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Vieta-Pell and Vieta-Pell-Lucas polynomials

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Abstract
In the present paper, we introduce the recurrence relation of Vieta-Pell and Vieta-Pell-Lucas polynomials. We obtain the Binet form and generating functions of Vieta-Pell and Vieta-Pell-Lucas polynomials and define their associated sequences. Moreover, we present some differentiation rules and finite summation formulas.
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1 Introduction

Andre-Jeannin [1] introduced a class of polynomials $U_n(p,q;x)$ defined by

$$U_n(p,q;x) = (x+p)U_{n-1}(p,q;x) - qU_{n-2}(p,q;x), \quad n \geq 2,$$

with the initial values $U_0(p,q;x) = 0$ and $U_1(p,q;x) = 1$.

Vieta-Lucas polynomials were studied as Vieta polynomials by Robbins [2]. Vieta-Fibonacci and Vieta-Lucas polynomials are defined by

$$V_n(x) = xV_{n-1}(x) - V_{n-2}(x), \quad n \geq 2,$$
$$v_n(x) = xv_{n-1}(x) - v_{n-2}(x), \quad n \geq 2,$$

respectively, where $V_0(x) = 0$, $V_1(x) = 1$ and $v_0(x) = 2$, $v_1(x) = x$ [3]. The recursive properties of Vieta-Fibonacci and Vieta-Lucas polynomials were given by Horadam [3].

For $p = 0$ and $q = 1$, Vieta-Fibonacci polynomials are a special case of the polynomials $U_n(p,q;x)$ in [1]. Further, $U_{n,m}(p,q;x)$ in [4] for $p = 0$, $q = 1$, $m = 2$ gives Vieta-Fibonacci polynomials.

Chebyshev polynomials are a sequence of orthogonal polynomials which can be defined recursively. Recall that the nth Chebyshev polynomials of the first kind and second kind are denoted by $T_n(x)$ and $U_n(x)$, respectively.

It is well known that the Chebyshev polynomials of the first kind and second kind are closely related to Vieta-Fibonacci and Vieta-Lucas polynomials. So, in [5] Vitula and Slota redefined Vieta polynomials as modified Chebyshev polynomials. The related features of Vieta and Chebyshev polynomials are given as

$$V_n(x) = U_n\left(\frac{1}{2}x\right) \quad [3],$$
$$v_n(x) = 2T_n\left(\frac{1}{2}x\right) \quad \text{(see [2, 6])}.$$

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For $|x| > 1$, we consider $t_n(x)$ and $s_n(x)$ polynomials by the following recurrence relations:

$$t_n(x) = 2xt_{n-1}(x) - t_{n-2}(x), \quad n \geq 2,$$
$$s_n(x) = 2xs_{n-1}(x) - s_{n-2}(x), \quad n \geq 2,$$

where $t_0(x) = 0$, $t_1(x) = 1$ and $s_0(x) = 2$, $s_1(x) = 2x$. We call $t_n(x)$ the nth Vieta-Pell polynomial and $s_n(x)$ the nth Vieta-Pell-Lucas polynomial.

The relations below are obvious

$$s_n(x) = 2T_n(x),$$
$$t_{n+1}(x) = U_n(x).$$

The first few terms of $t_n(x)$ and $s_n(x)$ are as follows:

$$\begin{aligned}
&t_2(x) = 2x, && s_2(x) = 4x^2 - 2,\\
&t_3(x) = 4x^2 - 1, && s_3(x) = 8x^3 - 6x,\\
&t_4(x) = 8x^3 - 4x, && s_4(x) = 16x^4 - 16x^2 + 2,\\
&t_5(x) = 16x^4 - 12x^2 + 1, && s_5(x) = 32x^5 - 40x^3 + 10x,\\
&t_6(x) = 32x^5 - 32x^3 + 6x, && s_6(x) = 64x^6 - 96x^4 + 36x^2 - 2,\\
&t_7(x) = 64x^6 - 80x^4 + 24x^2 - 1, && s_7(x) = 128x^7 - 224x^5 + 112x^3 - 14x.
\end{aligned}$$

The aim of this paper is to determine the recursive key features of Vieta-Pell and Vieta-Pell-Lucas polynomials. In conjunction with these properties, we examine their interrelations and define their associated sequences. Furthermore, we present some differentiation rules and summation formulas.

