Research Article

Combinatorial Interpretation of General Eulerian Numbers

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Since the 1950s, mathematicians have successfully interpreted the traditional Eulerian numbers and q-Eulerian numbers combinatorially. In this paper, the authors give a combinatorial interpretation to the general Eulerian numbers defined on general arithmetic progressions \( \{a, a+d, a+2d, \ldots \} \).

1. Introduction

Definition 1. Given a positive integer \( n \), define \( \Omega_n \) as the set of all permutations of \( \{1, 2, 3, \ldots, n\} \). For a permutation \( \pi = p_1 p_2 p_3 \ldots p_n \in \Omega_n \), \( \pi \) is called an ascent of \( \pi \) if \( p_i < p_{i+1} \); it is called a weak exceedance of \( \pi \) if \( p_i \geq i \).

It is well known that a traditional Eulerian number \( A_{n,k} \) is the number of permutations \( \pi \in \Omega_n \) that have \( k \) weak exceedances [1, page 215]. And \( A_{n,k} \) satisfies the recurrence:
\[
A_{n+1,k} = \binom{n+1}{k} A_{n,k} + (n+1-k) A_{n,k-1} \quad (1 \leq k \leq n) \tag{1}
\]

Besides the recursive formula (1), \( A_{n,k} \) can be calculated directly by the following analytic formula [2, page 8]:
\[
A_{n,k} = \sum_{i=0}^{k-1} (-1)^i (k-i)^n \binom{n+1}{i} \quad (1 \leq k \leq n) \tag{2}
\]

Definition 2. Given a permutation \( \pi = p_1 p_2 p_3 \ldots p_n \in \Omega_n \), define functions
\[
\text{maj} \pi = \sum_{i=0}^{n-1} j, \quad a(n,k,i) = \# \{ \pi \mid \text{maj} \pi = i \land \pi \text{ has } k \text{ ascents} \} \quad (3)
\]

Under Carlitz's definition, the \( q \)-Eulerian numbers \( \Lambda_{n,k}(q) \) are given by
\[
\Lambda_{n,k}(q) = q^{(n-k+1)(n-k)/2} \sum_{i=0}^{k(n-k-1)} a(n,k-i) q^i \tag{4}
\]

where functions \( a(n,k,i) \) are as defined in Definition 2.

In [5], instead of studying \( q \) sequences, the authors have generalized Eulerian numbers to any general arithmetic progression \( \{a, a+d, a+2d, a+3d, \ldots \} \).

Under the new definition, and given an arithmetic progression as defined in (5), the general Eulerian numbers \( \Lambda_{n,k}(a,d) \) can be calculated directly by the following equation [5, Lemma 2.6]:
\[
\Lambda_{n,k}(a,d) = \sum_{i=0}^{k} (-1)^i (k-i+1)^d a^i \binom{n+1}{i} \tag{6}
\]

Interested readers can find more results about the general Eulerian numbers and even general Eulerian polynomials in [5].

2. Combinatorial Interpretation of General Eulerian Numbers

The following concepts and properties will be heavily used in this section.
Definition 3. Let $W_{nk}$ be the set of $n$-permutations with $k$ weak exceedences. Then $|W_{nk}| = A_{nk}$. Furthermore, given a permutation $\pi = p_1p_2p_3\ldots p_n$, let $Q_n(\pi) = i$, where $p_i = n$.

Given a permutation $\pi \in \Omega_n$, it is known that $\pi$ can be written as a one-line form like $\pi = p_1p_2p_3\ldots p_n$ or $\pi$ can be written in a disunion of distinct cycles. For $\pi$ written in a cycle form, we can use a standard representation by writing (a) each cycle starting with its largest element and (b) the cycles in increasing order of their largest element. Moreover, given a permutation $\pi$ written in a standard representation cycle form, define a function $f$ as $f(\pi)$ to be the permutation obtained from $\pi$ by erasing the parentheses. Then $f$ is known as the fundamental bijection from $\Omega_n$ to itself [6, page 30]. Indeed, the inverse map $f^{-1}$ of the fundamental bijection function $f$ is also famous in illustrating the relation between the ascents and weak exceedances as follows [2, page 98].

