Applied Mathematics and Computation 219 (2013) 5564–5568

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Some identities for the generalized Fibonacci numbers and the generalized Lucas numbers

Andrzej Włoch

Rzeszów University of Technology, Faculty of Mathematics and Applied Physics, al. Powstańców Warszawy 12, 35-359 Rzeszów, Poland

ARTICLE INFO

Keywords:
Fibonacci number
Lucas number
k-Independent set
Counting

ABSTRACT

In this paper we study some properties of the generalized Fibonacci numbers and the generalized Lucas numbers. These numbers are equal to the total numbers of k-independent sets in special graphs. We give some identities for the generalized Fibonacci numbers and the generalized Lucas numbers, which can be useful also in problems of counting of k-independent sets in graphs.

1. Introduction

In general we use the standard terminology and notation of graph theory and combinatorics, see [1,2]. By a graph G we mean a finite, undirected, connected, simple graph with the vertex set $V(G)$ and the edge set $E(G)$. Let $P_n, n \geqslant 1$, and $C_n, n \geqslant 3$, denote a path and a cycle on n vertices, respectively. Let $k \geqslant 2$ be integer. A subset $S \subseteq V(G)$ is a k-independent set of G if for any two distinct vertices $x, y \in S$, $d_G(x,y) \geqslant k$. Moreover a subset containing only one vertex and the empty set also are k-independent. If $k = 2$, then this definition gives the definition of independent set in the classical sense. Let $NI_k(G)$ denote the number of k-independent sets in G and for $k = 2$, $NI_2(G) = NI(G)$.

The parameter $NI(G)$ first appears in the mathematical literature in a paper of Prodinger and Tichy, see [9], and this paper gave impetus for counting independent sets in graphs. They called this number as the Fibonacci number of a graph in view of the following facts:

Fact 1.1. $NI(P_n) = F_n$, where F_n is the nth Fibonacci number defined by $F_0 = 1,\ F_1 = 2$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$.

Fact 1.2. $NI(C_n) = L_n$, where L_n is the nth Lucas number defined by $L_0 = 2,\ L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$.

The interest begun by Prodinger and Tichy was multiplied by fact that independently Merrifield and Simmons introduced the parameter $NI(G)$ (which they called σ-index) to the chemical literature, see [8]. They showed the correlation between this index and some physicochemical properties of a molecular graph. The literature includes many papers dealing with the theory of counting of independent sets in graphs, the last survey written by Gutman and Wagner [5] collects and classifies these results, most of them are obtained quite recently.

Besides the usual Fibonacci and Lucas numbers many kinds of generalizations of these numbers have been presented in the literature. In [7] Kwaśnik and Włoch introduced more generalized concept, namely the generalized Fibonacci numbers $F(k,n)$ and the generalized Lucas numbers $L(k,n)$ which give the number of all k-independent sets in graphs P_n and C_n, respectively.

In [7] it was proved that for integers $k \geqslant 2,\ n \geqslant 0$ the numbers $F(k,n)$ satisfy the following recurrence

E-mail address: awloch@prz.edu.pl

0096-3003/$ - see front matter © 2012 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2012.11.030

Fact 1.3 [7]. $F(k,n) = n+1$ for $n = 0,1,\dots,k-1$ and $F(k,n) = F(k,n-1) + F(k,n-k)$ for $n \geqslant k$.

Moreover the recurrence formula for the Lucas numbers $L(k,n)$ was given. For integers $k \geqslant 2,\ n \geqslant 0$ the numbers $L(k,n)$ satisfy the following recurrence.

Fact 1.4 [7]. $L(k,n) = n+1$ for $n = 0,1,\dots,2k-1$ and $L(k,n) = (k-1)F(k,n-(2k-1)) + F(k,n-(k-1))$ for $n \geqslant 2k$.

For $n \geqslant 0$ we have that $F(2,n) = F_n$ and for $n \geqslant 3$, $L(2,n) = L_n$.

In the literature other generalizations of the number of Fibonacci type are studied, see [3,6,11,12,15].

