Some Congruences for Central Binomial Sums Involving Fibonacci and Lucas Numbers

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Abstract

We present several polynomial congruences about sums with central binomial coefficients and harmonic numbers. In the final section we collect some new congruences involving Fibonacci and Lucas numbers.

1 Introduction

Recently, the following identity was proposed by Knuth in the problem section of the American Mathematical Monthly [3]:

\[
\left( \sum_{k=1}^{\infty} \binom{2k}{k} \frac{x^k}{k} \right)^2 = 4 \sum_{k=1}^{\infty} \binom{2k}{k} (H_{2k-1} - H_k) \frac{x^k}{k},
\]

where \(H_n = \sum_{k=1}^{n} 1/k\) is the \(n\)-th harmonic number. Playing around with this formula, we realized that there is a corresponding polynomial congruence, namely, for all prime numbers \(p\),

\[
\left( \sum_{k=1}^{p-1} \binom{2k}{k} \frac{x^k}{k} \right)^2 \equiv 4 \sum_{k=1}^{p-1} \binom{2k}{k} (H_{2k-1} - H_k) \frac{x^k}{k} \pmod{p}.
\]
By using this congruence together with some previous results given in [5, 6], we find that for all prime numbers \( p > 3 \),

\[
\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k} x^k}{k} \equiv (2x - \alpha)^p \mathcal{L}_2(-\beta/\alpha) + 2\alpha^p \mathcal{L}_2(\beta/\alpha) \quad (\text{mod } p)
\]  

(3)

where \( \mathcal{L}_2(x) = \sum_{k=1}^{p-1} \frac{x^k}{k} \) is the finite dilogarithm and

\[
\alpha = \frac{1}{2} \left( 1 + \sqrt{1 - 4x} \right) \quad \text{and} \quad \beta = \frac{1}{2} \left( 1 - \sqrt{1 - 4x} \right).
\]

These kind of congruences have been actively investigated and many interesting formulas have been discovered (see the references in [5, 6]). For example, by letting \( x = 1 \) in (3), we recover the congruence

\[
\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k}}{k} \equiv \frac{7}{12} \left( \frac{p}{3} \right) B_{p-2} \left( 1/3 \right) \quad (\text{mod } p)
\]  

(4)

which appeared in [4], where \( \left( \frac{x}{y} \right) \) denotes the Legendre symbol, and \( B_n(x) \) is the \( n \)-th Bernoulli polynomial. Moreover, we show several congruences involving Fibonacci numbers \( F_n \) and Lucas numbers \( L_n \). Two of them are as follows: for all prime numbers \( p > 5 \),

\[
\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{2k}{k} H_{2k} F_{3k} \equiv \frac{13}{10} \left( \frac{p}{5} \right) q_L^2 \quad (\text{mod } p),
\]  

(5)

\[
\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{2k}{k} H_{2k} L_{3k} \equiv \frac{5}{2} q_L^2 \quad (\text{mod } p),
\]  

(6)

where \( q_L = (L_p - 1)/p \) is the so-called Lucas quotient.

The paper is organized into four sections. The next section is devoted to a brief introduction to the finite polylogarithm. In Section 3 we present the proofs of the main theorems about the polynomial congruences and in the final section we establish various congruences involving Fibonacci numbers.

## 2 The finite polylogarithm

The classical polylogarithm function is defined for complex \(|z| < 1\) and all positive integers \( d \) by the power series

\[
\text{Li}_d(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^d}.
\]
It is well known that the polylogarithm can be extended analytically to a wider range of \( z \) and it satisfies several remarkable identities such as the two reflection properties,

\[
\text{Li}_2(z) + \text{Li}_2(1/z) = -\frac{\pi^2}{6} - \ln^2(-z) \
\text{and} \quad \text{Li}_2(z) + \text{Li}_2(1-z) = -\frac{\pi^2}{6} - \ln(z) \ln(1-z).
\]

These identities allow the explicit evaluation of the polylogarithm at some special values, such as

\[
\text{Li}_2(1) = \zeta(2) = -\frac{\pi^2}{6}, \quad \text{Li}_2(1/2) = \frac{\pi^2}{12}, \quad \text{Li}_2(\phi_-) = -\frac{\pi^2}{15} + \frac{\ln^2(\phi_+)}{2}.
\]

where \( \phi_\pm = (1 \pm \sqrt{5})/2 \).

