Note

A factorization of the symmetric Pascal matrix involving the Fibonacci matrix

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Abstract

In this short note, we give a factorization of the Pascal matrix. This result was apparently missed by Lee et al. [Some combinatorial identities via Fibonacci numbers, Discrete Appl. Math. 130 (2003) 527–534].

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1. Introduction

For a fixed $n$, the $n \times n$ lower triangular Pascal matrix, $P_n = [p_{i,j}]_{i,j=1,2,\ldots,n}$, (see [1,6]), is defined by

$$p_{i,j} = \begin{cases} \binom{i-1}{j-1} & \text{if } i \geq j, \\ 0 & \text{otherwise}. \end{cases} \quad (1)$$

Let $F_n$ be the $n$th Fibonacci number with the generating series $\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$. The $n \times n$ Fibonacci matrix $\mathcal{F}_n = [f_{i,j}]_{i,j=1,2,\ldots,n}$ is the unipotent lower triangular Toeplitz matrix defined by

$$f_{i,j} = \begin{cases} F_{i-j+1} & \text{if } i-j+1 \geq 0, \\ 0 & \text{if } i-j+1 < 0. \end{cases} \quad (2)$$

In [4], Lee et al. discussed the factorizations of Fibonacci matrix $\mathcal{F}_n$ and the eigenvalues of symmetric Fibonacci matrices $\mathcal{F}_n \mathcal{F}_n^T$. The inverse of $\mathcal{F}_n$ was also given as follows:

$$\mathcal{F}_n^{-1} = [f'_{i,j}]_{i,j=1,2,\ldots,n} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i-2 \leq j \leq i-1, \\ 0 & \text{otherwise}. \end{cases} \quad (3)$$

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In fact, formula (3) is an immediate consequence of the isomorphism between lower formal power series and lower triangular Toeplitz matrices.

In [5], Lee et al. obtained the following result:

\[ P_n = \mathcal{F}_n \mathcal{L}_n, \]  

(4)

where \( \mathcal{L}_n = [l_{i,j}]_{i,j=1,2,\ldots,n} \) is defined by

\[ l_{i,j} = \binom{i-1}{j-1} - \binom{i-2}{j-1} - \binom{i-3}{j-1}. \]

In this short note, we give a second factorization of the Pascal matrix which was apparently missed by the authors in [5].

2. The main results

First, we define an \( n \times n \) matrix \( \mathcal{R}_n = [r_{i,j}]_{i,j=1,2,\ldots,n} \) as follows:

\[ r_{i,j} = \binom{i-1}{j-1} - \binom{i-2}{j} - \binom{i-3}{j+1}. \]

From the definition of \( \mathcal{R}_n \), it is easy to see that \( \mathcal{R}_n \) is unipotent lower triangular. It satisfies \( r_{i,1} = -\frac{1}{2}(i + 1)(i - 2) \) for \( i \geq 2 \) and \( r_{i,j} = r_{i-1,j} + r_{i-1,j-1} \) for \( i, j \geq 2 \).

Next we give the following factorization of the Pascal matrix.

**Theorem 2.1.** We have

\[ P_n = \mathcal{R}_n \mathcal{F}_n. \]

(6)

**Proof.** It suffices to prove \( P_n \mathcal{F}_n^{-1} = \mathcal{R}_n \). For \( i \geq 1 \) we have \( \sum_{k=1}^{i} p_{i,k} f_{k,1}' = p_{i,1} f_{1,1}' + p_{i,2} f_{2,1}' + p_{i,3} f_{3,1}' = 1 + \binom{i-1}{1}(-1) + \binom{i-1}{2}(-1) = -\frac{1}{2}(i + 1)(i - 2) = r_{i,1} \), and for \( i \geq 1, j \geq 2 \), we have \( \sum_{k=1}^{n} p_{i,k} f_{k,j}' = p_{i,j} f_{j,j}' + p_{i,j+1} f_{j+1,j} + p_{i,j+2} f_{j+2,j}' = \binom{i-1}{j-1} - \binom{i-1}{j} - \binom{i-1}{j+1} = r_{i,j} \), which implies that \( P_n \mathcal{F}_n^{-1} = \mathcal{R}_n \), as desired. \( \square \)

**Example.**

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
1 & 3 & 3 & 1 & 0 \\
1 & 4 & 6 & 4 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\-2 & 1 & 1 & 0 & 0 \\
\-5 & -1 & 2 & 1 & 0 \\
\-9 & \-6 & 1 & 3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 \\
3 & 2 & 1 & 1 & 0 \\
5 & 3 & 2 & 1 & 1
\end{pmatrix}.
\]

From the theorem, we have the following combinatorial identity involving the Fibonacci numbers.

