ON THE PROBLÈME DES MÉNAGES

MAX WYMAN AND LEO MOSER

Introduction. The classical problème des ménages asks for the number of ways of seating at a circular table $n$ married couples, husbands and wives alternating, so that no husband is next to his own wife.

An outline of the history of the problem to 1946 was given by Kaplansky and Riordan (11). They also presented a bibliography, which is augmented and brought up to date in the bibliography of the present paper.

The first explicit solution of the problem is due to Touchard (23) and the simplest derivation of Touchard's formula is due to Kaplansky (9). In the present paper a new explicit solution to the problem is obtained, via an exponential generating function for certain numbers closely related to the ménage numbers and introduced by Cayley (4). Although the new explicit expression is quite complicated, it does lead to some new and deep results concerning the ménage numbers. In particular, it is shown that the usual asymptotic formula for these numbers can actually be used to compute the numbers exactly.

Several other new explicit expressions for the ménage numbers are obtained and one of these suggests a strong conjecture concerning Latin rectangles for which some evidence is presented.

The most extensive published tables of the ménage numbers are those given by Lucas (13). These go up to $n = 25$. In the present paper we present tables which give the numbers up to $n = 65$. These were computed by F. L. Miksa, using a recursion formula of Cayley (4), and checked by means of congruences due to Riordan (20).

1. A Generating Function. Rather than deal directly with the ménage numbers $M_n$ many authors introduce the number $U_n$ defined by

$$M_n = 2 \ (n!) \ U_n.$$

Further, Cayley (4) introduced an auxiliary sequence $q_n$ defined by

$$U_n = q_n - q_{n-2},$$

and showed that the $q_n$ satisfy the recurrence relation

$$q_n = n \ q_{n-1} + q_{n-2} + (-1)^{n-1} (n - 2).$$

If we introduce the generating function $F(t)$ by

$$F(t) = \sum_{n=0}^{\infty} \frac{q_n t^n}{n!},$$

Received September 24, 1957.

468
then it is easily shown that $F(t)$ is the solution of

$$F(t) = (1 - t) F' - 2F - F = t e^{-t},$$

$$F(0) = F'(0) = 0,$$

where the "dot" means differentiation with respect to $t$.

The substitution

$$F = (1 - t)^{3/2}y, x = 2(1 - t)^{1/2}$$

makes (1.5) take the form

$$y'' + x^{-1} y' - (1 + x^{-2})y = \frac{1}{2}x(1 - \frac{1}{4}x^2) e^{(x^2/4 - 1)},$$

$$y(2) = y'(2) = 0,$$

where the prime denotes differentiation with respect to $x$. The homogeneous equation is well known and the complementary function can be expressed in terms of the modified Bessel functions as

$$A I_1(x) + B K_1(x),$$

where $A, B$ are constants.

In order to determine a particular integral $P(x)$ of (1.7), we assume a series solution of the form

$$P(x) = \sum_{n=0}^{\infty} a_n x^{n+3}.$$ Substituting into (1.7) we immediately are led to

$$a_0 = e^{-1/16}, a_{2n+1} = 0,$$

$$4a_{2n}(n + 1)(n + 2) - a_{2n-2} = e^{-1}(1 - n)/2^{2n+1} n!$$

This recurrence relation is easily solved and our particular solution can be put into the form

$$P(x) = e^{-1/2} \left[ I_1(x) - \frac{1}{2}x e^{x^2/4} + 2 \sum_{n=1}^{\infty} b_n \left( \frac{1}{2}x \right)^{2n+1} \right],$$

where

$$b_n = \left( \sum_{s=1}^{n} s! \right)/n!(n + 1)!.$$ Replacing $s!$ by

$$\int_0^{\infty} e^{-z} z^s dz,$$

we find

$$P(x) = e^{-1/2} \left[ I_1(x) - \frac{1}{2}x e^{x^2/4} + 2 \int_0^{\infty} F(x, z) dz \right],$$

where $F(x, z) = z e^{-z} (I_1(x) - z^{1/2} I_1(xz^{-1}))/ (1 - z)$.