In the graph terminology for an arbitrary $k \geqslant 2$ the number $F(k,n)$ is equal to the total number of k-independent sets in graph P_n, i.e.

$$F(k,n) = NI_k(P_n), \quad n \geqslant 1.$$

For $k \geqslant 2$ the number $L(k,n)$ is equal to the total number of k-independent sets in graph C_n, i.e.

$$L(k,n) = NI_k(C_n), \quad n \geqslant 3.$$

Note that for $n = 0,1,2$ and $k \geqslant 2$ the numbers $L(k,n)$ does not have the graph interpretation with respect to the number of k-independent sets.

The generalized Fibonacci numbers and the generalized Lucas numbers were studied in many papers, mainly with respect to their connections with the number of k-independent sets in graphs, see for example [14]. In special graph product the number of k-independent sets is expressed using concept of the generalized Fibonacci polynomial of graph, see [10,13]. The concept of k-independent sets in graphs is studied in the literature in many papers, also with concept of (k,l)-kernels in graphs, see [4,10].

Table 1 gives the initial words of the generalized Fibonacci numbers and the generalized Lucas numbers for special case of n and k.

In this paper we present some identities for $F(k,n)$ and $L(k,n)$ for an arbitrary $k \geqslant 2$. These identities can be useful for counting of k-independent sets in graphs.

2. Main results

Firstly we prove the basic recurrence relations for the number $L(k,n)$.

Theorem 2.1. *Let $n \geqslant 0,\ k \geqslant 2$ be integers. Then $L(k,n) = n+1$ for $n = 0,1,\dots,2k-1$ and for $n \geqslant 2k$, $L(k,n) = L(k,n-1) + L(k,n-k)$.*

Proof. If $n = 0,1,\dots,2k-1$, then the Theorem immediately follows from Fact 1.4. Let $n \geqslant 2k$. Applying the formulas from Facts 1.3 and 1.4 we obtain that

$$\begin{aligned}
L(k,n) &\overset{\text{by } 1.4}{=} (k-1)F(k,n-(2k-1)) + F(k,n-(k-1)) \overset{by 1.3}{=} (k-1)(F(k,n-(2k-1)-1) + F(k,n-(2k-1)-k)) \\
&+ F(k,n-(k-1)-1) + F(k,n-(k-1)-k) = (k-1)F(k,n-(2k-1)-1) + (k-1)F(k,n-(2k-1)-k) \\
&+ F(k,n-(k-1)-1) + F(k,n-(k-1)-k) = (k-1)F(k,n-2k) + (k-1)F(k,n-3k+1) \\
&+ F(k,n-k) + F(k,n-2k+1) = (k-1)F(k,n-1-(2k-1)) + F(k,n-1-(k-1)) + (k-1)F(k,n-k \\
&-(2k-1)) + F(k,n-k-(k-1)) \overset{\text{by } 1.4}{=} L(k,n-1) + L(k,n-k).
\end{aligned}$$

Finally $L(k,n) = L(k,n-1) + L(k,n-k)$, which ends the proof. $\square$

If $n \geqslant 2$ and $k = 2$, then we obtain known recurrence relation for L_n, namely $L_2 = 3,\ L_3 = 4$ and for $n \geqslant 4,\ L_n = L_{n-1} + L_{n-2}$.

Now we present the sequence of identities for the generalized Fibonacci numbers $F(k,n)$.

