Note

On a connection between the Pascal, Stirling and Vandermonde matrices

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Received 24 February 2004; received in revised form 15 March 2007; accepted 14 May 2007
Available online 24 May 2007

Abstract

In this paper, we are going to study some additional relations between the Stirling matrix \(S_n\) and the Pascal matrix \(P_n\). Also the representation for the matrix \(T_n\) and \(T_n^{-1}\) in terms of \(s_n\) and \(S_n\) will be considered. Consequently, this will give an answer to an open problem proposed by EI-Mikkawy [On a connection between the Pascal, Vandermonde and Stirling matrices—II, Appl. Math. Comput. 146 (2003) 759–769].

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Keywords: Stirling number; Stirling matrix; Vandermonde matrix; Pascal matrix

1. Introduction

The lower triangular Pascal matrix \(P_n\) and the symmetric Pascal matrix \(Q_n\) which derived naturally from the Pascal triangle were studied by many authors [1–5] in recent years. The Stirling matrix of the first kind \(s_n\) and the Stirling matrix of the second kind \(S_n\) obtained from the Stirling numbers of the first kind \(s(i, j)\) and of the second kind \(S(i, j)\), respectively, are also introduced [6–8]. In [9], the author investigated a connection between the Pascal, Vandermonde and Stirling matrices, and showed by using MAPLE that a stochastic matrix \(T_n\) links together these matrices. In [10], the author raised that to generate the elements of the matrix \(T_n\) for any arbitray \(n\) using only one or two recurrence relations is an open question. In this paper, we obtain some relations between the Stirling matrix \(S_n\) and the Pascal matrix \(P_n\), and give a representation for the matrices \(T_n\) and \(T_n^{-1}\) by the using the Stirling matrices \(s_n\) and \(S_n\), the recurrence relations of the elements of the matrices \(T_n\) and \(T_n^{-1}\) are also obtained, hence we answer the open problem proposed by EI-Mikkawy [10]. As a consequence we obtain some combinatorial identities related to the Stirling numbers.

2. Preliminary results

Let \(n, k\) be nonnegative integers and \(n \geq k\), the Stirling numbers of the first kind \(s(n, k)\) and of the second kind \(S(n, k)\) can be defined as the coefficients in the following expansion of a variable \(x\) : \( (x)_n = \sum_{k=0}^{n} s(n, k) x^k \), and \( x^n = \sum_{k=0}^{n} S(n, k) (x)_k \), where \((x)_k = x(x-1)(x-2) \cdots (x-k+1)\) for any integer \(k > 0\), and \((x)_0 = 1\). \( s(k, k) = S(k, k) = 1\) for \(k \geq 0\), and \( s(n, 0) = S(n, 0) = 0\) for \(n > 0\).

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0166-218X/S - see front matter © 2007 Published by Elsevier B.V.
doi:10.1016/j.dam.2007.05.017
It is known that the Stirling numbers have the following recurrence relations (see [11]):
\[
s(n, k) = s(n - 1, k - 1) - (n - 1)s(n - 1, k), \tag{1}
\]
\[
S(n, k) = S(n - 1, k - 1) + kS(n - 1, k). \tag{2}
\]

The \( n \times n \) Pascal matrix \( P_n \) is defined by (see [4,5]) \( P_n = \left[ \binom{i-1}{j-1} \right]_{1 \leq i, j \leq n} \), where \( \binom{i}{j} = 0 \), if \( i < j \). It is known that \( P_n = \left[ (-1)^{i+j} \binom{i-1}{j-1} \right]_{1 \leq i, j \leq n} \). The Stirling matrix of the first kind \( s_n \) and the Stirling matrix of the second kind \( S_n \) are defined, respectively, by \( s_n = [s(i, j)]_{1 \leq i, j \leq n} \), \( S_n = [S(i, j)]_{1 \leq i, j \leq n} \), where \( s(i, j) = 0, S(i, j) = 0 \) if \( i < j \).

For example,
\[
S_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 3 & 1 & 0 \\
1 & 7 & 6 & 1
\end{pmatrix}, \quad S_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 3 & 1 & 0 \\
1 & 7 & 6 & 1
\end{pmatrix}.
\]

It is easy to see that \( S_n s_n = I_n, S_n^{-1} = s_n \).

