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ScienceDirect
Discrete Applied Mathematics 155 (2007) 2025 – 2030
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Note

On a connection between the Pascal, Stirling and Vandermonde matrices

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Received 24 February 2004; received in revised form 15 March 2007; accepted 14 May 2007
Available online 24 May 2007

Abstract

In this paper, we are going to study some additional relations between the Stirling matrix S_n and the Pascal matrix P_n. Also the representation for the matrix T_n and T_n^{-1} in terms of s_n and S_n will be considered. Consequently, this will give an answer to an open problem proposed by EI-Mikkawy [On a connection between the Pascal, Vandermonde and Stirling matrices—II, Appl. Math. Comput. 146 (2003) 759–769].

Keywords: Stirling number; Stirling matrix; Vandermonde matrix; Pascal matrix

1. Introduction

The lower triangular Pascal matrix P_n and the symmetric Pascal matrix Q_n which derived naturally from the Pascal triangle were studied by many authors [1–5] in recent years. The Stirling matrix of the first kind s_n and the Stirling matrix of the second kind S_n obtained from the Stirling numbers of the first kind $s(i, j)$ and of the second kind $S(i, j)$, respectively, are also introduced [6–8]. In [9], the author investigated a connection between the Pascal, Vandermonde and Stirling matrices, and showed by using MAPLE that a stochastic matrix T_n links together these matrices. In [10], the author raised that to generate the elements of the matrix T_n for any arbitray n using only one or two recurrence relations is an open question. In this paper, we obtain some relations between the Stirling matrix S_n and the Pascal matrix P_n, and give a representation for the matrices T_n and T_n^{-1} by the using the Stirling matrices S_n and s_n, the recurrence relations of the elements of the matrices T_n and T_n^{-1} are also obtained, hence we answer the open problem proposed by EI-Mikkawy [10]. As a consequence we obtain some combinatorial identities related to the Stirling numbers.

2. Preliminary results

Let n, k be nonnegative integers and $n \geqslant k$, the Stirling numbers of the first kind $s(n, k)$ and of the second kind $S(n, k)$ can be defined as the coefficients in the following expansion of a variable x : $(x)_n = \sum_{k=0}^{n} s(n, k)x^k$, and $x^n = \sum_{k=0}^{n} S(n, k)(x)_k$, where $(x)_k = x(x-1)(x-2)\cdots(x-k+1)$ for any integer $k > 0$, and $(x)_0 = 1$. $s(k, k) = S(k, k) = 1$ for $k \geqslant 0$, and $s(n, 0) = S(n, 0) = 0$ for $n > 0$.

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0166-218X/$ - see front matter © 2007 Published by Elsevier B.V.
doi:10.1016/j.dam.2007.05.017

It is known that the Stirling numbers have the following recurrence relations (see [11]):

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), \tag{1}$$

$$S(n,k) = S(n-1,k-1) + kS(n-1,k). \tag{2}$$

The $n \times n$ Pascal matrix P_n is defined by (see [4,5]) $P_n = \left[\binom{i-1}{j-1}\right]_{1 \leqslant i,j \leqslant n}$, where $\binom{i}{j} = 0$, if $i < j$. It is known that $P_n = \left[(-1)^{i+j}\binom{i-1}{j-1}\right]_{1 \leqslant i,j \leqslant n}$. The Stirling matrix of the first kind s_n and the Stirling matrix of the second kind S_n are defined, respectively, by $s_n = [s(i,j)]_{1 \leqslant i,j \leqslant n}$, $S_n = [S(i,j)]_{1 \leqslant i,j \leqslant n}$, where $s(i,j) = 0$, $S(i,j) = 0$ if $i < j$.

For example,

$$s_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix}.$$

It is easy to see that $S_n s_n = I_n$, $S_n^{-1} = s_n$.

Lemma 1 (*Cheon and Kim [6]*). $S_n = P_n([1] \oplus S_{n-1})$; $s_n = ([1] \oplus s_{n-1})P_n^{-1}$.