Proposition 4. The function $f^{-1}$ gives a bijection between the set of permutations on $\{n\}$ with $k$ ascents and the set $W_{nk+1}$.

Example 5. The standard representation of permutation $\pi = 52437616k(2)43(7615) \in \Omega_n$ and $f(\pi) = 2437615k\pi(\pi) = 5; \pi = 5243761$ has $3$ ascents, while $f^{-1}(\pi) = (5243)(7615) = 6453271 \in W_{nk}$ has $3 + 1 = 4$ weak exceedances because $p_1 = 6 > 1,p_2 = 4 > 2,p_3 = 5 > 3$, and $p_k = 7 > 6$.

Now suppose we want to construct a sequence consisting of $k$ vertical bars and the first $n$ positive integers. Then the $k$ vertical bars divide these numbers into $k + 1$ compartments. In each compartment, there is either an odd number or all the numbers are listed in a decreasing order. The following definition is analogous to the definition of [2, page 8].

Definition 6. A bar in the above construction is called extraneous if either

(a) it is immediately followed by another bar; or

(b) each of the rest compartment is either empty or consists of integers in a decreasing order if this bar is removed.

Example 7. Suppose $n = 7$, $k = 4$; then in the following arrangement

\[
\begin{array}{c}
32117654
\end{array}
\]

(7)

the 1st, 2nd, and 4th bars are extraneous.

Now we are ready to give combinatorial interpretations to the general Eulerian numbers $A_{nk}(a,d)$. First note that (6) implies that $A_{nk}(a,d)$ is a homogeneous polynomial of degree with respect to $a$ and $d$. Indeed,

\[
A_{nk}(a,d) = \sum_{i=0}^{k} (-1)^i [(k + 1 - i)d - a]^i \binom{n + 1}{i}
\]

(8)

where

\[
c_{nk}(j) = \sum_{i=0}^{k} (-1)^i (k + 1 - i)^{n+1 - j} \binom{n+1}{i},
\]

(9)

0 \leq j \leq n.

The following theorem gives combinatorial interpretations to the coefficients $c_{nk}(j), 0 \leq j \leq n$.

Theorem 8. Let the general Eulerian numbers $A_{nk}(a,d)$ be written as in (8). Then

\[
c_{nk}(j) = \# \{ \pi \in W_{nk+1} : j < Q_n(\pi) \leq n \}
\]

(10)

+ \# \{ \pi \in W_{nk+1} : 1 \leq Q_n(\pi) \leq j \}.

Proof. We can check the result in (10) for two special values $j = 0$ and $j = n$ quickly. By (2),

when $j = 0$, $c_{nk}(0) = \sum_{i=0}^{k} (-1)^i (k + 1 - i)^{n+1} = A_{nk+1}$;

when $j = n$, $c_{nk}(n) = \sum_{i=0}^{k} (-1)^i (k + 1 - i)^{n+1} = A_{nk}$. Therefore, (10) is true for $j = 0$ and $j = n$.

Generally, for $1 \leq j \leq n-1$, we write down $k$ bars with $k + 1$ compartments in between. Place each element of $\{n\}$ in a compartment. If none of the $k$ bars is extraneous, then the arrangement corresponds to a permutation with $k$ ascents. Let $B_{j}$ be the set of arrangements with at most one extraneous bar at the end and none of integers $\{1,2,\ldots,j\}$ locating in the last compartment. We will show that $c_{nk}(j) = |B_{j}|$.

To achieve that goal, we use the Principle of Inclusion and Exclusion. There are $(k + 1)^{n+1}/k^j$ ways to put $n$ numbers into $k + 1$ compartments with elements $\{1,2,\ldots,j\}$ avoiding the last compartments.