**Lemma 1** (Cheon and Kim [6]). \( S_n = P_n([1] \oplus S_{n-1}); s_n = ([1] \oplus s_{n-1})P_n^{-1}. \)

**Lemma 2** (Cheon and Kim [6]). Define \( V_n \) be the \( n \times n \) Vandermonde matrix by \( V_n(i, j) = j^{i-1}, 1 \leq i, j \leq n. \) Then \( V_n = S_n D_n P_n^T, \) where \( D_n = \text{diag}(0!, 1!, 2!, \ldots, (n-1)!). \)

**Lemma 3** (El-Mikkawy [9]). Let \( Q_n = \left[ \binom{i+j-2}{j-1} \right]_{1 \leq i, j \leq n} \) be the \( n \times n \) symmetric Pascal matrix, then the matrix \( T_n \) links \( Q_n \) and the Vandermonde matrix \( V_n \) by \( Q_n = T_n V_n, \) where \( T_n = P_n D_n^{-1} s_n = P_n D_n^{-1} ([1] \oplus s_{n-1}) P_n^{-1}, \) and \( T_n^{-1} = S_n D_n P_n^{-1} = P_n ([1] \oplus S_{n-1}) D_n P_n^{-1}. \)

**Example 1.**
\[
S_4 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 \\
1 & 2^2 & 3^2 & 4^2 \\
1 & 2^3 & 3^3 & 4^3
\end{pmatrix}, \quad V_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 3 & 1 & 0 \\
1 & 7 & 6 & 1
\end{pmatrix}, \quad S_4 D_4 P_4^T = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 3
\end{pmatrix},
\]
\[
Q_4 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 \\
1 & 3 & 6 & 10 \\
1 & 4 & 10 & 20
\end{pmatrix}, \quad V_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
1 & 3 & 3 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6}
\end{pmatrix} = P_4 D_4^{-1} s_4 V_4.
\]

3. The main results

**Lemma 4** (Cheon and Kim [7]). \( V_n = ([1] \oplus S_{n-1}) D_n \Delta_n P_n^T, \) where \( \Delta_n \) is the \( n \times n \) lower triangular matrix whose \( (i, j) \)-entry is \( \begin{pmatrix} 1 \\ \frac{1}{i-j} \end{pmatrix} \) if \( i \geq j \) and otherwise 0.
Lemma 5. \( A_n P_n^{-1} = ([1] \oplus P_n^{-1})^{-1}. \)

Proof. \((A_n P_n^{-1})(i, j) = A_n(i, i, i, j)P_n^{-1}(i - 1, j) + A_n(i, i)P_n^{-1}(i, j) = 1 \cdot (-1)^{j-i-1} \left( \frac{j-2}{j-1} \right) + 1 \cdot (-1)^{i-j} \left( \frac{i-1}{j-1} \right) = (-1)^{i-j} \left( \left( \frac{j-2}{j-1} \right) - \left( \frac{i-2}{j-1} \right) \right) = (-1)^{i-j} \left( \frac{j-2}{j-1} \right) = ([1] \oplus P_n^{-1})(i, j). \)

Lemma 6. \( T_n^{-1} = ([1] \oplus S_n^{-1}) D_n([1] \oplus P_n^{-1}). \)

Proof. By Lemmas 3–5, we have \( T_n^{-1} = V_n Q_n^{-1} = ([1] \oplus S_n^{-1}) D_n A_n P_n^T (P_n^T)^{-1} P_n^{-1} = ([1] \oplus S_n^{-1}) D_n A_n P_n^{-1} = ([1] \oplus S_n^{-1}) D_n([1] \oplus P_n^{-1}). \)

Lemma 7. For each \( i, j = 1, 2, 3, \ldots, n, \ i \geq j \), we have

\[
S(i, j)j! = \sum_{k=j}^{i} (-1)^{i-k} S(i, k)k! \left( \frac{k-1}{j-1} \right). \tag{3}
\]