Lemma 2 (*Cheon and Kim [6]*). *Define V_n be the $n \times n$ Vandermonde matrix by $V_n(i,j) = j^{i-1}$, $1 \leqslant i, j \leqslant n$. Then $V_n = S_n D_n P_n^{\mathrm{T}}$, where $D_n = diag(0!, 1!, 2!, \ldots, (n-1)!)$.*

Lemma 3 (*EI-Mikkawy [9]*). *Let $Q_n = \left[\binom{i+j-2}{j-1}\right]_{1 \leqslant i,j \leqslant n}$ be the $n \times n$ symmetric Pascal matrix, then the matrix T_n links Q_n and the Vandermonde matrix V_n by $Q_n = T_n V_n$, where $T_n = P_n D_n^{-1} s_n = P_n D_n^{-1}([1] \oplus s_{n-1})P_n^{-1}$, and $T_n^{-1} = S_n D_n P_n^{-1} = P_n([1] \oplus S_{n-1})D_n P_n^{-1}$.*

Example 1.

$$S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 3 & 1 \end{pmatrix} = P_4([1] \oplus S_3)$$

$$V_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = S_4 D_4 P_4^{\mathrm{T}},$$

$$Q_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{6} \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \end{pmatrix} = P_4 D_4^{-1} s_4 V_4.$$

3. The main results

Lemma 4 (*Cheon and Kim [7]*). $V_n = ([1] \oplus S_{n-1})D_n \Delta_n P_n^{\mathrm{T}}$, *where Δ_n is the $n \times n$ lower triangular matrix whose (i,j)-entry is $\binom{1}{i-j}$ if $i \geqslant j$ and otherwise* 0.

Lemma 5. $\Delta_n P_n^{-1} = ([1] \oplus P_{n-1}^{-1}) = ([1] \oplus P_{n-1})^{-1}$.

Proof. $(\Delta_n P_n^{-1})(i, j) = \Delta_n(i, i-1) P_n^{-1}(i-1, j) + \Delta_n(i, i) P_n^{-1}(i, j) = 1 \cdot (-1)^{i-j-1} \binom{i-2}{j-1} + 1 \cdot (-1)^{i-j} \binom{i-1}{j-1} = (-1)^{i-j} \left(\binom{i-1}{j-1} - \binom{i-2}{j-1} \right) = (-1)^{i-j} \binom{i-2}{j-2} = ([1] \oplus P_{n-1}^{-1})(i, j)$. □

Lemma 6. $T_n^{-1} = ([1] \oplus S_{n-1}) D_n ([1] \oplus P_{n-1}^{-1})$.

Proof. By Lemmas 3–5, we have $T_n^{-1} = V_n Q_n^{-1} = (([1] \oplus S_{n-1}) D_n \Delta_n P_n^{\mathrm{T}})((P_n^{\mathrm{T}})^{-1} P_n^{-1}) = ([1] \oplus S_{n-1}) D_n \Delta_n P_n^{-1} = ([1] \oplus S_{n-1}) D_n ([1] \oplus P_{n-1}^{-1})$.

Lemma 7. *For each* $i, j = 1, 2, 3, \ldots, n,\ i \geqslant j$, *we have*

$$S(i, j) j! = \sum_{k=j}^{i} (-1)^{i-k} S(i, k) k! \binom{k-1}{j-1}. \tag{3}$$

Proof. For a fixed positive integer j, we prove the statement by induction on i. For $i = j$, the statement holds since the right hand of (3) equals $(-1)^{j-j} S(j, j) j! \binom{j-1}{j-1} = S(j, j) j!$, it is exactly the left hand of (3). Suppose it holds for $\leqslant i$, and we want to prove it for $i + 1$. Using the recurrence relation (2) and the induction hypothesis we obtain

$$\begin{aligned} S(i+1, j) j! &= j!(S(i, j-1) + j S(i, j)) \\ &= j! \left(\frac{1}{(j-1)!} \sum_{k=j-1}^{i} (-1)^{i-k} S(i, k) k! \binom{k-1}{j-2} + \frac{j}{j!} \sum_{k=j}^{i} (-1)^{i-k} S(i, k) k! \binom{k-1}{j-1} \right) \\ &= j \sum_{k=j-1}^{i} (-1)^{i-k} S(i, k) k! \binom{k-1}{j-2} + j \sum_{k=j}^{i} (-1)^{i-k} S(i, k) k! \binom{k-1}{j-1}, \end{aligned}$$

that is $S(i+1, j) j! = j \sum_{k=j}^{i} (-1)^{i-k} S(i, k) k! \binom{k-1}{j-1} + j \sum_{k=j-1}^{i} (-1)^{i-k} S(i, k) k! \binom{k-1}{j-2}$.