Let $B_{j}$ be the number of arrangements with the following features:

(1) none of $\{1,2,\ldots,j\}$ sits in the last compartment;

(2) each arrangement in $B_{j}$ has at least $i$ extraneous bars.

(3) in each arrangement in $B_{j}$, any two extraneous bars are not located right next to each other.
Then the Principle of Inclusion and Exclusion shows that

$$|\mathcal{B}| = (k + 1)^{n} - |B| + B_{1} - B_{2} + \cdots - (-1)^{k}B_{k}. \quad (11)$$

Now we consider the value of $B_{i}$ where $1 \leq i \leq k$. Suppose that we have $(k + 1 - i)$ compartments with $k - i$ bars in between. There are $(k + 1 - i)^{n - i}(k - i)^{i}$ ways to insert $n$ numbers into these $k + 1 - i$ compartments with first $j$ integers avoiding the last compartment and list integers in each component in a decreasing order. Then insert $i$ separating extraneous bars into $n + 1$ positions. So we get

$$B_{i} = (k + 1 - i)^{n - i}(k - i)^{i} \binom{n + 1}{i}. \quad (12)$$

Plug formula (12) into (11); we have $G_{n,k}(j) = |\mathcal{B}|$.

Given an arrangement $\pi \in \mathcal{B}$ if we remove the bars, then we obtain a permutation $\pi \in \Omega_{n}$. So without confusion, we just use the same notation $\pi$ to represent an arrangement in set $\mathcal{B}$ and a permutation on $[n]$. Now for each $\pi \in B$, $\pi$ either

- (case 1) has no extraneous bar and none of $\{1, 2, \ldots, j\}$ locates in the last compartment or
- (case 2) has only one extraneous bar at the end.

If $\pi$ is in case 1, then $\pi$ has $k$ ascents since each bar is non-extraneous. And the last compartment of $\pi$ is nonempty. Therefore the last cycle of $f^{-1}(\pi)$ has to be $(n \ldots p_{k})$. In other words, $Q_{n}(f^{-1}(\pi)) = p_{k}$ since none of $\{1, 2, \ldots, j\}$ locates in the last compartment. And by Proposition 4, $f^{-1}(\pi) \in W_{n,k}$. If $\pi$ is in case 2, then $\pi$ has $k - 1$ ascents since only the last bar is extraneous. Note that in this case, the arrangement with no elements of $\{1, 2, \ldots, j\}$ in the compartment second to the last or the last nonempty compartment has been removed by the Principle of Inclusion and Exclusion. Equivalently, at least one number of $\{1, 2, \ldots, j\}$ has to be in the compartment second to the last. So the last cycle of $f^{-1}(\pi)$ has to be $(n \ldots p_{k})$, and $Q_{n}(f^{-1}(\pi)) = p_{k} \leq j$. Also by Proposition 4, $f^{-1}(\pi) \in W_{n,k}$.

Combining all the results above, statement (10) is correct.

The next Theorem describes some interesting properties of the coefficients $G_{n,k}$.

**Theorem 9.** Let the coefficients $G_{n,k}$ be as described in Theorem 8. Then,

1. $\sum_{k=0}^{n} G_{n,k}(j) = n!$ for any $0 \leq j \leq n$;
2. $G_{n,k}(j) = c_{n,n-k}(n-j)$, for all $0 \leq j, k \leq n$.

Before we can prove Theorem 9, we need the following lemma which is also interesting by itself.

**Lemma 10.** Given a positive integer $n$, then

$$\# \{\pi \in W_{n,k} \& Q_{n}(\pi) = j\} = \# \{\pi \in W_{n,n-k} \& Q_{n}(\pi) = n - 1 - j\} \quad (13)$$

for any $1 \leq k, j \leq n$.

**Proof.** First of all, given a positive integer $n$, we define a function $g: \Omega_{n} \rightarrow \Omega_{n}$ as follows:

$$g(\pi) = (n + 1 - p_{1})(n + 1 - p_{2})\ldots(n + 1 - p_{k}). \quad (14)$$

For instance, for $\pi = 53214 \in \Omega_{5}$, $g(\pi) = 13452$. $g$ is obviously a bijection of $\Omega_{n}$ to itself.

