquet matrix* $\Psi(x_0)$ as

$$\Psi(x_0) := \Phi\big(\tau(x_0), x_0\big)\,.$$

The stability of the origin for our system depends on the asymptotic behaviour of the operator norm of the iterates of $\Psi(x_0)$:

Theorem 4.1 (stability in terms of Floquet iterate asymptotics). *The origin is a stable equilibrium for the system $\ddot{x} = -xf(x)$, $\ddot{y} = -yg(x)$ if and only if*

$$\limsup_{x_0 \to 0} \Big(\sup_{n \in \mathbb{N}} \big\| \Psi(x_0)^n \big\| \Big) < +\infty\,.$$

5 General results on the Floquet matrix

Proposition 5.1. *If f and g are of class C^n for some $n \geq 1$ then also the function $x_0 \mapsto \Psi(x_0)$ is of class C^n, and*

$$\Psi(0) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

where $\alpha = 2\pi\sqrt{g(0)/f(0)}$. The determinant of $\Psi(x_0)$ is constantly 1, and the two diagonal entries of $\Psi(x_0)$ coincide.

In particular, $\Psi(x_0)$ has the form

$$\Psi(x_0) = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$$

with $a^2 - bc = 1$.

Proposition 5.2 (iterate asymptotics for a fixed matrix). *Consider a real matrix* $A := \left(\begin{smallmatrix} a & b \\ c & a \end{smallmatrix}\right)$ *with* $\det A = 1$. *Then four cases are possible:*

i. $|a| > 1$ *and* $bc \neq 0$*: then* $\|A^n\| \to +\infty$ *as* $n \to +\infty$ *with exponential growth.*

ii. $|a| = 1$ *and either* $b \neq 0$ *or* $c \neq 0$*: then* $\|A^n\| \to +\infty$ *as* $n \to +\infty$ *with linear growth.*

iii. $|a| = 1$, $b = c = 0$*: then* $n \mapsto A^n$ *is constant.*

iv. $|a| < 1$ *and* $bc < 0$ *and: then for all* $n \in \mathbb{N}$

$$\min\left\{\sqrt{-b/c}, \sqrt{-c/b}\right\} \leq \|A^n\| \leq$$
$$\leq \max\left\{\sqrt{-b/c}, \sqrt{-c/b}\right\}$$

and

$$\limsup_{n\to+\infty} \|A^n\| \geq \frac{1}{2} \max\left\{\sqrt{-b/c}, \sqrt{-c/b}\right\}.$$

6 n-decidability

Definition 6.1. *We will say that the system is n-decidable if the stability or instability of the origin can be decided from the values of the derivatives of $f(x)$ and $g(x)$ at $x = 0$ of orders 0 to n.*

It is known that not all of our systems are *finitely decidable* (n-decidable for some n). However, finite decidability is generic:

Theorem 6.1 (0-decidability). *If $4g(0)/f(0)$ is not the square of a nonzero integer, then the origin is a stable equilibrium for the system $\ddot{x} = -xf(x)$, $\ddot{y} = -yg(y)$.*

The example $f(x) \equiv 1$, $g(x) \equiv 2$ that we have seen is then confirmed to be stable, because $4g(0)/f(0) = 8$ is not a square integer.

When $4g(0)/f(0)$ is the square of an integer, we may decide stability if we know enough about the asymptotic expansion of $\Psi(x_0)$ as $x_0 \to 0$.

Theorem 6.2 (stability from asymptotic expansion of Ψ). *Suppose that $\Psi(x_0)$ as $x_0 \to 0$ has the expansion*

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & bx_0^n \\ cx_0^m & 0 \end{pmatrix} + \\ + \begin{pmatrix} o(1) & o(x_0^n) \\ o(x_0^m) & o(1) \end{pmatrix}$$

for some $n, m \in \mathbb{N} \setminus \{0\}$, $b, c \in \mathbb{R}$.

- *if $bc > 0$ the origin is exponentially unstable: any neighbourhood of the origin of $\mathbb{R}^4$ has an initial point $(x_0, \dot{x}_0, y_0, \dot{y}_0)$ for which $y(t)$ is unbounded (exponentially) as $t \to +\infty$.*
- *If $bc < 0$ and $n \neq m$ then the origin is unstable with bounded trajectories: every solution starting near the origin is bounded, but there exists a sequence of initial points converging to zero for which $\sup_{t \geq 0} \|y_n(t)\|$ diverges to $+\infty$ as $n \to +\infty$.*

- *If* $bc < 0$ *and* $n = m$ *the origin is stable.*
- *If* $n \geq m$, $b \neq 0$ *and* $c = 0$ *the origin is unstable, but we can't say of which kind.*

We want to compute the asymptotic expansion of the Floquet matrix $\Psi(x_0)$ as $x_0 \to 0$.

