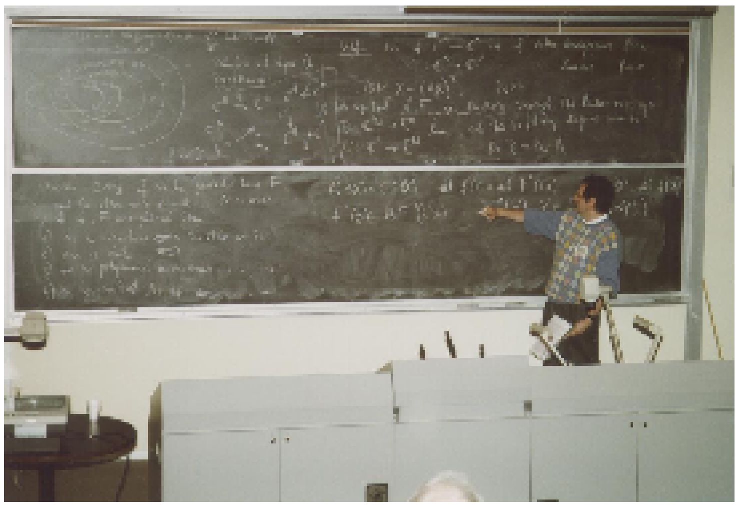
A NONLINEARIZABLE CUBIC-LINEAR MAPPING

GIANLUCA GORNI
Università di Udine
208, via delle Scienze
33100 Udine
Italy



Arno van den Essen in his paper [3] produces, among other things, the following cubic-homogeneous polynomial automorphism of $\mathbb{C}^5$:

$$f\colon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} x_2 x_5^2 \\ x_1^2 x_5 - x_4 x_5^2 \\ x_2^2 x_5 \\ 2x_1 x_2 x_5 - x_3 x_5^2 \\ 0 \end{pmatrix} \tag{0.1}$$

that has the following property: for all $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$ there exists no analytic automorphism $k_\lambda\colon \mathbb{C}^5 \to \mathbb{C}^5$ that linearizes λf, in the sense that $\lambda f(k_\lambda(x)) = k_\lambda(\lambda x)$ for all $x \in \mathbb{C}^5$. It is therefore a counterexample to a conjecture that originated in [1].

The same paper in theorem 2.3 states that there exists a cubic-linear polynomial automorphism F of $\mathbb{C}^{17}$ with the same property: for all $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$ there exists no analytic automorphism $K_\lambda\colon \mathbb{C}^{17} \to \mathbb{C}^{17}$ such that $\lambda F(K_\lambda(X)) = K_\lambda(\lambda X)$ for all $X \in \mathbb{C}^{17}$. This F is then a counterexample to a more special conjecture that

*The author is very glad to have been able to attend the conference in honour of Prof. Gary Meisters and is grateful to Gary himself, Bo Deng, Roger Wiegand and their families for their kindness and hospitality.

was advanced in [5]. However the F is not actually exhibited, but only a hint at its construction is given, following the general method described in [4].

Now we have carried out all the calculations and here are the results. The map F is defined as $F(X) := X - (AX)^{*3}$, where the exponent means the component-wise cubic power, and the matrix A is given as

$$A = \frac{1}{12}\begin{pmatrix}
0 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 2 & 4 & -2 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -12 \\
4 & 0 & 4 & -2 & 0 & -2 & 0 & 0 & -2 & -4 & 2 & 0 & -2 & 0 & 0 & -2 & -12 \\
0 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 2 & 4 & -2 & 0 & 0 & 0 & 0 & 2 & -12 \\
4 & 0 & -4 & 2 & 0 & -2 & 0 & 0 & 2 & 4 & -2 & 0 & -2 & 0 & 0 & 2 & -12 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & -12 \\
0 & -4 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & -12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \\
4 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 12 \\
4 & 0 & 4 & -2 & 0 & -2 & 0 & 0 & -2 & -4 & 2 & 0 & -2 & 0 & 0 & -2 & 12 \\
0 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 2 & 4 & -2 & 0 & 0 & 0 & 0 & 2 & 12 \\
4 & 0 & -4 & 2 & 0 & -2 & 0 & 0 & 2 & 4 & -2 & 0 & -2 & 0 & 0 & 2 & 12 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 12 \\
0 & -4 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