The finite polylogarithm function is the partial sum of the above series over the range \( 0 < k < p \) where \( p \) is a prime

\[
\mathcal{L}_d(x) = \sum_{k=1}^{p-1} x^k.
\]

It satisfies some nice properties that resemble the ones satisfied by the classical polylogarithm.

Here we restrict our attention to \( \mathcal{L}_2(x) \) (see [5] for more details): for all prime numbers \( p > 3 \),

\[
\mathcal{L}_2(x) \equiv x^p \mathcal{L}_2(1/x) \pmod{p},
\]

\[
\mathcal{L}_1(1-x) \equiv -Q_p(x) - p \mathcal{L}_2(x) \pmod{p^2},
\]

\[
\mathcal{L}_2(x) \equiv \mathcal{L}_2(1-x) + x^p \mathcal{L}_2(1-1/x) \pmod{p},
\]

\[
x^p \mathcal{L}_2(x) + (1-x)^p \mathcal{L}_2(1-x) \equiv \frac{1}{2} Q_p^2(x) \pmod{p}.
\]

where

\[
Q_p(x) = xq_p(x) + (1-x)q_p(1-x), \quad \text{with} \quad q_p(x) = \frac{x^{p-1} - 1}{p}.
\]

Several congruences for special values of \( \mathcal{L}_2(x) \) are known:

\[
\mathcal{L}_2(1) \equiv \mathcal{L}_2(-1) \equiv 0, \quad \mathcal{L}_2(2) \equiv 2 \mathcal{L}_2(1/2) \equiv -q_p^2(2) \pmod{p}.
\]

Moreover

\[
\mathcal{L}_2((1\pm i)/2) \equiv -\frac{q_p^2(2)}{8} + \frac{1}{4} \left( \left( \frac{-1}{p} \right) \pm i \right) E_{p-3} \pmod{p},
\]

\[
\mathcal{L}_2(\omega_6^{\pm 1}) \equiv \frac{1}{8} \left( \left( \frac{p}{3} \right) \pm \frac{\sqrt{3}}{3} \right) B_{p-2}(1/3), \pmod{p}
\]
where \( \omega_6 = \left(1 \pm i \sqrt{3}\right)/2 \) and \( E_n \) is \( n \)-th Euler number. Finally, for all prime numbers \( p > 5 \) we have

\[
\mathcal{L}_2(\phi_\pm) \equiv \mp \frac{\sqrt{5}}{10} \left( \frac{p}{5} \right) q_L^2 \pmod{p}, \\
\mathcal{L}_2(\phi_\pm^2) \equiv -\frac{1}{2} \left(1 \pm \frac{\sqrt{5}}{5} \left( \frac{p}{5} \right) \right) q_L^2 \pmod{p}, \\
\mathcal{L}_2(-\phi_\pm) \equiv -\frac{1}{4} \left(1 \pm \frac{\sqrt{5}}{5} \left( \frac{p}{5} \right) \right) q_L^2 \pmod{p}.
\]

Notice that the Lucas quotient satisfies (see [7]),

\[
q_L = Q(\phi_\pm) \equiv \frac{5 F_{p-(\frac{p}{5})}}{2p} \pmod{p}.
\]