**Corollary 2.2.**

\[
\binom{n-1}{r-1} = F_{n-r+1} + (n - 2)F_{n-r} + \frac{1}{2}(n^2 - 5n + 2)F_{n-r-1}
\]

\[ + \sum_{k=r}^{n-3} \binom{n-1}{k-1} \left( 2 - \frac{n}{k} - \frac{(n-k)(n-k-1)}{k(k+1)} \right) F_{k-r+1}. \]

(7)
In particular,

\[ \sum_{k=1}^{n} \left( \binom{n-1}{k-1} - \binom{n-1}{k} - \binom{n-1}{k+1} \right) F_k = 1. \]  

(8)

Lemma 2.3.

\[ \sum_{k=3}^{i} \left\{ \binom{i-2}{k-2} - \binom{i-2}{k-1} - \binom{i-2}{k} \right\} F_k = \frac{1}{2} (i+1)(i-2). \]  

(9)

Proof. We argue by induction on \( i \). If \( i = 3, 4 \), then lemma is true, respectively. Suppose the lemma is true for \( i \geq 4 \). Then

\[
\sum_{k=3}^{i+1} \left\{ \binom{i-1}{k-2} - \binom{i-1}{k-1} - \binom{i-1}{k} \right\} F_k
= \sum_{k=3}^{i} \left\{ \binom{i-2}{k-2} - \binom{i-2}{k-1} - \binom{i-2}{k} \right\} F_k + \sum_{k=3}^{i+1} \left\{ \binom{i-2}{k-3} - \binom{i-2}{k-2} - \binom{i-2}{k-1} \right\} F_k
= \frac{1}{2} (i+1)(i-2) + \sum_{k=2}^{i} \left\{ \binom{i-2}{k-2} - \binom{i-2}{k-1} - \binom{i-2}{k} \right\} F_{k+1}
= \frac{1}{2} (i+1)(i-2) + \sum_{k=2}^{i} \left\{ \binom{i-2}{k-2} - \binom{i-2}{k-1} - \binom{i-2}{k} \right\} \{F_k + F_{k-1}\}
= \frac{1}{2} (i+1)(i-2) + 1 - (i-2) - \binom{i-2}{2} + \frac{1}{2} (i+1)(i-2)
+ \sum_{k=2}^{i} \left\{ \binom{i-2}{k-2} - \binom{i-2}{k-1} - \binom{i-2}{k} \right\} F_{k-1}
= (i+1)(i-2) + 1 - (i-2) - \binom{i-2}{2} + \sum_{k=1}^{i-1} \left\{ \binom{i-2}{k-1} - \binom{i-2}{k} - \binom{i-2}{k+1} \right\} F_k
= (i+1)(i-2) + 1 - (i-2) - \binom{i-2}{2} + 1
= \frac{1}{2} (i+2)(i-1).

Hence the lemma is also true for \( i + 1 \). By induction, we complete the proof. □

Note. Since \( \frac{1}{2} (i+1)(i-2) \) is a linear combination of \( \binom{i}{k} \) for \( k = 0, 1, 2 \) (or \( \binom{i+1}{k} \)), the referee pointed out that Lemma 2.3 follows also from Theorem 2.1.
We define the $n \times n$ matrices $U_n, \overline{U}_n$ and $R_n$ by

$$U_n = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ -F_3 & 1 & 1 & 0 & \ldots & 0 & 0 \\ -F_4 & 0 & 1 & 1 & \ldots & 0 & 0 \\ -F_5 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -F_n & 0 & 0 & 0 & \ldots & 1 & 1 \end{pmatrix},$$