It can be checked that $A^2 = 0$. Consider also the following two matrices B:

$$B = \frac{1}{12}\begin{pmatrix}
4 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 2 & 4 & -2 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12
\end{pmatrix}$$

and C (shown here transposed to save space):

$$C^T = \begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

It can be verified that the product BC is the 5×5 identity matrix, that A and B have the same kernel, and that if we define $F(X) := X - (AX)^{*3}$ for $X \in \mathbb{C}^{17}$, we have that $f(x) = BF(Cx)$ for all $x \in \mathbb{C}^5$. This means that the mappings f and F are "paired" in the sense of [4].

It follows that the mapping F is a cubic-linear automorphism of $\mathbb{C}^{17}$ such that λF is not linearizable for any $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$.

We have implemented the procedure for finding A, B, C from f, that was described in [4], as the following routine written in the *Mathematica* programming language, version 3.0, by Wolfram Research Inc.

```
makePairing[cubicHomogeneous_Function]:=
  Module[{dimension,var,toCubeCombination,recombined,
      monomialList,combinationList,temp,comb,
      d0,b0,b,d,c,m,cm,dAnd0,a,answer},
dimension=Length[Last[cubicHomogeneous]];
var=Table[x[i],{i,dimension}];
SetAttributes[toCubeCombination,Listable];
toCubeCombination[a_+b_]:=toCubeCombination[a]+
  toCubeCombination[b];
toCubeCombination[0]:=0;
toCubeCombination[(num_.)*(a_)*(b_)*(c_)]:=
```

```
(num/24)*(a+b+c)^3+(num/24)*(a-b-c)^3-(num/24)*(a+b-c)^3-
     (num/24)*(a-b+c)^3/;
        NumberQ[num]&&!NumberQ[a]&&!NumberQ[b]&&!NumberQ[c];
toCubeCombination[(num_.)*(a_)*(b_)^2]:=
  (num/6)*(a+b)^3+(num/6)*(a-b)^3-(num/3)*a^3/;
        NumberQ[num]&&!NumberQ[a]&&!NumberQ[b]&&OrderedQ[{a,b}];
toCubeCombination[(num_.)*(a_)*(b_)^2]:=
  (num/6)*(a+b)^3-(num/6)*(b-a)^3-(num/3)*a^3/;
        NumberQ[num]&&!NumberQ[a]&&!NumberQ[b]&&!OrderedQ[{a,b}];
toCubeCombination[(num_.)*(a_)^3]:=
      num*a^3/;NumberQ[num]&&!NumberQ[a];
recombined=toCubeCombination[Evaluate[
      Expand[cubicHomogeneous@@var-var]]];
monomialList=Map[If[Head[#]===Plus,List@@#,{#}]&,
      recombined]//Flatten;
combinationList=
      Union[Select[monomialList,
            MatchQ[#1, (num_.)*(lin_)^3] & ]/.
            (num_.)*(lin_)^3:>lin/;NumberQ[num]];
temp=recombined/.
        Table[combinationList[[i]]^3->comb[i],
        {i,Length[combinationList]}];
    d0=Table[Coefficient[combinationList[[i]],x[j]],

{i,Length[combinationList]},{j,dimension}];
b0=-Table[
          Coefficient[temp[[i]],comb[j]],{i,dimension},{j,
            Length[combinationList]}];
For[i=0,Union[Flatten[Minors[b0,dimension]]]=={0},i++,
b0=Transpose[
          Join[Transpose[b0],
             {IdentityMatrix[dimension][[dimension - i]]}]];
d0=Join[d0,{Table[0,{dimension}]}]];
b=b0;
d=d0;
c= Module[{c,mat},
  mat=Array[c,{Length[b[[1]]],dimension}];
   mat/.
   Solve[b.mat==IdentityMatrix[dimension],
       Flatten[mat]][[1]]/.c[i_,j_]->0];
m= Transpose[NullSpace[b]];
m=m*LCM @@ Denominator[Union[Flatten[m]]];
cm=Transpose[Join[Transpose[c], Transpose[m]]];
dAnd0 =Transpose[
```

```
        Join[Transpose[d],
          Transpose[Table[0,{i,Length[d]},
          {j,Length[First[m]]}]]]]];
a=dAnd0.Inverse[cm];
answer=(b.c==IdentityMatrix[dimension])&&(
          Union[Expand[cubicHomogeneous@@var-var+
              b.(a.c.var)^3]]=={0})&&(
          Union[Flatten[a.Transpose[NullSpace[b]]]]=={0})&&

(Union[Flatten[b.Transpose[NullSpace[a]]]]=={0});
If[answer,{a,b,c},Print["Something is wrong"]]];
```

