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On the LU factorization of the Vandermonde matrix

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Abstract

In this paper, the author gives a simpler alternative approach to the LU factorization and 1-banded factorization of the Vandermonde matrix, and obtains explicit formulas of the triangular factors and 1-banded matrices by using symmetric functions.

Keywords: Vandermonde matrix; Symmetric function; LU factorization

1. Introduction

Expressing a matrix as a product of a lower triangular matrix L and an upper triangular matrix U is called an LU factorization. Such a factorization is typically obtained by Gaussian elimination. If L is a lower triangular with unit main diagonal and U is an upper triangular, the LU factorization of a matrix is unique. Using symmetric functions, Oruç and Phillips [1] established the LU factorization of Vandermonde matrix, and the lower and upper triangular matrices are, in turn, factored into 1-banded matrices, and thus expressed the Vandermonde matrix as a product of 1-banded matrices. In this paper, using symmetric functions and linear algebra, we find a shorter proof of the result on the LU factorization of the transpose of the Vandermonde matrix. This leads to greater simplification of the further factorization into 1-banded matrices.

Let n be a positive integer. For integers $1 \leqslant r \leqslant (n+1)$, the rth elementary symmetric function on the variables set $\{x_0, x_1, \ldots, x_n\}$ is defined by

$$e_r(x_0, x_1, \ldots, x_n) = \sum_{0 \leqslant k_1 < k_2 < \cdots < k_r \leqslant n} x_{k_1} x_{k_2} \cdots x_{k_r}, \tag{1}$$

and the rth complete symmetric function on the variables set $\{x_0, x_1, \ldots, x_n\}$ is defined by

$$h_r(x_0, x_1, \ldots, x_n) = \sum_{0 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_r \leqslant n} x_{k_1} x_{k_2} \cdots x_{k_r}. \tag{2}$$

We set $e_0(x_0, x_1, \ldots, x_n) = 1$, $h_0(x_0, x_1, \ldots, x_n) = 1$. It is not hard to check that the elementry and complete symmetric functions satisfy the following recurrence relations (see [1]):

$$e_r(x_0, x_1, \ldots, x_n) = e_r(x_0, x_1, \ldots, x_{n-1}) + x_n e_{r-1}(x_0, x_1, \ldots, x_{n-1}), \tag{3}$$

$$h_r(x_0, x_1, \ldots, x_n) = h_r(x_0, x_1, \ldots, x_{n-1}) + x_n h_{r-1}(x_0, x_1, \ldots, x_n). \tag{4}$$

Let $R_n[x]$ be the vector space of polynomials in x over the real number field R of degree at most n, and let $x_0, x_1, \ldots, x_{n-1}$ be arbitrary real numbers. Then the sets $B_1 = \{1, x, x^2, \ldots, x^n\}$ and $B_2 = \{[x]_0, [x]_1, \ldots, [x]_n\}$ are both bases for $R_n[x]$, where

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$[x]_r=(x-x_0)(x-x_1)\cdots(x-x_{r-1})$ for $1\leqslant r\leqslant n$, and $[x]_0=1$. Considering the relation between the two bases, we can easily get the following theorem.

Theorem 1. *For* $m=0,1,\ldots,n$,

$$[x]_m=\sum_{k=0}^{m}(-1)^{m-k}e_{m-k}(x_0,x_1,\ldots,x_{m-1})x^k, \tag{5}$$

$$x^m=\sum_{k=0}^{m}h_{m-k}(x_0,x_1,\ldots,x_k)[x]_k. \tag{6}$$

The $(n+1)\times(n+1)$ *matrices* $Q_n=Q_n[x_0,x_1,\ldots,x_{n-1}]$ *and* $L_n=L_n[x_0,x_1,\ldots,x_{n-1}]$ *are defined as*

$$Q_n(i,j)=\begin{cases}1 & \text{if } i=j,\\ (-1)^{i-j}e_{i-j}(x_0,x_1,\ldots,x_{i-1}) & \text{if } i>j\geqslant 0,\\ 0 & \text{otherwise},\end{cases}$$
$$L_n(i,j)=\begin{cases}1 & \text{if } i=j,\\ h_{i-j}(x_0,x_1,\ldots,x_j) & \text{if } i>j\geqslant 0,\\ 0 & \text{otherwise}.\end{cases}$$

Corollary 1. *The transition matrix from the basis* $B_1=\{1,x,x^2,\ldots,x^n\}$ *of* $R_n[x]$ *to the basis* $B_2=\{[x]_0,[x]_1,\ldots,[x]_n\}$ *of* $R_n[x]$ *is the matrix* Q_n. *The transition matrix from the basis* B_2 *back to the basis* B_1 *is the matrix* L_n.