3 Polynomial congruences for central binomial sums

In [5, 6], we studied various sum involving the central binomial coefficients. In particular, it has been shown that for all prime numbers \( p > 3 \),

\[
\sum_{k=1}^{p-1} \binom{2k}{k} x^k \equiv \sum_{k=1}^{p-1} \binom{p-1}{2k} (-4x)^k \equiv (1 - 4x)^{p-1/2} \pmod{p}, \tag{7}
\]

\[
\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} x^k \equiv \mathcal{L}_1(\alpha) + \mathcal{L}_1(\beta) \pmod{p}, \tag{8}
\]

\[
\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2} x^k \equiv 2 \mathcal{L}_2(\alpha) + 2 \mathcal{L}_2(\beta) \pmod{p}, \tag{9}
\]

\[
\sum_{k=1}^{p-1} \binom{2k}{k} H_k^{(2)} x^k \equiv \frac{2(\mathcal{L}_2(\beta) - \mathcal{L}_2(\alpha))}{\sqrt{1 - 4x}} \pmod{p}. \tag{10}
\]

where \( H_n^{(2)} = \sum_{k=1}^{n} 1/k^2 \).

In [1, Proposition 5], Boyadzhiev used the following Euler-type series transformation formula to handle series with central binomial coefficients: if \( a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_k \) then in a neighborhood of \( x = 0 \),

\[
\sum_{k=0}^{\infty} \binom{2k}{k} a_k x^k = \frac{1}{\sqrt{1 - 4x}} \sum_{j=0}^{\infty} \binom{2j}{j} b_j \left(\frac{-x}{1 - 4x}\right)^j.
\]
Moreover, it turns out that something similar holds for finite sum congruences:

\[ \sum_{k=0}^{p-1} \binom{2k}{k} a_k x^k \equiv \sum_{k=0}^{(p-1)/2} \binom{p-1}{2} k \binom{(p-1)/2}{k} (-4x)^k = \sum_{k=0}^{(p-1)/2} \binom{(p-1)/2}{k} (-4x)^k \sum_{j=0}^{k} \binom{k}{j} (-1)^j b_j \]

Hence, by (11.2)

\[ \sum_{j=0}^{(p-1)/2} (-1)^j b_j \sum_{k=j}^{(p-1)/2} \binom{(p-1)/2}{k} (-4x)^k \]

\[ \equiv \sum_{j=0}^{(p-1)/2} (-1)^j b_j \binom{(p-1)/2}{j} (-4x)^j (1 - 4x)^{p-1-j} \]

\[ \equiv (1 - 4x)^{p-1} \sum_{j=0}^{(p-1)/2} \binom{(p-1)/2}{j} b_j \left( \frac{-x}{1-4x} \right)^j \pmod{p}. \] (11)

In the next theorem we apply the above transformation.

**Theorem 1.** For all prime numbers \( p > 3, \)

\[ \sum_{k=1}^{p-1} \binom{2k}{k} H_k x^k \equiv -2(1 - 4x)^{\frac{p-1}{2}} \mathcal{L}_1 \left( -\frac{\beta}{\sqrt{1-4x}} \right) \pmod{p}, \] (12)

\[ \sum_{k=1}^{p-1} \binom{2k}{k} H_k x^k \equiv 2(1 - 4x)^{\frac{p-1}{2}} \left( \mathcal{L}_2 \left( \frac{\alpha}{\sqrt{1-4x}} \right) - \mathcal{L}_2 \left( -\frac{\beta}{\sqrt{1-4x}} \right) \right) \pmod{p}. \] (13)

**Proof.** It is easy to verify by induction that

\[ \sum_{k=1}^{n} (-1)^k \binom{n}{k} H_k(1) = -\frac{1}{n} \quad \text{and} \quad \sum_{k=1}^{n} (-1)^k \binom{n}{k} H_k(2) = -\frac{H_n}{n}. \]

Moreover

\[ \alpha \left( \frac{-x}{1-4x} \right) = \alpha \frac{\sqrt{1-4x}}{\sqrt{1-4x}} \quad \text{and} \quad \beta \left( \frac{-x}{1-4x} \right) = -\frac{\beta}{\sqrt{1-4x}}. \]