7 Variations equations for x

Let $t \mapsto x(t;x_0)$ be the solution of

$$\ddot{x} = -xf(x)\,, \qquad x(0) = x_0\,, \quad \dot{x}(0) = 0\,.$$

Setting

$$\mu_n(t) := \frac{\partial^n x}{\partial x_0^n}(t;0)\,.$$

we get the equations

$$\ddot{\mu}_0 + f(\mu_0)\mu_0 = 0\,, \qquad \mu_0(0) = 0\,, \quad \dot{\mu}_0(0) = 0\,,$$

$$\ddot{\mu}_1 + f(0)\mu_1 = 0\,, \qquad \mu_1(0) = 1\,, \quad \dot{\mu}_1(0) = 0\,,$$

$$\ddot{\mu}_2 + f(0)\mu_2 = -2f'(0)\mu_1^2\,, \qquad \mu_2(0) = 0\,, \quad \dot{\mu}_2(0) = 0\,,$$

This is an elementary triangular system. For example, with $\omega_0 := \sqrt{f(0)}$ the solution is

$$\mu_0(t) = 0\,, \qquad \mu_1(t) = \cos\omega_0 t\,,$$

$$\mu_2(t) = \frac{f'(0)}{3\omega_0^2}\big(2\cos\omega_0 t + \cos 2\omega_0 t - 3\big)\,.$$

8 The derivatives of the period $\tau(x_0)$

We know that $\tau(0) = 2\pi$ and that τ is smooth near the origin if f, g are smooth. The computation of $\tau'(0), \tau''(0) \ldots$ starts from the periodicity equations

$$x\big(\tau(x_0), x_0\big) = x_0 \qquad \dot{x}\big(\tau(x_0), x_0\big) = 0$$

Differentiating *twice* the *second* equation and setting $x_0 = 0$, we get that

$$2\ddot{\mu}_1(2\pi/\omega)\tau'(0) + \ddot{\mu}_2(2\pi/\omega) = 0\,.$$

Using the explicit formulas for μ_1, μ_2 we deduce that

$$\tau'(0) = 0\,.$$

Taking more derivatives of the equation we get

$$\tau''(0) = \frac{\pi}{12 f(0)^{5/2}}\big(20 f'(0)^2 - 9 f(0) f''(0)\big)$$

and so on.

9 1-decidability

For $n = 1$ the derivative $\Psi^{(n)}(0)$ can be found by hand (for $n = 2$ with quite some effort). For higher orders we have written a computer algebra program.

Proposition 9.1 (order 1, $n = 1$). *If $g(0) = f(0)/4$ then*

$$\Psi(0) = (-1)^1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Psi'(0) = \frac{2\pi g'(0)}{f(0)^{3/2}} \begin{pmatrix} 0 & 1 \\ g(0) & 0 \end{pmatrix}.$$

If $g'(0) \neq 0$ the origin is unstable of the exponential kind.

Proposition 9.2 (Order 1, $n \geq 2$). *If $g(0) = n^2 f(0)/4$ with $n \geq 2$ then*

$$\Psi(0) = (-1)^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Psi'(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This case is not 1-decidable.

10 Application to central force

Barone-Cesar-Zampieri's old sufficient condition for instability in the central force case ($f \equiv g$) is that $4f'(0)^2 - 3f(0)f''(0) \neq 0$. Using the formulas for $\Psi^{(n)}(0)$ for $n = 1, \ldots, 4$ we can say something when $4f'(0)^2 - 3f(0)f''(0) = 0$:

Proposition 10.1 (4-decidable instability with central force). *If $f \equiv g$ (central force) and*

$$\begin{cases} 4f'(0)^2 - 3f(0)f''(0) = 0 \\ 40f'(0)^4 - 24f(0)^2 f'(0) f^{(3)}(0) + \\ \quad + 3f(0)^3 f^{(4)}(0) \neq 0 \end{cases}$$

then the origin is an unstable equilibrium, with linear growth.