If we introduce the principal value of the integral at $z = 1$ we can rearrange the terms so that
\[ P(x) = e^{-1} \left[ L I_1(x) - \frac{1}{2}x e^{x^2/4} + 2 \int_0^\infty G(x, z) dz \right], \]

where
\[ L = 2 \int_0^\infty \frac{e^{-z}}{1 - z} dz - 1, \quad G(x, z) = \frac{z^2 e^{-z} I_1(xz^4)}{z - 1}. \]

Thus the general solution of (1.7) must be of the form
\[ y = A I_1(x) + B K_1(x) + P(x), \]
where the constants \( A, B \) must be chosen to satisfy \( y(2) = y'(2) = 0 \).

The analysis so far is straightforward and it seems likely that it has been carried thus far before. The major difficulty is in the evaluation of the constants \( A \) and \( B \). In view of the complexity of the functions involved it is, indeed, remarkable that these constants can be evaluated in a tractable form. The evaluation of the constants is given in the next section.

2. Evaluation of the constants. If \( f_1(x), f_2(x) \) denote two functions of \( x \) we introduce the usual Wronskian notation \( W(f_1, f_2) \) by
\[ W(f_1, f_2) = f_1 f_2' - f_2 f_1'. \]

In order to satisfy the boundary conditions \( y(2) = y'(2) = 0 \) we have
\[ A I_1(2) + B K_1(2) + P(2) = 0 \]
\[ A I_1'(2) + B K_1'(2) + P'(2) = 0. \]

Since it is well known that \( W(I_1(2), K_1(2)) = -\frac{1}{2} \) we have
\[ A = 2 W(P(2), K_1(2)), \quad B = 2 W(I_1(2), P(2)). \]

We evaluate these Wronskians, by the usual procedure, from the differential equations satisfied by \( P(x) \) and \( I_1(x) \). These differential equations are
\[ x P'' + P' - (x + x^{-1}) P = \frac{1}{4}x^2(1 - \frac{1}{4}x^2) \exp (\frac{1}{4}x^2 - 1), \]
\[ x I_1'' + I_1' - (x + x^{-1}) I_1 = 0. \]

We multiply (2.4) by \( I_1 \) and (2.5) by \( P \). By subtraction of the resulting equations and integration from \( x = 0 \) to \( x = 2 \) we obtain
\[ 2 W(I_1(2), P(2)) = \frac{9}{2} e^{-1} \int_0^2 x^2(1 - \frac{1}{4}x^2) e^{x^2/4} I_1(x) dx. \]

Hence
\[ B = \frac{9}{2} e^{-1} \int_0^2 x^2(1 - \frac{1}{4}x^2) e^{x^2/4} I_1(x) dx, \]
and similarly
\[ A = -\frac{9}{4} e^{-1} \int_0^2 x^2(1 - \frac{1}{4}x^2) e^{x^2/4} K_1(x) dx. \]
In order to evaluate (2.7) we write (2.5) in the form

\[ I_1'' + (x^{-1} I_1)' - I_1 = 0. \tag{2.9} \]

Multiplying (2.9) by \( \exp(x^2/4) \) and integrating from 0 to 2 we can show, by integrating by parts, that

\[ \int_0^2 e^{x^2/4} \left( \frac{1}{2} x^2 - 1 \right) I_1(x) \, dx = 1 - e I_1'(2) + \frac{1}{2} e I_1(2). \tag{2.10} \]

Similarly by multiplying the differential equation by \( x^2 \exp(x^2/4) \) and repeating the process we find

\[ \int_0^2 e^{x^2/4} (x^2 + \frac{1}{2} x^4) I_1(x) \, dx = 6 e I_1(2) - 4 e I_1'(2). \tag{2.11} \]

Multiplying (2.10) by eight and subtracting (2.11) we obtain

\[ \int_0^2 e^{x^2/4} (x^2 - \frac{1}{2} x^4) I_1(x) \, dx = 8 - 4 e I_1(2) - 2 e I_1'(2) + 8 \int_0^2 e^{x^2/4} I_1(x) \, dx \tag{2.12} \]

From the known recurrence relations of the modified Bessel functions we have

\[ 2 I_1'(2) + I_1(2) = 2 I_0(2). \tag{2.13} \]