On the other hand,

$$\begin{aligned} &\sum_{k=j}^{i+1} (-1)^{i+1-k} S(i+1, k) k! \binom{k-1}{j-1} \\ &\quad = \sum_{k=j}^{i+1} (-1)^{i+1-k} (k S(i, k) + S(i, k-1)) k! \binom{k-1}{j-1} \\ &\quad = \sum_{k=j}^{i+1} x (-1)^{i+1-k} S(i, k) k! k \binom{k-1}{j-1} + \sum_{k=j}^{i+1} (-1)^{i+1-k} S(i, k-1) k! \binom{k-1}{j-1} \\ &\quad = \sum_{k=j}^{i} (-1)^{i+1-k} S(i, k) k! k \binom{k-1}{j-1} + \sum_{k=j}^{i+1} x (-1)^{i+1-k} S(i, k-1) k! \binom{k-1}{j-1} \\ &\quad = -\sum_{k=j}^{i} (-1)^{i-k} S(i, k) k! k \binom{k-1}{j-1} + \sum_{t=j-1}^{i} (-1)^{i-t} S(i, t) t! (t+1) \binom{t}{j-1} \\ &\quad = j \sum_{k=j}^{i} (-1)^{i-k} S(i, k) k! \binom{k-1}{j-1} - \sum_{k=j}^{i} (-1)^{i-k} S(i, k) k! (k+j) \binom{k-1}{j-1} \end{aligned}$$

$$
\begin{aligned}
&+(-1)^{i-j+1}S(i,j-1)j!+\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!(k+1)\binom{k}{j-1}\\
&=j\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-1}+(-1)^{i-j+1}S(i,j-1)j!\\
&+\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!\left((k+1)\binom{k}{j-1}-(k+j)\binom{k-1}{j-1}\right)\\
&=j\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-1}+(-1)^{i-j+1}S(i,j-1)j!+\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!j\binom{k-1}{j-2}\\
&=j\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-1}+j\sum_{k=j-1}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-2},
\end{aligned}
$$

therefore, $S(i+1,j)j! = \sum_{k=j}^{i+1} x(-1)^{i+1-k}S(i+1,k)k!\binom{k-1}{j-1}$, this completes the proof. □

Using Lemma 7 and considering the matrix equality, we obtain the following results immediately:

Theorem 1. $\widetilde{S_n} = J_n \widetilde{S_n} J_n P_n$, *and* $\widetilde{S_n} P_n^{-1} = J_n \widetilde{S_n} J_n$, *where* $\widetilde{S_n} = S_n\, diag(1, 2!, \cdots, n!)$.

Theorem 2. *The matrix* T_n^{-1} *has the following decomposition and properties*:

(a) $T_n^{-1} = J_n([1] \oplus \widetilde{S}_{n-1})J_n$;
(b) $T_n^{-1}(i,j) = (-1)^{i-j}S(i-1,j-1)(j-1)!$;
(c) $T_n^{-1}(i,j) = [T_n^{-1}(i-1,j-1) - T_n^{-1}(i-1,j)](j-1)$.

Proof. (a) Using Lemma 6 and Theorem 1, we have $T_n^{-1} = ([1] \oplus S_{n-1})D_n([1] \oplus P_{n-1}^{-1}) = ([1] \oplus \widetilde{S}_{n-1})([1] \oplus P_{n-1}^{-1}) = [1] \oplus (\widetilde{S}_{n-1}P_{n-1}^{-1}) = [1] \oplus (J_{n-1}\widetilde{S}_{n-1}J_{n-1}) = J_n([1] \oplus \widetilde{S}_{n-1})J_n$.