2. The LU factorization of the Vandermonde matrix

Let $V_n=V_n[x_0,x_1,\ldots,x_n]=(x_j^i)_{0\leqslant i,j\leqslant n}$ be a $(n+1)\times(n+1)$ Vandermonde matrix with distinct $x_0,x_1,\ldots,x_n\in R$. In [1], the symmetric functions have been used for the LU factorization of the transpose of the Vandermonde matrix V_n. From Eq. (6) we have $x_j^i=\sum_{k=0}^{i}h_{i-k}(x_0,x_1,\ldots,x_k)[x_j]_k$, $i,j=0,1,2,\ldots n$. Thus we have the following modification of the Theorem 2.1 of [1].

Theorem 2. *The* $(n+1)\times(n+1)$ *Vandermonde matrix* V_n *can be factorized as* $V_n=L_nU_n$, *where* $L_n=L_n[x_0,x_1,\ldots,x_n]$ *is a lower triangular matrix with units on its main diagonal, the* (i,j)*-entry of* L_n *is* $L_n(i,j)=h_{i-j}(x_0,x_1,\ldots,x_j)$, $i\geqslant j$, *and* $U_n=U_n[x_0,x_1,\ldots,x_n]$ *is an upper triangular matrix, and the* (i,j)*-entry of* U_n *satisfy* $U_n(i,j)=[x_j]_i=(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{i-1})$, $i\leqslant j$.

Example 1. For $n=3$ we have

$$V_3=\begin{pmatrix}1&1&1&1\\ x_0&x_1&x_2&x_3\\ x_0^2&x_1^2&x_2^2&x_3^2\\ x_0^3&x_1^3&x_2^3&x_3^3\end{pmatrix}\quad\text{and}\quad V_3=L_3U_3,$$

where

$$L_3=\begin{pmatrix}h_0(x_0)&0&0&0\\ h_1(x_0)&h_0(x_0,x_1)&0&0\\ h_2(x_0)&h_1(x_0,x_1)&h_0(x_0,x_1,x_2)&0\\ h_3(x_0)&h_2(x_0,x_1)&h_1(x_0,x_1,x_2)&h_0(x_0,x_1,x_2,x_3)\end{pmatrix}$$
$$=\begin{pmatrix}1&0&0&0\\ x_0&1&0&0\\ x_0^2&x_0+x_1&1&0\\ x_0^3&x_0^2+x_0x_1+x_1^2&x_0+x_1+x_2&1\end{pmatrix},$$

$$U_3 = \begin{pmatrix} [x_0]_0 & [x_1]_0 & [x_2]_0 & [x_3]_0 \\ 0 & [x_1]_1 & [x_2]_1 & [x_3]_1 \\ 0 & 0 & [x_2]_2 & [x_3]_2 \\ 0 & 0 & 0 & [x_3]_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & x_1 - x_0 & x_2 - x_0 & x_3 - x_0 \\ 0 & 0 & (x_2 - x_0)(x_2 - x_1) & (x_3 - x_0)(x_3 - x_1) \\ 0 & 0 & 0 & (x_3 - x_0)(x_3 - x_1)(x_3 - x_2) \end{pmatrix}.$$

We define the $(n+1) \times (n+1)$ matrices $H_n[x_0, x_1, \ldots, x_{n-1}]$, $T_n[x_0, x_1, \ldots, x_n]$, $\overline{L_{n-1}}[x_0, x_1, \ldots, x_{n-2}]$, $\overline{U_{n-1}}[x_1, x_2, \ldots, x_n]$ by

$$H_n[x_0, x_1, \ldots, x_{n-1}](i, j) = \begin{cases} 1 & \text{if } j = i \\ x_j & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$T_n[x_0, x_1, \ldots, x_n](i, j) = \begin{cases} 1 & \text{if } j = i = 0 \text{ or } j = i + 1, \\ x_i - x_0 & \text{if } j = i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\overline{L_{n-1}} = \begin{pmatrix} 1 & 0 \\ 0 & L_{n-1} \end{pmatrix}, \quad \overline{U_{n-1}} = \begin{pmatrix} 1 & 0 \\ 0 & U_{n-1}[x_1, \ldots, x_n] \end{pmatrix}.$$

Lemma 1. (a) $L_n = \overline{L_{n-1}} H_n$; (b) $U_n = T_n \overline{U_{n-1}}$.

Proof. (a) We show that the (i, j) entries on both sides are equal. Since the product of the two lower triangular matrices is again lower triangular, it suffices to consider $i \geqslant j$.

If $i = 0$, then $(\overline{L_{n-1}} H_n)(0, 0) = 1 = L_n(0, 0)$.