Hence, by (11) and (8),

\[ \sum_{k=1}^{p-1} \binom{2k}{k} H_k x^k \equiv -(1 - 4x)^{\frac{p-1}{2}} \sum_{j=1}^{p-1} \binom{2k}{k} \frac{1}{k} \left( \frac{-x}{1-4x} \right)^k \]

\[ \equiv -(1 - 4x)^{\frac{p-1}{2}} \left( \mathcal{L}_1 \left( \frac{\alpha}{\sqrt{1-4x}} \right) + \mathcal{L}_1 \left( -\frac{\beta}{\sqrt{1-4x}} \right) \right) \]

\[ \equiv -(1 - 4x)^{\frac{p-1}{2}} \left( \mathcal{L}_1 \left( 1 - \frac{\alpha}{\sqrt{1-4x}} \right) + \mathcal{L}_1 \left( -\frac{\beta}{\sqrt{1-4x}} \right) \right) \]

\[ \equiv -2(1 - 4x)^{\frac{p-1}{2}} \mathcal{L}_1 \left( -\frac{\beta}{\sqrt{1-4x}} \right) \pmod{p}, \]

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where we also used $L_1(x) \equiv L_1(1 - x)$. Thus the proof of (12) is complete.

As regards (13), Eqns. (11) and (10) imply

$$
\sum_{k=1}^{p-1} \left( \binom{2k}{k} \frac{H_k x^k}{k} \right) \equiv -(1 - 4x)^{\frac{p-1}{2}} \sum_{j=1}^{p-1} \left( \binom{2k}{k} \frac{H_k^{(2)}}{x} \right) \left( \frac{-x}{1 - 4x} \right)^k \\
\equiv 2(1 - 4x)^{\frac{p-1}{2}} \left( L_2 \left( \frac{\alpha}{\sqrt{1 - 4x}} \right) - L_2 \left( -\frac{\beta}{\sqrt{1 - 4x}} \right) \right) \pmod{p}.
$$

In the next theorem, we establish (2), the analogous congruence for the series (1).

**Theorem 2.** For all prime numbers $p > 3$,

$$
\left( \sum_{k=1}^{p-1} \left( \binom{2k}{k} \cdot x^k \right) \right) \left( \sum_{k=1}^{p-1} \left( \binom{2k}{k} \cdot \frac{x^k}{k} \right) \right) \equiv 2 \sum_{k=1}^{p-1} \left( \binom{2k}{k} \right) \left( H_{2k-1} - H_k \right) x^k \pmod{p},
$$

(14)

$$
\left( \sum_{k=1}^{p-1} \left( \binom{2k}{k} \cdot x^k \right) \right)^2 \equiv 4 \sum_{k=1}^{p-1} \left( \binom{2k}{k} \right) \left( H_{2k-1} - H_k \right) \frac{x^k}{k} \pmod{p}.
$$

(15)

**Proof.** Since $p$ divides $\binom{2k}{k}$ for $(p - 1)/2 < k < p$, it follows that

$$
\left( \sum_{k=1}^{p-1} \left( \binom{2k}{k} \cdot x^k \right) \right) \left( \sum_{k=1}^{p-1} \left( \binom{2k}{k} \cdot \frac{x^k}{k} \right) \right) \equiv \sum_{n=1}^{p-1} x^n \sum_{k=1}^{n-1} \left( \frac{1}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \right) \pmod{p}.
$$

In a similar way,

$$
\left( \sum_{k=1}^{p-1} \left( \binom{2k}{k} \cdot \frac{x^k}{k} \right) \right)^2 \equiv \sum_{n=1}^{p-1} x^n \sum_{k=1}^{n-1} \left( \frac{1}{k(n-k)} \binom{2k}{k} \binom{2(n-k)}{n-k} \right) \pmod{p}.
$$

Therefore, it suffices to show by induction that

$$
\sum_{k=1}^{n-1} F(n, k) = 2(H_{2n-1} - H_n) \quad \text{where} \quad F(n, k) = \frac{1}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \binom{2n}{n}^{-1}.
$$