Hence

\[ \int_0^2 e^{x^2/4} (x^2 - \frac{1}{2} x^4) I_1(x) \, dx = 8 - 4 e I_0(2) + 8 \int_0^2 e^{x^2/4} I_1(x) \, dx. \tag{2.14} \]

Let us now consider the integral

\[ J = \int_0^2 e^{x^2/4} I_1(x) \, dx. \]

The substitution \( x = 2u^4 \) transforms \( J \) into

\[ J = \int_0^1 e^u I_1(2u^4) u^{-1} \, du \tag{2.15} \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n! (n + 1)!} \frac{1}{\int_0^1 e^u u^n du} \]

\[ = \sum_{n=0}^{\infty} \frac{(1 - n + n(n - 1) \cdots (-1)^n n! e + (-1)^{n+1} n!}{n! (n + 1)!} \]

\[ = e \left[ I_1(2) - I_2(2) + I_3(2) \cdots \right] + e^{-1} - 1 \]

\[ = e \sum_{n=1}^{\infty} (-1)^{n+1} I_n(2) + e^{-1} - 1. \]

However, from the generating function for \( I_n(x) \) we can prove that

\[ e^{-2} = I_0(2) + 2 \sum_{n=1}^{\infty} (-1)^n I_n(2). \tag{2.16} \]

Thus

\[ J = \frac{1}{2} e^{-1} + \frac{1}{2} e I_0(2) - 1 \tag{2.17} \]
and, from (2.14),

\[
(2.18) \quad \int_0^2 e^{x^4/4} (x^2 - \frac{1}{2}x^4) \, I_1(x) \, dx = 4e^{-1}.
\]

Finally from (2.7), (2.18) we have that the constant B is given by

\[
(2.19) \quad B = 2e^{-2}.
\]

The evaluation of the constant A can also be carried out with the help of the integral representation.

\[
(2.20) \quad 2K_1(2u^\frac{1}{2}) \, u^{-\frac{1}{2}} = \int_0^\infty \exp(-uz - z^{-1}) \, dz.
\]

The final result is that

\[
(2.21) \quad A = e^{-1} + 2e^{-1} \int_0^\infty e^{-z}/(z - 1) \, dz.
\]

These results imply that the desired solution of (1.7) is

\[
(2.22) \quad y = 2e^{-2}K_1(x) - \frac{1}{2}e^{-1}x^2 + 2e^{-1} \int_0^\infty \frac{z^\frac{1}{2}e^{-z}I_1(x(z)^{\frac{1}{2}})}{1 - z} \, dz
\]

and that the generating function \(F(t)\), for \(q_n\) is given by

\[
(2.23) \quad F(t) = 2e^{-2}(1 - t)^{-1}K_1(2(1 - t)^{\frac{1}{2}}) - e^{-t} - 2e^{-1} \int_0^\infty H(z, t) \, dz
\]

where

\[
H(z, t) = z^\frac{1}{2} e^{-z} I_1(2(z - st)^{\frac{1}{2}})/(1 - z)(1 - t)^{\frac{3}{2}}.
\]

The modified Bessel functions satisfy the well known differentiation formulae

\[
(2.24) \quad \left(\frac{d}{dz}\right)^m z^{-\alpha} I_\alpha(z) = z^{-\alpha-m} I_{\alpha+m}(z),
\]

\[
(2.25) \quad \left(\frac{d}{dz}\right)^m z^{-\alpha} K_\alpha(z) = (-1)^m z^{-\alpha-m} K_{\alpha+m}(z).
\]

Hence

\[
(2.26) \quad q_n = F^{(n)}(0) = 2e^{-2}K_{n+1}(2) + (-1)^{n+1} + 2(-1)^{n+1}e^{-1} \int_0^\infty M_{n+1}(z) \, dz,
\]

where

\[
M_{n+1}(z) = z^{\frac{1}{2}(n+1)} e^{-z} I_{n+1}(2z^{\frac{1}{2}})/(1 - z).
\]

Since the ménage numbers \(U_n\) are given by \(U_n = q_n - q_{n-2}\) we find that

\[
(2.27) \quad U_n = 2e^{-2}nK_n(2) + 2(-1)^n + 2n(-1)^n e^{-1} \int_0^\infty M_n(z) \, dz.
\]