Now for some fixed $1 \leq k, j \leq n$, suppose $S_{k,j} = \{\pi \in W_{n,k} \& Q_{n}(\pi) = j\}$, and $T_{k,j} = \{\pi \in W_{n,n-k} \& Q_{n}(\pi) = n - 1 - j\}$. For any $\pi \in S_{k,j}$, we write $\pi$ in the standard representation cycle form. So $\pi = (p_{1}, \ldots, (n \ldots j)$ and $f(\pi) = (n \ldots p_{k} \ldots j)$ has $k - 1$ ascents by Proposition 4. Now we compose $f(\pi)$ with the bijection function $g$ as just defined. Then $g(f(\pi)) = n + 1 - p_{1} \ldots n + 1 - j$ has $n - k$ ascents, which implies that $f^{-1}(g(f(\pi)))$ has $n + 1 - k$ weak exceedances. So $f^{-1}(g(f(\pi))) \in W_{n,n-k}$. Note that the last cycle of $f^{-1}(g(f(\pi)))$ has to be $(n \ldots n + 1 - j)$. Therefore, $f^{-1}(g(f(\pi))) \in T_{k,j}$. Since both $f$ and $g$ are bijection functions, $f^{-1}g$ gives a bijection between $S_{k,j}$ and $T_{k,j}$.

Now we are ready to prove Theorem 9.

**Proof of Theorem 9.** For part 1, by Theorem 8,

$$\sum_{k=0}^{n} G_{n,k}(j) = \sum_{k=0}^{n} \# \{\pi \in W_{n,k+1} \& j < Q_{n}(\pi) \leq n\}$$

$$+ \sum_{k=0}^{n} \# \{\pi \in W_{n,k} \& 1 \leq Q_{n}(\pi) \leq j\} \quad (15)$$

$$= \sum_{k=0}^{n} \# \{\pi \in W_{n,k} \& |\Omega_{n}| = n!.$$ For part 2, also by Theorem 8,

$$G_{n,k}(j) = \sum_{i=j+1}^{n} \# \{\pi \in W_{n,k+1} \& Q_{n}(\pi) = i\}$$

$$+ \sum_{i=1}^{j} \# \{\pi \in W_{n,k} \& Q_{n}(\pi) = i\} \quad (16)$$

$$= \sum_{i=j+1}^{n} \# \{\pi \in W_{n,n-k} \& Q_{n}(\pi) = n + 1 - i\}$$

$$+ \sum_{i=1}^{j} \# \{\pi \in W_{n,n-k} \& Q_{n}(\pi) = n + 1 - i\}$$

$$= c_{n,n-k}(n-j). \quad (16)$$
Remark 12. Using the analytic formula of \( c_{n,k}(j) \) as in (9), part 2 of Theorem 9 implies the following identity:

\[
\sum_{j=0}^{k} (-1)^{j}(k + 1 - \delta)^{j} \binom{n+1}{i} = \sum_{j=0}^{k} (-1)^{j}(n + 1 - k - \delta)^{j} \binom{n+1}{i},
\]

(17)

where \( \delta \) is a positive integer, and \( 0 \leq j, k \leq n \).

3. Another Combinatorial Interpretation of \( c_{n,k}(1) \) and \( c_{n,k}(n-1) \)

In pursuing the combinatorial meanings of the coefficients \( c_{n,k} \), the authors have found some other interesting properties about permutations. The results in this section will reveal close connections between the traditional Eulerian numbers \( A_{n,k} \) and \( c_{n,k}(j) \), where \( j = 1 \) or \( j = n-1 \).

One fundamental concept of permutation combinatorics is inversion. A pair \((p_{i}, p_{j})\) is called an inversion of the permutation \( \pi = p_{1}p_{2}\ldots p_{n} \) if \( i < j \) and \( p_{i} > p_{j} \) [6, page 36]. The following definition gives the main concepts of this section.