(10)

$$\overline{U}_k = I_{n-k} \oplus U_k$$ and $$R_n = [1] \oplus \overline{A}_n - 1,$$ i.e., $A$ is the matrix $A$ shifted one row down and one column to the right with first column given by $(1, 0, 0, \ldots)$. From the definition of $U_k$, we have $U_1 = U_2 = I_n$ and $U_n = U_n$. Hence

Lemma 2.4.

$$R_n = \overline{U}_n U_n.$$  \hfill (11)

Proof. The $(i, j)$ element of $R_n$ is $r_{i-1,j-1}$, $(i, j = 2, 3, \ldots, n)$, or $1 (i = 1, j = 1)$, or $0 (i \neq 1, j = 1$ or $i = 1, j \neq 1$).

Let $\overline{U}_n U_n = (D_{i,j})$ and $U_n = (u_{i,j})$. Obviously, $D_{1,1} = 1 = r_{1,1}, D_{2,1} = 0 = r_{2,1}$ and $D_{i,j} = 0 (i < j)$. For $i \geq 3$, by Lemma 2.3, we have

$$D_{i,1} = \sum_{k=1}^{i} r_{i-1,k-1} u_{k,1}$$

$$= - \sum_{k=3}^{i} \left\{ \binom{i-2}{k-2} - \binom{i-2}{k-1} - \binom{i-2}{k} \right\} F_k$$

$$= - \frac{1}{2} (i+1)(i-2)$$

$$= r_{i,1}.$$ (12)

When $i \geq j \geq 2$, we have

$$D_{i,j} = \sum_{k=1}^{i} r_{i-1,k-1} u_{k,j} = r_{i-1,j-1} + r_{i-1,j} = r_{i,j}.$$ (13)

Thus, $R_n = \overline{U}_n U_n$. \hfill \Box

Example.

$$\overline{U}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ -5 & -1 & 2 & 1 & 0 \\ -9 & -6 & 1 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 1 & 0 \\ -5 & 0 & 0 & 1 & 1 \end{pmatrix} = \overline{U}_5 U_5.$$

An immediate consequence of Lemma 2.4 and the definition of the $\overline{U}_k$ is
Theorem 2.5.

\[ R_n = U_1 U_2 \ldots U_{n-1} U_n. \]  

Example.

\[ R_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ -5 & -1 & 2 & 1 & 0 \\ -9 & -6 & 1 & 3 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ -3 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 1 \\ -5 & 0 & 0 & 1 & 1 \end{pmatrix}. \]

Let

\[ S_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad S_{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \]

\[ S_k = S_0 \oplus I_k, \text{ for } k \in \mathbb{N}, \quad F_n = [1] \oplus F_{n-1}, \quad G_1 = I_n, \quad G_2 = I_{n-3} \oplus S_{-1}, \text{ and } G_k = I_{n-k} \oplus S_{-k} \text{ for } k \geq 3. \]

In [4], the authors gave the following result:

\[ F_n = G_1 G_2 \ldots G_n. \]

Hence we have:

Theorem 2.6.

\[ P_n = \overline{U}_1 \overline{U}_2 \ldots \overline{U}_{n-1} U_n G_1 G_2 \ldots G_n. \]  

Example.

\[ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \]
3. A remark

In this note, all matrix-identities are expressed using finite matrices. Since all matrix-identities involve lower-triangular matrices, they have an analogue for infinite matrices. We state them briefly as follows.

Let $P$, $F$, $L$, $U$ and $R$ are the infinite cases of the matrices $P_n$, $F_n$, $L_n$, $U_n$ and $R_n$, respectively. Furthermore, define

$$U^{(k)} = I_k \oplus U$$

and

$$R^{(k)} = I_k \oplus R.$$ 

Then $P = FL = R$ (cf. (4) and Theorem 2.1), $R = U^{(1)}$ (cf. Lemma 2.4) and $R = R^{(t+1)} U^{(t)} \ldots U^{(2)} U^{(1)} U$, where $t$ is an arbitrary nonnegative integer (cf. Theorem 2.5).

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