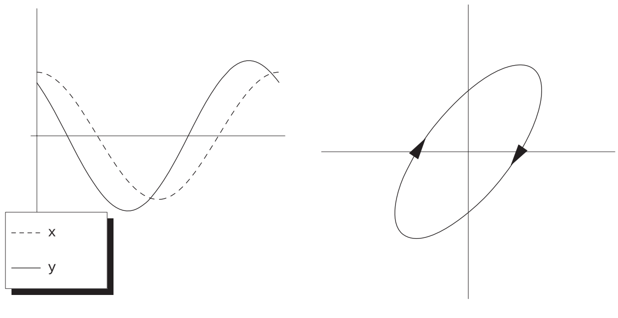
An *example* of 4-decidable instability with central forces:

$$f(x) := g(x) := 1 + 48x^4 \,.$$

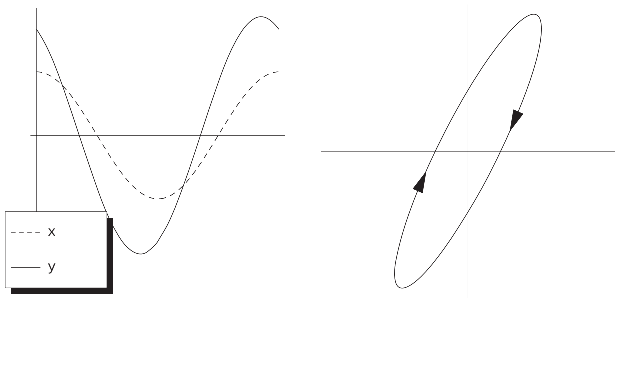
The asymptotic expansion of the Floquet matrix $\Psi(x)$ as $x \to 0$ is

$$\Psi(x) = \begin{pmatrix} 1 & -24\pi x^4 + o(x^4) \\ 0 & 1 \end{pmatrix} \,.$$

Here is a graph of a $\tau(x_0)$-cycle:



and here is the next 133th cycle:



11 Notable special cases

The following classes of equations are special cases of our main problem:

$$\ddot{y} = -y \cdot (a + bx_0 \cos t) \quad \text{(Mathieu's equation)}$$

$$\ddot{y} = -y \cdot (a + bx_0 \cos t + cx_0^2 \cos^2 t)$$

(Whitakker's equation).

For example, for Whittaker's equation we have the following 2-decidable cases:

- if $a > 0$ and $2\sqrt{a} \notin \mathbb{N}$ the origin is stable;
- if $a = 1/4$ and $b \neq 0$ it is unstable;
- if $a = 1/4$, $b = 0$ and $c \neq 0$ it is stable;
- if $a = 1$, $b \neq 0$ and $-1/3 \leq c/b^2 \leq 5/9$ it is unstable;
- if $a = 1$ and either $-b^2/3 > c$ or $c > 5b^2/9$ it is stable;
- for $a = n^2/4$ with $n \geq 3$ and $b^2 \neq c(n^2 - 1)$ it is stable.

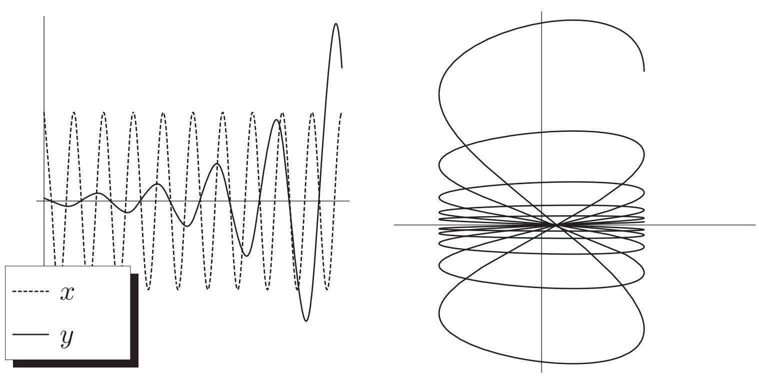
12 A 1-decidable instability example with exponential growth

$$f(x) \equiv 4 \qquad g(x) := 1 + x$$

The Floquet matrix $\Psi(x)$ expands as

$$\Psi(x) = \begin{pmatrix} -1 & -\frac{\pi}{4}x \\ -\frac{\pi}{4}x & -1 \end{pmatrix} + o(x).$$

Here are some cycles:



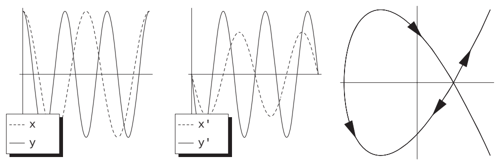
13 A 2-decidable stable example

$$f(x) := 4 + \frac{576}{19}x^2\,, \quad g(x) := 9 + \frac{576}{19}x^2\,.$$

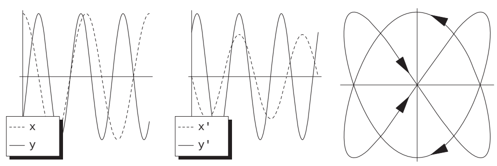
The Floquet matrix $\Psi(x)$ expands as

$$\Psi(x) = \begin{pmatrix} -1 & 2\pi x^2 \\ -18\pi x^2 & -1 \end{pmatrix} + o(x^2).$$

