generating function of Laguerre polynomials

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We start from the definition of Laguerre polynomials via their Rodrigues formula

\[ L_n(z) := e^z \frac{d^n}{dz^n} e^{-z} z^n \quad (n = 0, 1, 2, \ldots). \]  

(1)

The consequence

\[ f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \]  

(2)

of Cauchy integral formula allows to write (1) as the complex integral

\[ L_n(z) = \frac{n!}{2\pi i} \oint_C e^z e^{-\zeta} z^n \frac{d\zeta}{(\zeta - z) n+1} = \frac{n!}{2\pi i} \oint_C e^{z-\zeta} \frac{d\zeta}{(1-\zeta)^n(\zeta-z)}, \]

where \( C \) is any closed contour around the point \( z \) and the direction is anticlockwise. The substitution

\[ \zeta - z := \frac{zt}{1-t}, \quad \zeta = \frac{z}{1-t}, \quad t = 1 - \frac{z}{\zeta} \quad d\zeta = \frac{z dt}{(1-t)^2} \]

here yields

\[ L_n(z) = \frac{n!}{2\pi i} \oint_{C'} \frac{e^{-\frac{zt}{1-t}} z dt}{(1-t)^2 t^n} = \frac{n!}{2\pi i} \oint_{C'} \frac{e^{-\frac{zt}{1-t}}}{(1-t)^{n+1}} dt \]

where the contour \( C' \) goes round the origin. Accordingly, by (2) we can infer that

\[ L_n(z) = \left[ \left. \frac{d^n e^{-\frac{zt}{1-t}}}{dt^n} \right|_{t=0} \right]. \]
whence we have found the generating function
\[ \frac{e^{-\frac{zt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(z)}{n!} t^n \]
of the Laguerre polynomials.