If we replace \(K_n(2), I_n(2 z^{\frac{1}{2}})\) by their known series expansions we can obtain an explicit series expression for \(U_n\) in terms of \(n\). This expression is very complicated. However (2.27) is a useful expression in that one can derive many of
the known results directly without resorting to the series expression. For example, it is readily shown from (2.27) that

\[
\sum_{n=2}^{\infty} U_n I_n(2t) = e^{-2t/(1-t)} - I_0(2t) + I_1(2t).
\]

Hence, by redefining \( U_0, U_1 \), to be 1 and \(-1\) respectively we obtain Touchard’s result (24):

\[
\sum_{n=0}^{\infty} U_n I_n(2t) = e^{-2t/(1-t)}.
\]

In the next section we shall use (2.27) to derive some new results for the ménage numbers.

3. New results. It has been shown (11) that an asymptotic expansion for \( U_n \) is given by

\[
U_n \sim e^{-2} n! \left[ 1 - \frac{1}{(n-1)} + \frac{1}{2!(n-1)(n-2)} \ldots \right].
\]

By means of (2.27) we shall prove a much deeper result.

To prove this result we write (2.27) in the form

\[
U_n = 2e^{-2}n K_n(2) + J_n,
\]

where

\[
J_n = 2(-1)^n \left\{ 1 + n e^{-1} \int_0^{\infty} \frac{z^{n/2}e^{-z} I_n(2z)}{1-z} \, dz \right\}.
\]

In (3.3) we replace the first term of the bracket by means of

\[
1 = e^{-1} \sum_{m=0}^{\infty} 1/m!
\]
and \( I_n(2z) \) by its series expression

\[
I_n(2z) = z^n \sum_{m=0}^{\infty} \frac{z^m}{m!(m+n)!}.
\]

Hence \( J_n \) takes the form

\[
J_n = 2(-1)^n e^{-1} \left[ \sum_{m=0}^{\infty} \left\{ (1/m!) + n \int_0^\infty \frac{e^{-z}}{1-z} \sum_{m=0}^{\infty} \frac{z^{m+n}}{m!(m+n)!} \, dz \right\} \right].
\]

This can be put in the form

\[
J_n = 2(-1)^n e^{-1} \left\{ Cn I_n(2) + \sum_{m=0}^{\infty} \frac{b_{mn}}{m!(m+n)!} \right\},
\]

where

\[
C = \int_0^\infty \frac{e^{-z}}{1-z} \, dz,
\]

\[
b_{mn} = (m+n)! - n\{(m+n-1)! + (m+n-2)! + \ldots + 1\}
= (m+n-1)! m - n\{(m+n-2)! + (m+n-3)! + \ldots + 1\}.
\]
It is trivial to show

\[ |C| < 4e^{-1}, \]

and

\[ |nI_n(2)| \leq e/(n - 1)!. \]

Hence

\[ |CnI_n(2)| \leq 4/(n - 1)!. \]

Let us consider the series term of (3.7) and write

\begin{align*}
(3.12) \quad H_n &= \sum_{m=0}^{\infty} \frac{b_{mn}}{m!(m + n)!} \\
&= \frac{n! - n\{(n - 1)! + \ldots + 1\}}{n!} \\
&\quad + \frac{(n + 1)! - n(n! + (n - 1)! + \ldots + 1)}{(n + 1)!} \\
&\quad + \sum_{m=2}^{\infty} \frac{b_{mn}}{m!(m + n)!} \\
&= \frac{(n - 2)! + (n - 3)! + \ldots + 1}{(n - 1)!} \left(1 + \frac{1}{n + 1}\right) \\
&\quad + \sum_{m=2}^{\infty} \frac{b_{mn}}{m!(m + n)!}.
\end{align*}

If \( n \geq 7 \) it is easily shown that

\[ \frac{(n - 2)! + (n - 3)! + \ldots + 1}{(n - 1)!} \left(1 + \frac{1}{n + 1}\right) \leq \frac{2}{n + 1} \]

and

\[ \left| \sum_{m=2}^{\infty} \frac{b_{mn}}{m!(m + n)!} \right| \leq \frac{2(e - 1)}{n + 1}. \]

Hence for \( n \geq 7 \),

\[ |H_n| \leq \frac{2e}{n + 1}. \]

Actually (3.15) is a very crude inequality. It is, however, sufficient for our purposes.