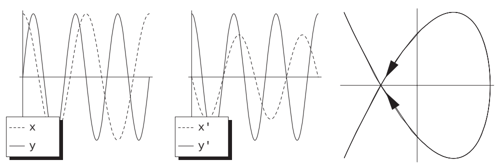
Here are the first two cycles with initial data $(x_0, \dot{x}_0) = (y_0, \dot{y}_0) = (1/100, 0)$:



415 cycles later:



after 833 cycles:



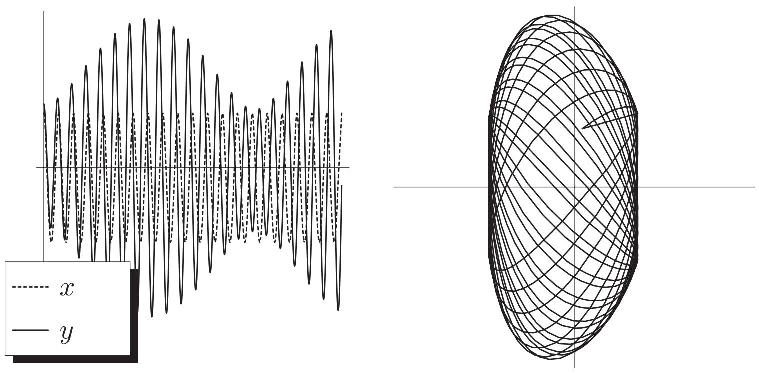
14 A 2-decidable stable example with beatings

$$f(x) := 1 + x + x^2 , \qquad g(x) := 1 + x^2 .$$

The Floquet matrix $\Psi(x)$ expands as

$$\Psi(x) = \begin{pmatrix} 1 & \frac{\pi}{3}x^2 \\ -\frac{5\pi}{6}x^2 & 1 \end{pmatrix} + o(x^2).$$

Here are some cycles:



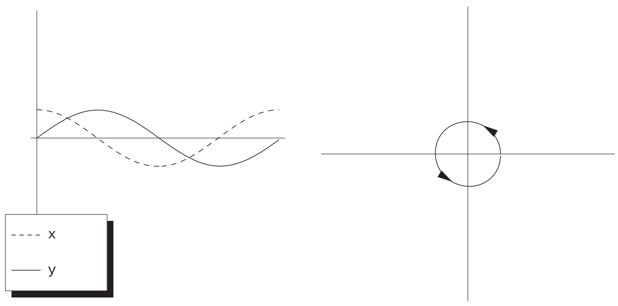
15 A 4-decidable unstable example with beatings

$$f(x) := 1 + x^2\,, \qquad g(x) := 1 + x^2 - x^4\,.$$

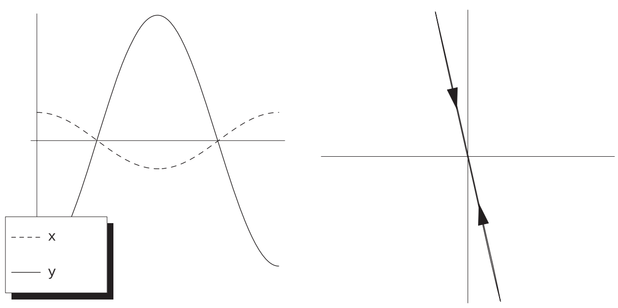
The Floquet matrix $\Psi(x)$ expands as

$$\Psi(x) = \begin{pmatrix} 1 & -\frac{\pi}{2!}x^2 + \frac{33\pi}{4!\cdot 4}x^4 \\ \frac{60\pi}{4!\cdot 4}x^4 & 1 \end{pmatrix} + o(x^4).$$