Definition 12. For a fixed positive integer \( k \), let \( AW_{n,k} = \{ \pi = p_{1}p_{2}\ldots p_{n} \mid \pi \in W_{n,k} \text{ and } p_{i} < p_{j} \text{ for } i < j \} \) and \( BW_{n,k} = \{ p_{1}p_{2}\ldots p_{n} \mid \pi \in W_{n,k} \text{ and } p_{i} > p_{j} \text{ for } i < j \} \). Given a permutation \( \pi = p_{1}p_{2}\ldots p_{n} \in W_{n,k} \), if \( p_{i} > p_{j} \) for \( i < j \), then \( \pi \in AW_{n,k} \), and if \( p_{i} < p_{j} \) for \( i < j \), then \( \pi \in BW_{n,k} \).

Our last result of this paper is the following theorem which reveals that both \( |AW_{n,k}| \) and \( |BW_{n,k}| \) take exactly the same recursive formula as the traditional Eulerian numbers \( A_{n,k} \) as shown in (9).

**Theorem 15.** For a fixed positive integer \( n \), let \( AW_{n,k} \) and \( BW_{n,k} \) be as defined in Definition 12, then

\[
k|AW_{n-1,k}| + (n + 1 - k)|AW_{n-1,k-1}| = |AW_{n,k}|,
\]

(18)

\[
k|BW_{n-1,k}| + (n + 1 - k)|BW_{n-1,k-1}| = |BW_{n,k}|.
\]

(19)

**Proof.** A computational proof can be obtained straightforward by using (9) and Theorem 13. But here we provide a proof in a flavor of combinatorics.

**Idea of the Proof.** For (18), given a permutation \( A_{1} = p_{1}p_{2}\ldots p_{n-1} \in AW_{n,k} \), for each position \( i \) with \( p_{i} \geq i \), we insert \( i \) into a certain place of \( A_{1} \), such that the new permutation \( A_{i} \) is in \( AW_{n,k} \). There are \( k \) such positions, so we can get \( k \) new permutations in \( AW_{n,k} \). Similarly, if \( A_{2} = p_{1}p_{2}\ldots p_{n-1} \in BW_{n,k} \), for each position \( i \) with \( p_{i} < i \), and the position at the end of \( A_{2} \), we insert \( i \) into a specific position of \( A_{2} \) and the resulting new permutation \( A_{i} \) is in \( BW_{n,k} \). There are \( n + 1 - k \) such positions, so we can get \( n + 1 - k \) new permutations in \( BW_{n,k} \). We will show that all the permutations obtained from the above constructions are distinct, and they have exhausted all the permutations in \( AW_{n,k} \) and \( BW_{n,k} \).

For any fixed \( A' = \pi_{1}\pi_{2}\ldots\pi_{n} \in AW_{n,k} \), then \( \pi_{1} < \pi_{n} \).

We classify \( A' \) into the following disjoint cases:

**Case a.** Consider that \( \pi_{i} = n \) with \( i < n \). So \( A' = n\pi_{1}\pi_{2}\ldots\pi_{i-1}\pi_{n}q_{1}\ldots q_{r} \).

\[
(\text{a1) } \pi_{1} < \pi_{n-1}, \text{ and } \pi_{n} \geq i;
\]

\[
(\text{a2) } \pi_{1} < \pi_{n-1}, \text{ and } \pi_{n} < i;
\]

\[
(\text{a3) } \pi_{1} > \pi_{n-1}, \pi_{n} < n - 1, \text{ and } \pi_{n} \geq i;
\]

\[
(\text{a4) } \pi_{1} > \pi_{n-1}, \pi_{n} < n - 1, \text{ and } \pi_{n} < i;
\]

\[
(\text{a5) } \pi_{1} > \pi_{n-1}, \text{ and } \pi_{n} = n - 1.
\]