It holds for $n = 1$, and it is straightforward to verify that

$$
F(n+1, k) - F(n, k) = G(n, k + 1) - G(n, k) \quad \text{with} \quad G(n, k) = -\frac{k^2(2n - 2k + 1)F(n, k)}{(n+1)(2n+1)(n+1-k)}.
$$
Hence, by the inductive assumption,
\[
\sum_{k=1}^{n} F(n+1,k) = \sum_{k=1}^{n} F(n,k) + \sum_{k=1}^{n} (G(n,k+1) - G(n,k))
\]
\[
= 2(H_{2n-1} - H_n) + F(n,n) + G(n,n+1) - G(n,1)
\]
\[
= 2(H_{2n-1} - H_n) + \frac{1}{n} + 0 + \frac{(2n-1)F(n,1)}{(n+1)(2n+1)n}
\]
\[
= 2(H_{2n-1} - H_n) + \frac{1}{n} + \frac{1}{(n+1)(2n+1)} = 2(H_{2n+1} - H_{n+1}).
\]

Now we are ready to show that our main result (3) and the congruence corresponding to the series [2, Theorem 6]: for \(|x| < 1/4,\)
\[
\sum_{k=1}^{\infty} \binom{2k}{k} H_{2k} x^k = \frac{1}{\sqrt{1-4x}} \left( \ln \left( \frac{1 + \sqrt{1 - 4x}}{2} \right) - 2 \ln(\sqrt{1 - 4x}) \right). \tag{16}
\]

**Theorem 3.** For all prime numbers \(p > 3,\)
\[
\sum_{k=1}^{p-1} \binom{2k}{k} H_{2k} x^k \equiv (1 - 4x)^{(p-1)/2} \left( L_1(\beta) - 2L_1 \left( -\frac{\beta}{\sqrt{1 - 4x}} \right) \right) \pmod{p} \tag{17}
\]
\[
\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k} x^k}{k} \equiv (2x - \alpha)^p L_2(-\beta/\alpha) + 2\alpha^p L_2(\beta/\alpha) \pmod{p}. \tag{18}
\]

**Proof.** As regards (17), since \(H_{2k} = \frac{1}{2k} + (H_{2k-1} - H_k) + H_k,\) it follows immediately that,
\[
\sum_{k=1}^{p-1} \binom{2k}{k} H_{2k} x^k = \frac{1}{2} \sum_{k=1}^{p-1} \binom{2k}{k} x^k + \sum_{k=1}^{p-1} \binom{2k}{k} (H_{2k-1} - H_k) x^k + \sum_{k=1}^{p-1} \binom{2k}{k} H_k x^k
\]
and we apply (7), (14), and (12). In a similar way, for (18),
\[
\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k} x^k}{k} = \frac{1}{2} \sum_{k=1}^{p-1} \binom{2k}{k} x^k + \sum_{k=1}^{p-1} \binom{2k}{k} (H_{2k-1} - H_k) x^k + \sum_{k=1}^{p-1} \binom{2k}{k} H_k x^k
\]
and then we use (9), (15), and (13).

As a remark, we point out that although the series (16) does not converge for \(x = 1/4,\) by letting \(f(x)\) be the left-hand side of (16) then
\[
\sum_{k=1}^{\infty} \binom{2k}{k} \frac{H_{2k}}{4^k k} = \int_{0}^{\frac{1}{4}} \frac{f(x)}{x} \, dx = \frac{5\pi^2}{12}.
\]
On the other hand, it can be verified that the congruence (18) holds even for \( x = 1/4 \), and for all prime numbers \( p > 3 \),
\[
\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k}}{4^k k} \equiv \mathcal{L}_2(1) \equiv 0 \pmod{p}.
\]