(b) From (a), we have $T_n^{-1}(i,j) = (J_n([1] \oplus \widetilde{S}_{n-1})J_n)(i,j) = (-1)^{i-j}S(i-1,j-1)(j-1)!$.

(c) From (b) and recurrence relation (2), we obtain $[T_n^{-1}(i-1,j-1) - T_n^{-1}(i-1,j)](j-1) = [(-1)^{i-j}S(i-2,j-2)(j-2)! - (-1)^{i-j-1}S(i-2,j-1)(j-1)!](j-1) = (-1)^{i-j}(j-2)![S(i-2,j-2) + S(i-2,j-1)(j-1)](j-1) = (-1)^{i-j}(j-1)!S(i-1,j-1) = T_n^{-1}(i,j)$. □

EI-Mikkawy [10] point out that to generate the elements of the matrix T_n for any arbitray n using only one or two recurrence relations is an open question. We are now in a position to give a answer to this problem.

Theorem 3. *The matrix* T_n *has the following decomposition and properties*:

(a) $T_n = J_n D_n^{-1}([1] \oplus s_{n-1})J_n$;
(b) $T_n(i,j) = (-1)^{i-j}s(i-1,j-1)/(i-1)!$;
(c) T_n *is a stochastic matrix*;
(d) $T_n(i,j) = (1/(i-1))T_n(i-1,j-1) + (i-2)/(i-1)T_n(i-1,j)$.

Proof. (a) From Theorem 2 (a), $T_n^{-1} = J_n([1] \oplus \widetilde{S}_{n-1})J_n$, hence $T_n = (J_n([1] \oplus \widetilde{S}_{n-1})J_n)^{-1} = (J_n([1] \oplus S_{n-1})D_nJ_n)^{-1} = J_nD_n^{-1}([1] \oplus s_{n-1})J_n$.

(b) By (a), we have $T_n(i,j) = (J_nD_n^{-1}([1] \oplus s_{n-1})J_n)(i,j) = (-1)^{i-j}s(i-1,j-1)/(i-1)!$.

(c) From (b), it is clear that the elements of the matrix T_n are all nonnegative. Since $\sum_{k=1}^{i}(-1)^k s(i,k)=(-1)^i i!$, we have $\sum_{j=1}^{i} T_n(i,j)=\sum_{j=1}^{i}(-1)^{i-j}s(i-1,j-1)(1/(i-1)!)=((-1)^i/(i-1)!)\sum_{j=1}^{i}(-1)^j s(i-1,j-1)=((-1)^i/(i-1)!)\sum_{k=1}^{i-1}(-1)^{k+1}s(i-1,k)=((-1)^i/(i-1)!)(-1)^i(i-1)!=1$, therefore, T_n is a stochastic matrix.

(d) From (b) and recurrence relation (1), we have $(1/(i-1))T_n(i-1,j-1)+(i-2)/(i-1)T_n(i-1,j))=(1/(i-1))[(-1)^{i-j}s(i-2,j-2)1/(i-2)!+(i-2)(-1)^{i-j-1}s(i-2,j-1)1/(i-2)!]=1/i-1(-1)^{i-j}(1/(i-2)!)[s(i-2,j-2)-(i-2)s(i-2,j-1)]=(-1)^{i-j}s(i-1,j-1)1/(i-1)!=T_n(i,j)$. $\square$

Example 2.

$$T_5=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&\frac12&\frac12&0&0\\0&\frac13&\frac12&\frac16&0\\0&\frac14&\frac{11}{24}&\frac14&\frac1{24}\end{pmatrix}=\begin{pmatrix}1&0&0&0&0\\0&-1&0&0&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&\frac12&0&0\\0&0&0&\frac16&0\\0&0&0&0&\frac1{24}\end{pmatrix}$$

$$\times\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&-1&1&0&0\\0&2&-3&1&0\\0&-6&11&-6&1\end{pmatrix}\begin{pmatrix}1&0&0&0&0\\0&-1&0&0&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&1\end{pmatrix}=J_5D_5^{-1}([1]\oplus s_4)J_5,$$