**Case b.** Consider that \( \pi_{n} = n \). So \( \pi_{i} = n - 1 \) for some \( i < n \) and \( A' = n\pi_{1}\pi_{2}\ldots\pi_{i-1}\pi_{n}p_{1}\ldots p_{r} \).

\[
(\text{b1) } \pi_{1} < \pi_{n-1};
\]

\[
(\text{b2) } \pi_{n-1} < \pi_{n} < n - 1, \text{ and } \pi_{n-1} \geq \pi_{i};
\]

\[
(\text{b3) } \pi_{n-1} < \pi_{n} < n - 1, \text{ and } \pi_{n-1} < \pi_{i};
\]

\[
(\text{b4) } \pi_{1} = n - 1.
\]

Based on the classifications listed above, we can construct a map \( f : [AW_{n-1,k}, BW_{n-1,k-1}] \rightarrow AW_{n,k} \) by applying the idea of the proof we have illustrated at the beginning of the proof. To save space, the map \( f \) is demonstrated in Table 1. From Table 1 we can see that in each case, the positions of inserting \( n \) are all different. So all the images obtained in a certain case are different. Since all the cases are disjoint, all the images \( A' \in AW_{n,k} \) are distinct.
Table 1: The map $f : \{AW_{n-1,k}, AW_{n-1,k-1}\} \rightarrow AW_{n,k}$.

<table>
<thead>
<tr>
<th>$A = p_1p_2 \cdots p_{n+1}$</th>
<th>Position $i$</th>
<th>Condition</th>
<th>$A' \in AW_{n,k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1 &lt; i \leq n-1$ and $p_i \geq i$</td>
<td>$p_i &gt; p_i$</td>
<td>$A' = p_1p_2 \cdots p_{i-1}p_{i+1} \cdots p_{n+1}$</td>
</tr>
<tr>
<td></td>
<td>$p_i &lt; p_i$ and $p_n \leq n-1$</td>
<td>$A' = p_1p_2 \cdots p_{i-1}p_{i+1} \cdots p_{n+1}$</td>
<td></td>
</tr>
<tr>
<td>$A \in AW_{n-1,k}$</td>
<td>$i = 1$</td>
<td>$p_1 = n-1$ and $j &lt; n-1$</td>
<td>$A' = p_1p_2 \cdots p_{n-1}p_{n}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$p_n = n-1$</td>
<td>$A' = p_1p_2 \cdots p_{n-2}p_{n-1}$</td>
</tr>
<tr>
<td>$A \in AW_{n-1,k-1}$</td>
<td>$1 &lt; i \leq n-1$ and $p_i &lt; i$</td>
<td>$p_i &gt; p_i$</td>
<td>$A' = p_1p_2 \cdots p_{i-1}p_{i+1} \cdots p_{n+1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$p_i &lt; p_i$ and $p_n \leq n-1$</td>
<td>$A' = p_1p_2 \cdots p_{i-1}p_{i+1} \cdots p_{n+1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A' = p_1p_2 \cdots p_i \cdot p_i p_{i+1} \cdots p_{n+1}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: The map $g : \{BW_{n-1,k}, BW_{n-1,k-1}\} \rightarrow BW_{n,k}$.