Proof. For a fixed positive integer \( j \), we prove the statement by induction on \( i \). For \( i = j \), the statement holds since the right hand of (3) equals \((-1)^{j-j} S(j, j)j! \left( \frac{j-1}{j-1} \right) = S(j, j)j! \), it is exactly the left hand of (3). Suppose it holds for \( \leq i \), and we want to prove it for \( i + 1 \). Using the recurrence relation (2) and the induction hypothesis we obtain

\[
S(i + 1, j)j! = j! (S(i, j - 1) + jS(i, j))
\]

\[
= j! \left( \frac{1}{(j-1)!} \sum_{k=j}^{i} (-1)^{i-k} S(i, k)k! \left( \frac{k-1}{j-2} \right) + \frac{j}{j!} \sum_{k=j}^{i} (-1)^{i-k} S(i, k)k! \left( \frac{k-1}{j-1} \right) \right)
\]

\[
= j \sum_{k=j}^{i} (-1)^{i-k} S(i, k)k! \left( \frac{k-1}{j-2} \right) + j \sum_{k=j}^{i} (-1)^{i-k} S(i, k)k! \left( \frac{k-1}{j-1} \right),
\]

that is \( S(i + 1, j)j! = j \sum_{k=j}^{i} (-1)^{i-k} S(i, k)k! \left( \frac{k-1}{j-1} \right) + j \sum_{k=j}^{i} (-1)^{i-k} S(i, k)k! \left( \frac{k-1}{j-2} \right). \)

On the other hand,

\[
\sum_{k=j}^{i+1} (-1)^{i+1-k} S(i + 1, k)k! \left( \frac{k-1}{j-1} \right)
\]

\[
= \sum_{k=j}^{i+1} (-1)^{i+1-k}(kS(i, k) + S(i, k - 1))k! \left( \frac{k-1}{j-1} \right)
\]

\[
= \sum_{k=j}^{i+1} x(-1)^{i+1-k} S(i, k)k! \left( \frac{k-1}{j-1} \right) + \sum_{k=j}^{i+1} (-1)^{i+1-k} S(i, k - 1)k! \left( \frac{k-1}{j-1} \right)
\]

\[
= \sum_{k=j}^{i} (-1)^{i+1-k} S(i, k)k! \left( \frac{k-1}{j-1} \right) + \sum_{k=j}^{i+1} \sum_{k=j}^{i+1} x(-1)^{i+1-k} S(i, k - 1)k! \left( \frac{k-1}{j-1} \right)
\]

\[
= - \sum_{k=j}^{i} (-1)^{i-k} S(i, k)k! \left( \frac{k-1}{j-1} \right) + \sum_{t=j-1}^{i} (-1)^{i-t} S(i, t)! \frac{t}{j-1} \left( \frac{t}{j-1} \right)
\]

\[
= j \sum_{k=j}^{i} (-1)^{i-k} S(i, k)k! \left( \frac{k-1}{j-1} \right) - \sum_{k=j}^{i} (-1)^{i-k} S(i, k)k! (k + j) \left( \frac{k-1}{j-1} \right)
\]

\[
= \sum_{k=j}^{i+1} (-1)^{i+1-k} S(i + 1, k)k! \left( \frac{k-1}{j-1} \right).
\]
Theorem 3. 

\[ + (-1)^{i-j+1}S(i, j-1)! + \sum_{k=j}^{i} (-1)^{i-k}S(i, k)k!(k+1) \binom{k}{j-1} \]

\[ = j \sum_{k=j}^{i} (-1)^{i-k}S(i, k)k! \binom{k-1}{j-1} + (-1)^{i-j+1}S(i, j-1)! \]

\[ + \sum_{k=j}^{i} (-1)^{i-k}S(i, k)k! (k+1) \binom{k}{j-1} - (k+j) \binom{k-1}{j-1} \]

\[ = j \sum_{k=j}^{i} (-1)^{i-k}S(i, k)k! \binom{k-1}{j-1} + (-1)^{i-j+1}S(i, j-1)! + \sum_{k=j}^{i} (-1)^{i-k}S(i, k)k!j \binom{k-1}{j-2} \]

\[ = j \sum_{k=j}^{i} (-1)^{i-k}S(i, k)k! \binom{k-1}{j-1} + j \sum_{k=j-1}^{i} (-1)^{i-k}S(i, k)k! \binom{k-1}{j-2} , \]

therefore, \( S(i+1, j)! = \sum_{k=j}^{i+1} x(-1)^{i+1-k}S(i+1, k)k! \binom{k-1}{j}, \) this completes the proof. \( \square \)

Using Lemma 7 and considering the matrix equality, we obtain the following results immediately:

**Theorem 1.** \( \tilde{S}_n = J_n \tilde{S}_n J_n P_n \), and \( \tilde{S}_n P_n^{-1} = J_n \tilde{S}_n J_n, \) where \( \tilde{S}_n = S_n \text{diag}(1, 2!, \ldots, n!) \).