Combining these results we have from (3.7)

\[ |J_n| \leq \frac{4}{n + 1} + \frac{8}{e(n - 1)!} \]

if \( n \geq 7 \).

Hence for \( n \geq 8 \) we have

\[ |J_n| \leq 0.45. \]

Let us now return to (3.2) and examine the series expression for \( K_n(2) \). This is given by
ON THE PROBLÈME DES MÉNAGES

\[ K_n(2) = \frac{1}{2} \sum_{m=0}^{n-1} \frac{(-1)^m (n - m - 1)!}{m!} + \frac{1}{2} (-1)^n \sum_{m=0}^{n} \frac{\Psi(n + m + 1) + \Psi(m + 1)}{m!(n + m)!}, \]

where

\[ \Psi(k + 1) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} - \gamma, \quad \Psi(1) = -\gamma \]

and \( \gamma \) is Euler's constant.

It is easily shown that

\[ \left| \sum_{m=0}^{\infty} \frac{\Psi(n + m + 1) + \Psi(m + 1)}{m!(n + m)!} \right| \leq \frac{e}{2(n - 1)!}. \]

This implies

\[ 2n K_n(2) = n \sum_{m=0}^{n-1} \frac{(-1)^m (n - m - 1)!}{m!} + R_n, \]

where the remainder satisfies \( |R_n| < n e/(n - 1)! \).

Combining the results of (3.2), (3.17) and (3.21) we obtain

\[ U_n = e^{-2} n \sum_{m=0}^{n-1} \frac{(-1)^m (n - m - 1)!}{m!} + R'_n, \]

where for \( n > 8 \) the remainder \( R'_n \) is definitely less than \( \frac{1}{2} \).

Using the notation \( \lfloor x \rfloor \) to denote the closest integer to \( x \), we have shown that, for \( n > 8 \)

\[ U_n = \left\{ e^{-2} n \sum_{m=0}^{n-1} \frac{(-1)^m (n - m - 1)!}{m!} \right\}. \]

It is easy to verify that (3.23) remains valid for \( 0 < n < 7 \). Hence we have proved the following theorem:

**Theorem.** For all values of \( n \) the ménage numbers \( U_n \) are given by (3.23).

It is thus seen that the asymptotic expansion obtained in (11) is much more than an asymptotic expansion.

In concluding this section we might remark that about half of the terms in (3.23) are redundant in that their sum adds up to less than \( \frac{1}{2} \). Further our analysis also implies that

\[ U_n = \{ 2e^{-2} n K_n(2) \}. \]

We shall make use of (3.24) in the next section to make an interesting conjecture.

**4. A Conjecture.** The modified Bessel function \( K_n(2) \) has the integral representation

\[ K_n(2) = \frac{1}{2} \int_0^\infty t^{n-1} e^{-t - 1} dt. \]
Hence (3.24) may be written

\[ U_n = \left\{ e^{-2} n \int_0^\infty t^{n-1} e^{-t-1} dt \right\}. \tag{4.2} \]

The discovery of (4.2) led us to re-examine some of the known results in Latin rectangles. The simplest problem in this class is the so-called "problème des rencontres." This asks for the number of ways \( R_n \) of writing a second line of integers 1, 2, \ldots, \( n \) which is discordant with a first line of integers written in their normal order. It is well known that

\[ R_n = \{ e^{-1} n! \} = \left\{ e^{-1} \int_0^\infty x^n e^{-x} dx \right\}. \tag{4.3} \]

Next in simplicity, in this class of problems, is the so-called reduced three line Latin rectangle problem. This asks for the number of ways \( P_n \) of having two lines of integers each of which is discordant with the first line of integers, written in normal order. For this case it was shown by Yamamoto (26) that

\[ P_n \sim e^{-3} (n!)^2 \left[ 1 + \frac{H_1(-\frac{1}{2})}{n} + \frac{H_2(-\frac{1}{2})}{n(n-1)} + \ldots \right], \tag{4.4} \]

where \( H_n(x) \) is a Hermite polynomial.