If $i > 0$,

when $j = 0$, $(\overline{L_{n-1}} H_n)(i, 0) = \overline{L_{n-1}}(i, 0) H_n(0, 0) + \overline{L_{n-1}}(i, 1) H_n(1, 0) = 0 + h_{i-1}(x_0) x_0 = h_i(x_0) = L_n(i, 0)$;

when $j = i$, $(\overline{L_{n-1}} H_n)(i, i) = \overline{L_{n-1}}(i, i) H_n(j, j) = 1 = L_n(i, i)$;

when $j < i$, $(\overline{L_{n-1}} H_n)(i, j) = \overline{L_{n-1}}(i, j) H_n(j, j) + \overline{L_{n-1}}(i, j+1) H_n(j+1, j) = h_{i-j}(x_0, x_1, \ldots, x_{j-1}) + h_{i-j-1}(x_0, x_1, \ldots, x_j) x_j = h_{i-j}(x_0, x_1, \ldots, x_j) = L_n(i, j)$.

(b) To show the equality of the (i, j) entries on both sides, it suffices to consider $i \leqslant j$. If $i = 0$, then $(T_n \overline{U_{n-1}})(0, j) = T_n(0, 0) \overline{U_{n-1}}(0, j) + T_n(0, 1) \overline{U_{n-1}}(1, j) = 1 = U_n(0, j)$.

If $i > 0$,

when $j = i$, $(T_n \overline{U_{n-1}})(i, i) = T_n(i, i) \overline{U_{n-1}}(i, i) = (x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1}) = U_n(i, i)$;

when $j > i$, $(T_n \overline{U_{n-1}})(i, j) = T_n(i, i) \overline{U_{n-1}}(i, j) + T_n(i, i+1) \overline{U_{n-1}}(i+1, j) = (x_i - x_0)(x_j - x_1)(x_j - x_2) \cdots (x_j - x_{i-1}) + (x_j - x_1)(x_j - x_2) \cdots (x_j - x_{i-1})(x_j - x_i) = (x_j - x_0)(x_j - x_1)(x_j - x_2) \cdots (x_j - x_{i-1}) = U_n(i, j)$. □

From Theorem 2 and Lemma 1, we have the following modification of the Theorem 3.1 of [1].

Theorem 3. *The Vandermonde matrix V_n can be factorized into n 1-lower banded matrices and n 1-upper banded matrices such that*

$V_n = L_n^{(1)} L_n^{(2)} \cdots L_n^{(n)} U_n^{(n)} \cdots U_n^{(2)} U_n^{(1)}$, *where, for* $1 \leqslant k \leqslant n$,

$$L_n^{(k)}(i, j) = \begin{cases} 1 & \textit{if } j = i, \\ x_{j-n+k} & \textit{if } i = j + 1, i \geqslant n - k + 1, \\ 0 & \textit{otherwise}. \end{cases}$$

$$U_n^{(k)}(i, j) = \begin{cases} 1 & \textit{if } j = i, j \leqslant n - k \textit{ or } j = i + 1, j \geqslant n - k + 1, \\ x_i - x_{n-k} & \textit{if } j = i, j > n - k, \\ 0 & \textit{otherwise}. \end{cases}$$

Thus, $L_n = L_n^{(1)} L_n^{(2)} \cdots L_n^{(n)}$, *and* $U_n = U_n^{(n)} \cdots U_n^{(2)} U_n^{(1)}$.

Example 2. $V_3 = L_3 U_3$, and L_3 is factorized into 1-lower banded matrices, $L_3 = L_3^{(1)} L_3^{(2)} L_3^{(3)}$, where

$$L_3^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & x_0 & 1 \end{pmatrix}, \quad L_3^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x_0 & 1 & 0 \\ 0 & 0 & x_1 & 1 \end{pmatrix}, \quad L_3^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_0 & 1 & 0 & 0 \\ 0 & x_1 & 1 & 0 \\ 0 & 0 & x_2 & 1 \end{pmatrix}.$$

Similarly, U_3 is factorized into 1-upper banded matrices, $U_3 = U_3^{(3)} U_3^{(2)} U_3^{(1)}$, where

$$U_3^{(3)} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & x_1 - x_0 & 1 & 0 \\ 0 & 0 & x_2 - x_0 & 1 \\ 0 & 0 & 0 & x_3 - x_0 \end{pmatrix}, \quad U_3^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & x_2 - x_1 & 1 \\ 0 & 0 & 0 & x_3 - x_1 \end{pmatrix},$$

$$U_3^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & x_3 - x_2 \end{pmatrix}.$$

Thus, $V_3 = L_3 U_3 = L^{(1)} L^{(2)} L^{(3)} U^{(3)} U^{(2)} U^{(1)}$.

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Reference

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