Theorem 2.2. *Let $k \geqslant 2$ be integer. Then for $n \geqslant k+1$*

$$\cdot \quad \sum_{i=0}^{n-k} F(k,i) = F(k,n) - k$$

Table 1
The values of $F(k,n)$ and $L(k,n)$ for special case of k and n.

n	0	1	2	3	4	5	6	7	8	9
F_n	1	2	3	5	8	13	21	34	55	89
$F(3,n)$	1	2	3	4	6	9	13	19	28	41
$F(4,n)$	1	2	3	4	5	7	10	14	19	26
$F(5,n)$	1	2	3	4	5	6	8	11	15	20
L_n	2	1	3	4	7	11	18	29	47	76
$L(3,n)$	1	2	3	4	5	6	10	15	21	31
$L(4,n)$	1	2	3	4	5	6	7	8	13	19

Proof (*By induction on n*). If $n = k+1$, then using the Fact 1.3 we have that $F(k,n) = F(k,k+1) = F(k,k) + F(k,1) = k+1+2 = k+3$. Hence $\sum_{i=0}^{k+1-k} F(k,i) = F(k,0) + F(k,1) = 3 = F(k,k+1) - k$.

Assume that the Theorem is true for an arbitrary $n = p$, so $\sum_{i=0}^{p-k} F(k,i) = F(k,p) - k$. We shall prove that the equality holds for $n = p+1$, so $\sum_{i=0}^{p+1-k} F(k,i) = F(k,p+1) - k$.

Using simple calculations we have that $\sum_{i=0}^{p+1-k} F(k,i) = \sum_{i=0}^{p-k} F(k,i) + F(k,p-k+1) = F(k,p) - k + F(k,p-(k-1)) \overset{\text{by } 1.3}{=} F(k,p+1) - k$, which ends the proof. □

If $k = 2$, then we obtain known equality for Fibonacci numbers, namely $\sum_{i=0}^{n-2} F_i = F_n - 2$ for $n \geqslant 2$.

Theorem 2.3. *Let $k \geqslant 2,\ n \geqslant k$ be integers. Then*

$$\sum_{i=1}^{n} F(k, ik-1) + 1 = F(k, nk).$$

Proof. Using the Fact 1.3 we have $F(k,n-1) = F(k,n) - F(k,n-k)$ for $n \geqslant k$.

And for integers $k-1, 2k-1, \ldots, nk-1$ we obtain

$$\begin{aligned}
&F(k,k-1) = F(k,k) - F(k,0),\\
&F(k,2k-1) = F(k,2k) - F(k,k),\\
&F(k,3k-1) = F(k,3k) - F(k,2k),\\
&\vdots\\
&F(k,nk-1) = F(k,nk) - F(k,(n-1)k).
\end{aligned}$$

Adding these equalities we obtain that

$$\sum_{i=1}^{n} F(k, ik-1) = F(k,nk) - F(k,0) = F(k,nk) - 1,$$

which ends the proof. □

If $k = 2$, then we obtain known equality for the Fibonacci numbers, namely

$$\sum_{i=1}^{n} F_{2i-1} + 1 = F_{2n}.$$

Theorem 2.4. *Let $k \geqslant 2,\ n \geqslant 2k-2$ be integers. Then*

$$F(k,n) = \sum_{i=0}^{k-1} F(k, n-(k-1)-i).$$

Proof. Let $n \geqslant 2k-2$. Then using the Fact 1.3 by $(k-1)$ times we obtain

$$\begin{aligned}
F(k,n) &= F(k,n-1) + F(k,n-k)\\
&= F(k,n-2) + F(k,n-k-1) + F(k,n-k)\\
&= F(k,n-3) + F(k,n-k-2) + F(k,n-k-1) + F(k,n-k)\vdots\\
&= F(k,n-(k-1)) + F(k,n-2(k-1)) + F(k,n-2(k-1)+1) + \cdots + F(k,n-k)\\
&= F(k,n-(k-1)) + F(k,n-k) + \cdots + F(k,n-2(k-1))\\
&= \sum_{i=0}^{k-1} F(k, n-(k-1)-i),
\end{aligned}$$

which ends the proof. □

If $k = 2$ and $n \geqslant 2$, then we obtain the basic equality for the Fibonacci numbers, $F_n = F_{n-1} + F_{n-2}$.