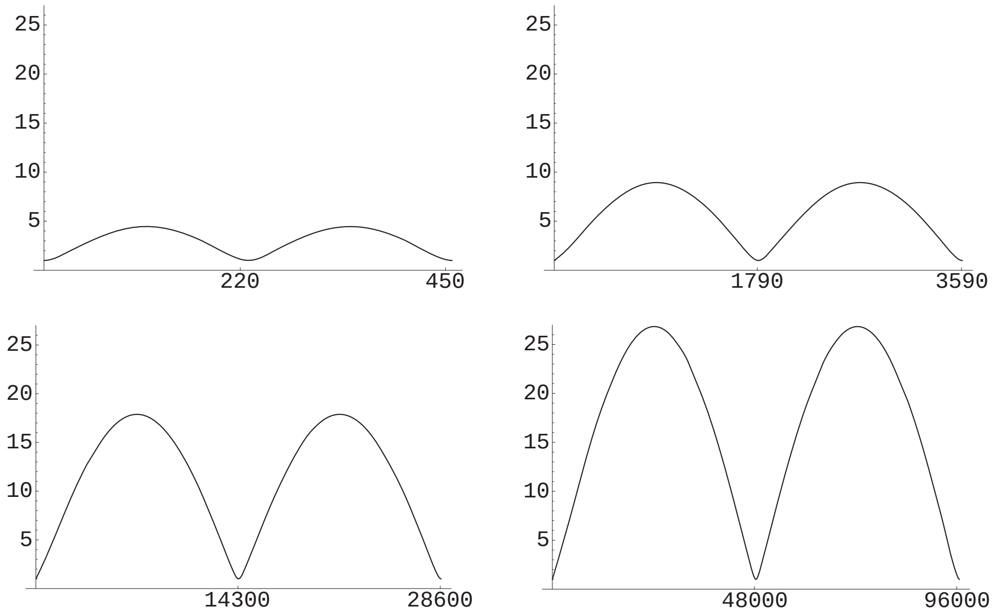
Here is a cycle:



and at its widest, 110 cycles later:



In these graphs the horizontal scale is measured in $\tau(x_0)$-cycles, while on the vertical axis there is the y-amplitude over the cycle for the initial values $x_0 = 1/5, 1/10, 1/20, 1/30$ respectively, and $(y_0, \dot{y}_0) = (0, 1)$.



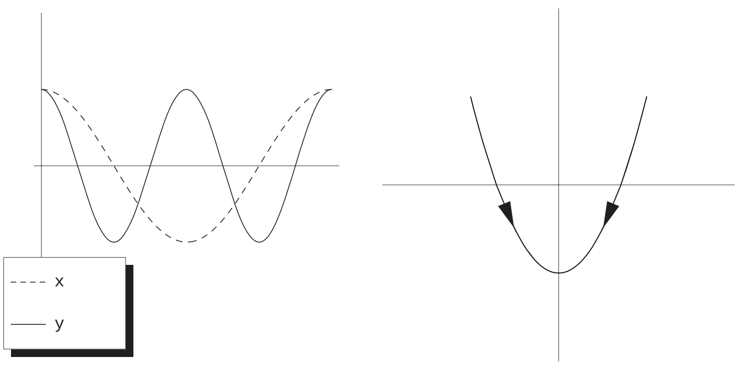
16 A 6-decidable unstable example with beatings

$$f(x) := 1 - x^4 + 26x^6\,, \quad g(x) := 4 - 8x^4 + 140x^6$$

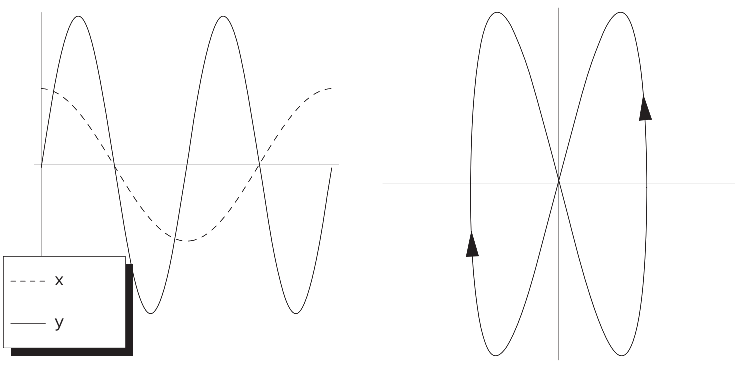
The Floquet matrix $\Psi(x)$ expands as

$$\Psi(x) = \begin{pmatrix} 1 + o(1) & -\frac{4725\pi}{6!}x^6 + o(x^6) \\ \frac{24\pi}{4!}x^4 + o(x^4) & 1 + o(1) \end{pmatrix}.$$

Here is a cycle:



and at its widest, 2000 cycles later:



In these graphs the horizontal scale is measured in $\tau(x_0)$-cycles, while on the vertical axis there is the y-amplitude over the cycle for the initial values $x_0 = 1/5, 1/10, 1/20, 1/30$ and $(y_0, \dot{y}_0) = (1, 0)$.