### 4 Congruences with Fibonacci and Lucas numbers

By looking at this table and by using the values of \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \), we can easily obtain the explicit values of the congruences established in the previous section.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \omega_6 )</td>
<td>( \omega_6^{-1} )</td>
</tr>
<tr>
<td>-1</td>
<td>( \phi_+ )</td>
<td>( \phi_- )</td>
</tr>
<tr>
<td>-2</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>1/2</td>
<td>( (1+i)/2 )</td>
<td>( (1-i)/2 )</td>
</tr>
<tr>
<td>1/3</td>
<td>( (1+\omega_6)/3 )</td>
<td>( (1+\omega_6^{-1})/3 )</td>
</tr>
<tr>
<td>1+i</td>
<td>1-i</td>
<td>i</td>
</tr>
<tr>
<td>1-i</td>
<td>1+i</td>
<td>-i</td>
</tr>
<tr>
<td>( \pm i\sqrt{3} )</td>
<td>( 1+\omega_6^{\mp 1} )</td>
<td>( -\omega_6^{\mp 1} )</td>
</tr>
<tr>
<td>( -\phi_3 )</td>
<td>( -\phi_- )</td>
<td>( \phi_-^2 )</td>
</tr>
<tr>
<td>( -\phi_3^+ )</td>
<td>( \phi_+^2 )</td>
<td>( -\phi_+ )</td>
</tr>
</tbody>
</table>

For example, for all prime numbers \( p > 3 \), by taking \( x = 1, 1/2, 1/3 \) in (18), we get respectively (4), and the next two congruences,
\[
\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k}}{2^k k} \equiv \frac{3}{16} \left( \frac{-1}{p} \right) B_{p-2}(1/4) \pmod{p},
\]
\[
\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_{2k}}{3^k k} \equiv \frac{2}{9} \left( \frac{p}{3} \right) B_{p-2}(1/3) \pmod{p}.
\]

To order to get the congruences with \( F_n \) and \( L_n \) we need consider the cases \( x = -\phi_3^+ \). If \( x = -\phi_3^+ \) then \( 2x - \alpha = -\phi_4^4 \) and
\[
\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{2k}{k} H_{2k} \phi_3^k \equiv (-\phi_-^4)^p \mathcal{L}_2(\phi_-) + 2(-\phi_-)^p \mathcal{L}_2(-\phi_-)
\]
\[
\equiv \frac{1}{2} \left( -7 + 3 \left( \frac{p}{5} \right) \sqrt{5} \right) \mathcal{L}_2(\phi_-) + \left( -1 + \left( \frac{p}{5} \right) \sqrt{5} \right) \mathcal{L}_2(-\phi_-)
\]
\[
\equiv \left( \frac{5}{4} - \frac{13}{20} \left( \frac{p}{5} \right) \sqrt{5} \right) q_L^2 \pmod{p}.
\]
where we used the fact that $2\phi_\pm^p \equiv 1 \pm \left(\frac{p}{5}\right) \sqrt{5} \pmod{p}$. In a similar way, we find that

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{2k}{k} H_{2k} \phi_\pm^{3k} \equiv \left(\frac{5}{4} \pm \frac{13}{20} \left(\frac{p}{5}\right) \sqrt{5}\right) q_L^2 \pmod{p},$$

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{2k}{k} H_k \phi_\pm^{3k} \equiv \left(\frac{1}{2} \pm \frac{3}{10} \left(\frac{p}{5}\right) \sqrt{5}\right) q_L^2 \pmod{p}.$$

Since $\sqrt{5} F_{3k} = \phi_+^{3k} - \phi_-^{3k}$ and $L_{3k} = \phi_+^{3k} + \phi_-^{3k}$, it follows that for $p > 5$, (5) and (6) hold and also we find that

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{2k}{k} H_k F_{3k} \equiv \frac{3}{5} \left(\frac{p}{5}\right) q_L^2 \pmod{p},$$

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{2k}{k} H_k L_{3k} \equiv q_L^2 \pmod{p}.$$

References


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