Theorem 2.5. *Let $k \geqslant 2,\ n \geqslant 2k-1$ be integers. Then*

$$F(k,n) = F(k,n-1) + F(k,n-2) - \sum_{i=0}^{k-3} F(k, n-(2k-1)+i).$$

Proof. Let $k \geqslant 2,\ n \geqslant 2k-1$. If $k=2$, then $F(2,n)=F(2,n-1)+F(2,n-2)$ and the Theorem follows by basic recurrence for the Fibonacci numbers. Assume now that $k \geqslant 3$. Then using Fact 1.3 and some calculations we have

$$\begin{aligned} F(k,n-1)+F(k,n-2)-\sum_{i=0}^{k-3}F(k,n-(2k-1)+i) &= F(k,n-1)+F(k,n-2)-F(k,n-2k+1)-F(k,n-2k+2)-\cdots-F(k,n-k-2)\\ &= F(k,n-1)+F(k,n-3)+F(k,n-k-2)-F(k,n-2k+1)\\ &\quad -F(k,n-2k+2)-\cdots-F(k,n-k-3)-F(k,n-k-2)\\ &= F(k,n-1)+F(k,n-3)-F(k,n-2k+1)-F(k,n-2k+2)-\cdots\\ &\quad -F(k,n-k-3). \end{aligned}$$

This operation we repeat $(k-2)$ times and we obtain the following dependences.

$$\begin{aligned} &F(k,n-1)+F(k,n-3)-F(k,n-2k+1)-\cdots-F(k,n-k-3)\\ &\quad = F(k,n-1)+F(k,n-4)-F(k,n-2k+1)-\cdots-F(k,n-k-4)\\ &\quad = F(k,n-1)+F(k,n-5)-F(k,n-2k+1)-\cdots-F(k,n-k-5)=\cdots \end{aligned}$$

Finally we obtain $F(k,n-1)+F(k,n-k)=F(k,n)$, which ends the proof. □

Now we present some identities for the generalized Lucas numbers.

Theorem 2.6. *Let $k \geqslant 2,\ n \geqslant 2k$ be integers. Then*

$$\sum_{i=2}^{n}L(k,ki)=L(k,nk+1)-(k+2).$$

Proof. By the Fact 1.4 we have $L(k,n-1)=L(k,n)-L(k,n-k)$, for $n \geqslant 2k$. Using this relation for integers $2k,3k,\ldots,nk$ we obtain

$$\begin{aligned} &L(k,2k)=L(k,2k+1)-L(k,k+1),\\ &L(k,3k)=L(k,3k+1)-L(k,2k+1),\\ &L(k,4k)=L(k,4k+1)-L(k,3k+1),\\ &\vdots\\ &L(k,nk)=L(k,nk+1)-L(k,(n-1)k+1). \end{aligned}$$

Adding these equalities we obtain, that

$$\begin{aligned} \sum_{i=2}^{n}L(k,ki) &= L(k,2k+1)-L(k,k+1)+L(k,3k+1)-L(k,2k+1)+L(k,4k+1)-L(k,3k+1)+\cdots+L(k,nk+1)\\ &\quad -L(k,(n-1)k+1)=L(k,nk+1)-L(k,k+1)=L(k,nk+1)-(k+2), \end{aligned}$$

which ends the proof. □

If $k=2$, then we obtain known equality for the Lucas numbers, namely $\sum_{i=2}^{n}L_{2i}=L_{2n-1}-3$.

Theorem 2.7. *Let $k \geqslant 2,\ n \geqslant 2k$ be integers. Then*

$$L(k,n)=kF(k,n-(2k-1))+F(k,n-k).$$

Proof. To prove this Theorem we apply the graph interpretation of the number $L(k,n)$. Because for an arbitrary $k \geqslant 2$ and $n \geqslant 3$, $L(k,n)=NI_k(C_n)$, so it suffices to calculate the number of k-independent sets in the graph C_n. Suppose that $n \geqslant 2k$ and let S be an arbitrary k-independent set of C_n with the vertex set $V(C_n)$ numbered in the natural fashion. Then the definition of k-independent set immediately gives that for each two vertices $x_i,x_j \in S,\ k \leqslant |i-j| \leqslant n-k$.