**Theorem 2.** The matrix \( T_n^{-1} \) has the following decomposition and properties:

(a) \( T_n^{-1} = J_n ([1] \oplus S_{n-1}) J_n \);

(b) \( T_n^{-1}(i, j) = (-1)^{i-j}S(i-1, j-1)(j-1)! \);

(c) \( T_n^{-1}(i, j) = [T_n^{-1}(i-1, j-1) - T_n^{-1}(i-1, j)](j-1) \).

**Proof.** (a) Using Lemma 6 and Theorem 1, we have \( T_n^{-1} = ([1] \oplus S_{n-1}) D_n ([1] \oplus P_n^{-1}) = ([1] \oplus S_{n-1}) ([1] \oplus P_n^{-1}) = [1] \oplus (S_n^{-1} P_n^{-1}) = [1] \oplus (J_n^{-1} S_{n-1} J_n) = J_n ([1] \oplus S_{n-1}) J_n \).

(b) From (a), we have \( T_n^{-1}(i, j) = (J_n ([1] \oplus S_{n-1}) J_n)(i, j) = (-1)^{i-j}S(i-1, j-1)(j-1)! \).

(c) From (b) and recurrence relation (2), we obtain \( [T_n^{-1}(i-1, j-1) - T_n^{-1}(i-1, j)](j-1) = [(-1)^{i-j}S(i-2, j-2)! + (-1)^{i-j-1}S(i-2, j-1)(j-1)!](j-1) = (-1)^{i-j}(j-2)! S(i-2, j-2) + S(i-2, j-1)(j-1) = (-1)^{i-j}(j-1)! S(i-1, j-1) = T_n^{-1}(i, j). \) \( \square \)

El-Mikkawy [10] point out that to generate the elements of the matrix \( T_n \) for any arbitrary \( n \) using only one or two recurrence relations is an open question. We are now in a position to give a answer to this problem.

**Theorem 3.** The matrix \( T_n \) has the following decomposition and properties:

(a) \( T_n = J_n D_n^{-1} ([1] \oplus s_{n-1}) J_n \);

(b) \( T_n(i, j) = (-1)^{i-j}S(i-1, j-1)/(i-1)! \);

(c) \( T_n \) is a stochastic matrix;

(d) \( T_n(i, j) = (1/(i-1))T_n(i-1, j-1) + (i-2)/(i-1)T_n(i-1, j) \).

**Proof.** (a) From Theorem 2 (a), \( T_n^{-1} = J_n ([1] \oplus S_{n-1}) J_n \), hence \( T_n = (J_n ([1] \oplus S_{n-1}) J_n)^{-1} = (J_n ([1] \oplus S_{n-1}) D_n J_n)^{-1} = J_n D_n^{-1} ([1] \oplus s_{n-1}) J_n \).

(b) By (a), we have \( T_n(i, j) = (J_n D_n^{-1} ([1] \oplus s_{n-1}) J_n)(i, j) = (-1)^{i-j}S(i-1, j-1)/(i-1)! \).
(c) From (b), it is clear that the elements of the matrix $T_n$ are all nonnegative. Since $\sum_{k=1}^{i} (-1)^k s(i, k) = (-1)^i i!$, we have $\sum_{j=1}^{i} T_n(i, j) = \sum_{j=1}^{i} (-1)^{i-j} s(i-1, j-1)/(i-1)! = ((-1)^i/(i-1)!) \sum_{j=1}^{i} (-1)^j s(i-1, j-1) = ((-1)^i/(i-1)!) \sum_{j=1}^{i} (-1)^{k+1} s(i-1, k) = ((-1)^i/(i-1)!)((-1)^i/i!) = 1$, therefore, $T_n$ is a stochastic matrix.