<table>
<thead>
<tr>
<th>$B = p_1p_2 \cdots p_{n+1}$</th>
<th>Position $i$</th>
<th>Condition</th>
<th>$B' \in BW_{n,k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1 &lt; i \leq n-1$ and $p_i \geq i$</td>
<td>$p_i &gt; p_i$</td>
<td>$B' = p_1p_2 \cdots p_{i-1}p_{i+1} \cdots p_{n+1}p_i$</td>
</tr>
<tr>
<td></td>
<td>$p_i &lt; p_i$ and $p_n &lt; n-1$</td>
<td>$B' = p_1p_2 \cdots p_{i-1}p_{i+1} \cdots p_{n+1}$</td>
<td></td>
</tr>
<tr>
<td>$B \in WB_{n-1,k}$</td>
<td>$i = 1$</td>
<td>$p_1 &gt; 1$</td>
<td>$B' = np_2 \cdots p_{n+1}p_i$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$B' = np_2 \cdots p_{n-1}p_{n}$</td>
<td></td>
</tr>
<tr>
<td>$B \in WB_{n-1,k-1}$</td>
<td>$1 &lt; i \leq n-1$ and $p_i &lt; i$</td>
<td>$p_i &gt; p_i$</td>
<td>$B' = p_1p_2 \cdots p_{i-1}p_{i+1} \cdots p_{n+1}p_i$</td>
</tr>
<tr>
<td></td>
<td>$p_i &lt; p_i$ and $p_n &lt; n-1$</td>
<td>$B' = p_1p_2 \cdots p_{i-1}p_{i+1} \cdots p_{n+1}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A' = p_1p_2 \cdots p_{i-1}p_{i+1} \cdots p_{n+1}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


Similarly, for each \( B' = \pi_1 \pi_2 \pi_3 \ldots \pi_n \in BW_{n,k}, \) then \( \pi_1 > \pi_{n-1} \). We classify \( B' \) into the following disjoint cases.

Case c. Consider that \( \pi_1 = n \) with \( 1 < i \leq n - 1 \). So \( B' = \pi_1 \pi_2 \ldots \pi_{n-1} \pi_{n-1} \cdot \pi_{n-1} \).

(1) \( \pi_1 > \pi_{n-1} \) and \( \pi_{n-1} \geq i \);
(2) \( \pi_1 > \pi_{n-1} \) and \( \pi_{n-1} < i \);
(3) \( \pi_1 < \pi_{n-1} < n-1, \pi_{n-1} \geq i \);
(4) \( \pi_1 < \pi_{n-1} < n-1, \pi_{n-1} < i \);
(5) \( \pi_{n-1} = n-1 \);
(6) \( \pi_{n-1} = n \).

Case d. Consider that \( \pi_1 = n \). So \( B' = n\pi_2 \ldots \pi_{n-2} \pi_{n-1} \).

(d1) \( \pi_{n-2} < \pi_{n-1} \); 
(d2) \( \pi_{n-2} > \pi_{n-1} \).

To prove (9), we use a similar idea of proof as shown above. If \( B_1 = \pi_1 \pi_2 \pi_3 \ldots \pi_{n-1} \in BW_{n-1,k} \), for each position \( i \) with \( p_i \geq i \), we insert \( n \) into a certain place of \( B_1 \) to get \( B'_1 \in AW_{n,k} \). If \( B_2 = \pi_1 \pi_2 \pi_3 \ldots \pi_{n-1} \in BW_{n-1,k} \), for each position \( i \) with \( p_i < i \), and the position \( i \) where \( p_i = n-1 \), we insert \( n \) into a specific position of \( B_2 \) to obtain \( B'_2 \in AW_{n,k} \). Such a map \( g : BW_{n-1,k} \rightarrow BW_{n,k} \) is illustrated in Table 2. And the distinct images under \( g \) exhaust all the permutations in \( BW_{n,k} \).

Here is a concrete example for the constructions illustrated in Table 2.

Example 16. Suppose \( n = 4, k = 2 \). We want to obtain \( BW_{4,2} = \{3412, 3412, 3421, 4132, 4213, 4312, 4321 \} \) from \( BW_{3,2} = \{321, 231 \} \) and \( BW_{3,1} = \{312 \} \). For \( 321 \in BW_{3,2}, \pi_1 = 3 \geq 1 \), then it corresponds to \( B' = 4213 \) which is case (d1) in Table 2; \( p_2 = 2 \geq 2 \), then it corresponds to \( B' = 3412 \) which is case (c1) in Table 2. Similarly, we can construct \( \{3412, 4321 \} \) from \( 231 \in BW_{3,2} \) and \( \{3421, 3142, 4132 \} \) from \( 312 \in BW_{3,1} \) using Table 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