Let i be a fixed integer, $1 \leqslant i \leqslant k$. Two cases can occur now.

Case 1. $x_i \notin S$ for $i=1,\ldots,k$.

If $\mathcal{F}_1$ is the family of all sets S containing the vertex $x_i, i=1,\ldots,k$, then its cardinality $|\mathcal{F}_1|$ is equal to the number of all k-independent sets of the graph $C_n-\bigcup_{i=1}^{k}\{x_i\}$ which is isomorphic to P_{n-k}. In other words $|\mathcal{F}_1|=NI_k(P_{n-k})=F(k,n-k)$.

Case 2. $x_i \in S$ for $1 \leqslant i \leqslant k$.

Then it is clear that if $x_j \in S$, then $k \leqslant |i-j| \leqslant n-k$, which is equivalent to $k \leqslant j-i \leqslant n-k$. Therefore $i+k \leqslant j \leqslant n-k+i$. This gives that $S = S^{*} \cup \{x_i\}$, where S^{*} is an arbitrary k-independent set of the graph $C_n - N_{C_n}^{k-1}[x_i]$ which is isomorphic to P_{n-2k+1}. If $\mathcal{F}_2$ denotes the family of all independent sets such that the case 2 holds, then $|\mathcal{F}_2| = NI_k(P_{n-2k+1}) = F(k, n-(2k-1))$. Since the vertex x_i can be choosen on k ways so, from the above cases we have that $L(k,n) = NI_k(C_n) = kF(k, n-(2k-1)) + F(k, n-k)$. Thus the Theorem is proved. $\square$

References

[1] C. Berge, Principles of Combinatorics, Academic Press, New York and London, 1971.
[2] R. Diestel, Graph Theory, Springer-Verlag, Heidelberg, New-York, 2005.
[3] S. Falcon, A. Plaza, On k-Fibonacci numbers of arithmetic indexes, Applied Mathematics and Computation 208 (2009) 180–185.
[4] H. Galeana-Sánchez, R. Gomez-Aiza, (k,l)-kernels, (k,l)-semikernels, k-Grundy functions and duality for state splitting, Discussiones Mathematicae – Graph Theory 27 (2) (2007) 358–373.
[5] I. Gutman, S. Wagner, Maxima and minima of the Hosoya index and the Merrifield–Simmons index. A survey of results and techniques, Acta Applicandae Mathematicae 112/3 (2010) 323–346.
[6] E. Kiliç, The Binet formula, sums and representations of generalized Fibonacci p-numbers, European Journal of Combinatorics 29 (2008) 701–711.
[7] M. Kwaśnik, I. Włoch, The total number of generalized stable sets and kernels of graphs, Ars Combinatoria 55 (2000) 139–146.
[8] R.E. Merrifield, H.E. Simmons, Topological Methods in Chemistry, John Wiley & Sons, New York, 1989.
[9] H. Prodinger, R.F. Tichy, Fibonacci numbers of graphs, The Fibonacci Quarterly 20 (1982) 16–21.
[10] W. Szumny, A. Włoch, I. Włoch, On the existence and on the number of (k,l)-kernels in lexicographic product of graphs, Discrete Mathematics 308 (2008) 4616–4624.
[11] D. Tasci, E. Kiliç, On the order-k generalized Lucas numbers, Applied Mathematics and Computation 155 (2004) 637–641.
[12] N. Tuglu, E.G. Kocer, A. Stakhov, Bivariate Fibonacci like p-polynomials, Applied Mathematics and Computation 217 (2011) 10239–10246.
[13] I. Włoch, Generalized Fibonacci polynomial of graphs, Ars Combinatoria 68 (2003) 49–55.
[14] A. Włoch, I. Włoch, Generalized sequences and k-independent sets in graphs, Discrete Applied Mathematics 158 (2010) 1966–1970.
[15] S. Yang, On the k-generalized Fibonacci numbers and high-order linear recurrence relations, Applied Mathematics and Computation 196 (2008) 850–857.