(d) From (b) and recurrence relation (1), we have $(1/(i-1))T_n(i-1, j-1) + (i-2)/(i-1)T_n(i-1, j) = (1/(i-1))((-1)^{i-1} s(i-2, j-1) + (i-2) + (1)^{i-1} s(i-2, j-1)1/(i-2)! = 1/i - 1((-1)^{i-1} (1/(i-2)!))[s(i-2, j-2) - (i-2)s(i-2, j-1)] = (-1)^{i-1} s(i-1, j-1)1/(i-1)! = T_n(i, j)$. □

Example 2.

$$T_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = J_5D_5^{-1}(1 \oplus s_4)J_5,$$

$$T_5^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & 1 & -6 & 6 & 0 \\ 0 & -1 & 14 & -36 & 24 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = J_5([1] \oplus s_4)D_5J_5.$$

4. Some combinatorial identities

Applying the two different representations: $T_n^{-1} = S_n D_n P_n^{-1}$, and $T_n^{-1} = J_n([1] \oplus \sim s_{n-1})J_n$, the following results hold:

$$J_n([1] \oplus \sim s_{n-1})J_n = S_n D_n P_n^{-1}, \quad S_n D_n = J_n([1] \oplus \sim s_{n-1})J_n P_n,$$

$$J_n D_n^{-1}([1] \oplus s_{n-1})J_n = P_n D_n^{-1} P_n, \quad D_n^{-1} s_n = P_n^{-1} J_n D_n^{-1}([1] \oplus s_{n-1})J_n,$$

$$Q_n = J_n D_n^{-1}([1] \oplus s_{n-1})Q_n \quad P_n = J_n D_n^{-1}([1] \oplus s_{n-1})J_n S_n D_n.$$

Considering the matrix equality (4), we have the following identities for the Stirling numbers of the second kind:

$$S(i - 1, j - 1)(j - 1)! = \sum_{k=j}^{i} (-1)^{i-k} S(i, k)(k-1)! \binom{k-1}{j-1},$$

$$S(i, j)(j - 1)! = \sum_{k=j}^{i} (-1)^{i-k} S(i, k, k-1)(k-1)! \binom{k-1}{j-1}.$$
Using (5) yields the following identities for the Stirling numbers of the first kind:

\[ (-1)^i j \frac{s(i - 1, j - 1)}{(i - 1)!} = \sum_{k=j}^{i} \binom{i-1}{k-1} \frac{s(k, j)}{(k-1)!}, \quad 2 \leq i \text{ and } 2 \leq j \leq i - 1, \tag{9} \]

\[ \frac{s(i, j)}{(i - 1)!} = \sum_{k=j}^{i} (-1)^{i-j} \binom{i-1}{k-1} \frac{s(k-1, j-1)}{(k-1)!}, \quad 2 \leq i \text{ and } 2 \leq j \leq i - 1. \tag{10} \]

From (6) we obtain the following identities:

\[ \binom{i + j - 2}{j - 1} = \frac{1}{(i - 1)!} \sum_{k=1}^{i} (-1)^{i-k} s(i - 1, k - 1) j^{k-1}, \tag{11} \]

\[ \binom{i - 1}{j - 1} = \frac{(j - 1)!}{(i - 1)!} \sum_{k=j}^{i} (-1)^{i-k} s(i - 1, k - 1) S(k, j). \tag{12} \]

In particular for \( j = 1, 2 \), the identity (3) gives

\[ \sum_{k=1}^{i} (-1)^{i-k} S(i, k) k! = 1, \tag{13} \]

\[ \sum_{k=2}^{i} (-1)^{i-k} S(i, k) k!(k-1) = 2^i - 2. \tag{14} \]

In particular for \( j = 1, 2 \), the identity (8) gives

\[ \sum_{k=1}^{i} (-1)^{i-k} S(i - 1, k - 1) (k - 1)! = 1, \tag{15} \]

\[ \sum_{k=2}^{i} (-1)^{i-k} S(i - 1, k - 1) (k - 1)! (k - 1) = 2^{i-1} - 1. \tag{16} \]

Acknowledgements

The authors wish to thank the helpful comments and suggestions of the referees. This work is supported by Development Program for Outstanding Young Teachers in Lanzhou University of Technology and NSF of Gansu Province of China.

References