We have been able to prove an equivalent formula, namely

\[ P_n \sim e^{-3} (n!) \int_0^\infty x^n e^{x-x^2-x^{-1-\frac{1}{2}}} dx. \tag{4.5} \]

Finally Erdős and Kaplansky (7) have shown that the number \( P_n^k \) of reduced (\( n \) by \( (k + 1) \)), Latin rectangles is given asymptotically by

\[ P_n^k \sim e^{-k(k-1)} (n!)^{k-1} \left[ 1 - \left( \frac{k}{3} \right)n^{-1} + \left( \frac{1}{3} \right) \left( \frac{k}{3} \right)^2 + \frac{1}{2} \left( \frac{k}{3} \right)(k-5) n^{-2} + \ldots \right] \tag{4.6} \]

for \( K \ll (\log n)^{3/2-\epsilon} \). The validity of the same formula was proved by Yamamoto (26) for \( k < n^{1/3-\delta} \). The structure of the formula suggests an integral representation of the type

\[ P_n^k \sim e^{-k(k-1)} (n!)^{k-2} \int_0^\infty x^n \exp \left( -x - \left( \frac{k}{3} \right) x^{-1} \right. \]
\[ \left. + \frac{1}{2} \left( \frac{k}{3} \right) (k-5) x^{-2} + \ldots \right) dx. \tag{4.7} \]

Formula (4.7) is, as we have seen, true for \( k = 2,3 \). If it were possible to prove an integral relation of this type then the asymptotic behavior of \( P_n^k \) could be determined for all values of \( k \).

5. **An exact expression for the ménage numbers.** The usual explicit expression given for the ménage numbers \( U_n \) is

\[ U_n = \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!. \tag{5.1} \]
In this section we shall derive a second expression from Touchard’s generating function (2.9)

\[
\sum_{n=0}^{\infty} U_n I_n(2t) = e^{-2t}/(1 - t).
\]

Touchard has remarked that (5.2) constitutes a Neumann expansion for the, function \( e^{-2t}/(1 - t) \) in terms of the modified Bessel functions \( I_n(2t) \). However as far as we are aware, (5.2) has never been inverted to give an explicit expression for the \( U_n \).

If we expand \( e^{-2t}/(1 - t) \) into a Maclaurin expansion of the form

\[
\frac{e^{-2t}}{1 - t} = \sum_{r=0}^{\infty} \frac{k_r t^r}{r!}.
\]

then

\[
k_r = \left[ \frac{d^r}{dt^r} \frac{e^{-2t}}{1 - t} \right]_{t=0} = r! \sum_{s=0}^{r} \frac{(-2)^s}{s!}.
\]

Further from the well formulae for the coefficients of a Neumann expansion, (5.2) gives

\[
U_n = \frac{2 (\pi)}{\pi} \int_{c} \frac{e^{-2t} O_n(2it)}{1 - t} dt,
\]

where \( c \) is any closed contour, enclosing \( t = 0 \), such that \( |t| < 1 \). \( O_n(z) \) are the so-called Neumann polynomials given explicitly by

\[
O_n(z) = \frac{1}{n!} \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{n(n - m - 1)! (\frac{1}{2} z)^{2m-n-1}}{m!}.
\]

It follows immediately from (5.4), (5.5) and (5.6) that

\[
U_n = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m n(n - m - 1)! k_{n-2m} \frac{m!(n - 2m)!}{m! (n - 2m)!}.
\]

If we use the umbral convention of replacing \( k_r \) by \( k^r \) we obtain the neat, mnemonic, formula

\[
U_n = 2 T_n(\frac{1}{2} k).
\]