$$T_5^{-1}=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&-1&2&0&0\\0&1&-6&6&0\\0&-1&14&-36&24\end{pmatrix}=\begin{pmatrix}1&0&0&0&0\\0&-1&0&0&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&1&1&0&0\\0&1&3&1&0\\0&1&7&6&1\end{pmatrix}$$

$$\times\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&2&0&0\\0&0&0&6&0\\0&0&0&0&24\end{pmatrix}\begin{pmatrix}1&0&0&0&0\\0&-1&0&0&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&1\end{pmatrix}=J_5([1]\oplus S_4)D_5J_5.$$

4. Some combinatorial identities

Applying the two different representations: $T_n^{-1}=S_nD_nP_n^{-1}$, and $T_n^{-1}=J_n([1]\oplus\widetilde{S}_{n-1})J_n$, the following results hold:

$$J_n([1]\oplus\widetilde{S}_{n-1})J_n=S_nD_nP_n^{-1},\quad S_nD_n=J_n([1]\oplus\widetilde{S}_{n-1})J_nP_n,\tag{4}$$

$$J_nD_n^{-1}([1]\oplus s_{n-1})J_n=P_nD_n^{-1}s_n,\quad D_n^{-1}s_n=P_n^{-1}J_nD_n^{-1}([1]\oplus s_{n-1})J_n,\tag{5}$$

$$Q_n=J_nD_n^{-1}([1]\oplus s_{n-1})J_nV_n,\quad P_n=J_nD_n^{-1}([1]\oplus s_{n-1})J_nS_nD_n.\tag{6}$$

Considering the matrix equality (4), we have the following identities for the Stirling numbers of the second kind:

$$S(i-1,j-1)(j-1)!=\sum_{k=j}^{i}(-1)^{i-k}S(i,k)(k-1)!\binom{k-1}{j-1},\tag{7}$$

$$S(i,j)(j-1)!=\sum_{k=j}^{i}(-1)^{i-k}S(i-1,k-1)(k-1)!\binom{k-1}{j-1}.\tag{8}$$

Using (5) yields the following identities for the Stirling numbers of the first kind:

$$(-1)^{i-j}\frac{s(i-1,j-1)}{(i-1)!}=\sum_{k=j}^{i}\binom{i-1}{k-1}\frac{s(k,j)}{(k-1)!},\quad 2\leqslant i \text{ and } 2\leqslant j\leqslant i-1, \tag{9}$$

$$\frac{s(i,j)}{(i-1)!}=\sum_{k=j}^{i}(-1)^{i-j}\binom{i-1}{k-1}\frac{s(k-1,j-1)}{(k-1)!},\quad 2\leqslant i \text{ and } 2\leqslant j\leqslant i-1. \tag{10}$$

From (6) we obtain the following identities:

$$\binom{i+j-2}{j-1}=\frac{1}{(i-1)!}\sum_{k=1}^{i}(-1)^{i-k}s(i-1,k-1)j^{k-1}, \tag{11}$$

$$\binom{i-1}{j-1}=\frac{(j-1)!}{(i-1)!}\sum_{k=j}^{i}(-1)^{i-k}s(i-1,k-1)S(k,j). \tag{12}$$

In particular for $j=1,2$, the identity (3) gives

$$\sum_{k=1}^{i}(-1)^{i-k}S(i,k)k!=1, \tag{13}$$

$$\sum_{k=2}^{i}(-1)^{i-k}S(i,k)k!(k-1)=2^{i}-2. \tag{14}$$

In particular for $j=1,2$, the identity (8) gives

$$\sum_{k=1}^{i}(-1)^{i-k}S(i-1,k-1)(k-1)!=1, \tag{15}$$

$$\sum_{k=2}^{i}(-1)^{i-k}S(i-1,k-1)(k-1)!(k-1)=2^{i-1}-1. \tag{16}$$

Acknowledgements

The authors wish to thank the helpful comments and suggestions of the referees. This work is supported by Development Program for Outstanding Young Teachers in Lanzhou University of Technology and NSF of Gansu Province of China.

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