The function makePairing takes a cubic-homogeneous function f from $\mathbb{C}^n$ to itself, with $n \geq 2$, and returns a triple of matrices A, B, C such that $\ker A = \ker B$, $BC = I_n$ and $f(x) = x - B(ACx)^{*3}$. If f has constant Jacobian determinant so has the mapping $X \to X - (AX)^{*3}$, and f is an automorphism if and only if F is (in the respective dimensions).

The routine has been tested only with the following three cubic-homogeneous examples, the first of which is the (0.1) above and the others are taken from [3] and [4]:

```
f = Function[{x1, x2, x3, x4, x5},
  {x1 + x2 * x5^2, x2 + x1^2 * x5 - x4 * x5^2, x3 + x2^2  *x5,
   x4 + 2 x1 * x2 * x5 - x3 * x5^2, x5}];
g = Function[{x1, x2, x3, x4},
  {x1 + (x3*x1 + x4 * x2) * x4,
   x2 - (x3*x1 + x4*x2) * x3, x3 + x4^3, x4}];
h = Function[{y1, y2, y3, y4, y5},
  {y1, y2, y3, y4, y5} + 3 * {0, 0,
    y1^2 * y2  +y1 * y2^2  +2 * y1 * y2 * y4 - 2 * y1^2 * y5,
    - y1^2 * y2 - y1 * y2^2  -2y1 * y2 * y3 - 2 * y1^2 * y5,
    - y2^2 * y3 - y2^2 * y4}];
```

We cannot guarantee that the algorithm will not run into bugs in other cases, but there is a built-in check that should warn if the results are not reliable. The command to give is simply makePairing[f].

An electronic copy of a *Mathematica* notebook containing the routine should be available on the Web together with these proceedings.

References

1. B. Deng, G. Meisters, G. Zampieri, *Conjugation for polynomial mappings*, Z, Angew. Math. Phys. 46 (1995).
2. A. van den Essen (ed.), *Automorphisms of Affine Spaces*, Proceedings of the conference "Invertible Polynomial Maps", held in Curaçao, July 4-8, 1994, under auspices of the Caribbean Mathematical Foundation, Kluwer Academic Publisher, Dordrecht/Boston/London, 1995.
3. A. van den Essen, *A counterexample to Meister's Cubic-Linear Linearization Conjecture*, Preprint, University of Nijmegen, to appear in Indagationes Mathematicae.

4. G. Gorni, G. Zampieri, *On cubic-linear polynomial mappings*, Research Report, University of Udine (August 1996), to appear in Indagationes Mathematicae.
5. G. Meisters, The Cubic-Linear Linearization Conjecture. This paper is available since November 1995 on the World-Wide-Web at the address `http://w.w.w.math.unl.edu/~gmeister`.