where \( T_n(k) \) is the Chebyshev polynomial.
Table of Ménage Numbers, $U_n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$U_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>13</td>
</tr>
<tr>
<td>6</td>
<td>80</td>
</tr>
<tr>
<td>7</td>
<td>579</td>
</tr>
<tr>
<td>8</td>
<td>4738</td>
</tr>
<tr>
<td>9</td>
<td>43387</td>
</tr>
<tr>
<td>10</td>
<td>439792</td>
</tr>
<tr>
<td>11</td>
<td>48</td>
</tr>
<tr>
<td>12</td>
<td>90741</td>
</tr>
<tr>
<td>13</td>
<td>16642</td>
</tr>
<tr>
<td>14</td>
<td>96313</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td>089274</td>
</tr>
<tr>
<td>17</td>
<td>34464</td>
</tr>
<tr>
<td>18</td>
<td>1648064</td>
</tr>
<tr>
<td>19</td>
<td>35783</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
</tr>
<tr>
<td>21</td>
<td>264</td>
</tr>
<tr>
<td>22</td>
<td>39014</td>
</tr>
<tr>
<td>23</td>
<td>60058</td>
</tr>
<tr>
<td>24</td>
<td>1</td>
</tr>
<tr>
<td>25</td>
<td>5</td>
</tr>
<tr>
<td>26</td>
<td>24412</td>
</tr>
<tr>
<td>27</td>
<td>1</td>
</tr>
<tr>
<td>28</td>
<td>3726</td>
</tr>
<tr>
<td>29</td>
<td>6554</td>
</tr>
<tr>
<td>30</td>
<td>11</td>
</tr>
<tr>
<td>31</td>
<td>1076</td>
</tr>
<tr>
<td>32</td>
<td>34481</td>
</tr>
<tr>
<td>33</td>
<td>11</td>
</tr>
<tr>
<td>34</td>
<td>387</td>
</tr>
<tr>
<td>35</td>
<td>13579</td>
</tr>
<tr>
<td>36</td>
<td>4</td>
</tr>
<tr>
<td>37</td>
<td>181</td>
</tr>
<tr>
<td>38</td>
<td>6889</td>
</tr>
<tr>
<td>39</td>
<td>2</td>
</tr>
<tr>
<td>40</td>
<td>107</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
U_0 & = 1 \\
U_1 & = 1 \\
U_2 & = 0 \\
U_3 & = 1 \\
U_4 & = 2 \\
U_5 & = 13 \\
U_6 & = 80 \\
U_7 & = 579 \\
U_8 & = 4738 \\
U_9 & = 43387 \\
U_{10} & = 439792
\end{align*}
\]
Table of Ménage Numbers, \( U_n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( U_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>41</td>
<td>4415</td>
</tr>
<tr>
<td>42</td>
<td>1 85566</td>
</tr>
<tr>
<td>43</td>
<td>79 83996</td>
</tr>
<tr>
<td>44</td>
<td>3514 90268</td>
</tr>
<tr>
<td>45</td>
<td>17445 46717</td>
</tr>
<tr>
<td>46</td>
<td>69590 77881</td>
</tr>
<tr>
<td>47</td>
<td>40098 53669</td>
</tr>
<tr>
<td>48</td>
<td>06041 87840</td>
</tr>
<tr>
<td>49</td>
<td>27218 60142</td>
</tr>
<tr>
<td>50</td>
<td>58890 36142</td>
</tr>
<tr>
<td>51</td>
<td>33778 81355</td>
</tr>
<tr>
<td>52</td>
<td>61744 28069</td>
</tr>
<tr>
<td>53</td>
<td>27158 19299</td>
</tr>
<tr>
<td>54</td>
<td>42199 94397</td>
</tr>
<tr>
<td>55</td>
<td>92576 17576</td>
</tr>
<tr>
<td>56</td>
<td>94695 41458</td>
</tr>
<tr>
<td>57</td>
<td>19035 75900</td>
</tr>
<tr>
<td>58</td>
<td>52151 42090</td>
</tr>
<tr>
<td>59</td>
<td>14181 37324</td>
</tr>
<tr>
<td>60</td>
<td>94487 31311</td>
</tr>
<tr>
<td>61</td>
<td>00422 00690</td>
</tr>
<tr>
<td>62</td>
<td>40473 50277</td>
</tr>
<tr>
<td>63</td>
<td>61742 01966</td>
</tr>
<tr>
<td>64</td>
<td>12623 19720</td>
</tr>
<tr>
<td>65</td>
<td>74686 00288</td>
</tr>
</tbody>
</table>

NOTE: \( U_n = 15825417445 \ldots \text{etc.} \)
References


University of Alberta