2 Main results

Some fundamental recursive properties of Vieta-Pell and Vieta Pell-Lucas polynomials are given in this section.

Characteristic equation

Vieta-Pell and Vieta-Pell-Lucas polynomials have the following characteristic equation:

$$\lambda^2 - 2x\lambda + 1 = 0$$

with the roots α and β

$$\alpha = x + \sqrt{x^2 - 1},$$
$$\beta = x - \sqrt{x^2 - 1}.$$

Also, α and β satisfy the following equations:

$$\begin{aligned}
&\alpha + \beta = 2x,\\
&\alpha\beta = 1,\\
&\alpha - \beta = \Delta = 2\sqrt{x^2 - 1}.
\end{aligned} \tag{1}$$

Binet form

By appropriate procedure, we can easily find the Binet forms as

$$t_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2}$$

$$s_n(x) = \alpha^n + \beta^n. \tag{3}$$

Generating function

Vieta-Pell and Vieta-Pell-Lucas polynomials can be defined by the following generating functions:

$$\sum_{n=0}^{\infty} t_n(x)y^n = y\left(1 - 2xy + y^2\right)^{-1},$$

$$\sum_{n=0}^{\infty} s_n(x)y^n = (2 - 2xy)\left(1 - 2xy + y^2\right)^{-1}.$$

Negative subscript

We can also extend the definition of $t_n(x)$ and $s_n(x)$ to the negative index

$$t_{-n}(x) = -t_n(x),$$

$$s_{-n}(x) = s_n(x).$$

Simson formulas

$$t_{n+1}(x)t_{n-1}(x) - t_n^2(x) = -1,$$

$$s_{n+1}(x)s_{n-1}(x) - s_n^2(x) = 4\left(x^2 - 1\right).$$

We arrange the first ten coefficients of $t_n(x)$ in Table 1. Let $T(n,j)$ denote the element in row n and column j, where $j \geq 0$, $n \geq 1$. As seen from the Table 1, it is obvious that

$$T(n,0) = 2T(n-2,0) + T(n-1,0)$$

Table 1 The first ten coefficients of $t_n(x)$

$n\backslash j$	0	1	2	3	4
0	0				
1	1				
2	2				
3	4	–1			
4	8	–4			
5	16	–12	1		
6	32	–32	6		
7	64	–80	24	–1	
8	128	–192	80	–8	
9	256	–448	240	–40	1

can be written like the coefficients of Pell polynomials in [7]. Moreover,

$$\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor} T(n,j) = n.$$

For example, for $n = 8$ we can find

$$\sum_{j=0}^{3} T(8,j) = T(8,0) + T(8,1) + T(8,2) + T(8,3) = 8.$$

Let P_n denote the nth Pell number, so we have

$$\sum_{j=1}^{\lfloor\frac{n-1}{2}\rfloor} \left|T(n,j)\right| = P_n.$$

2.1 Interrelations of $t_n(x)$ and $s_n(x)$

Most of the equations below can be obtained by using the Binet form and convenient routine operations

$$t_{n+1}(x) - t_{n-1}(x) = s_n(x) = 2xt_n(x) - 2t_{n-1}(x), \tag{4}$$

$$s_{n+1}(x) - s_{n-1}(x) = 4\left(x^2 - 1\right)t_n(x), \tag{5}$$

$$t_n(x)s_n(x) = t_{2n}(x),$$

$$s_n^2(x) + 4\left(x^2 - 1\right)t_n^2(x) = 2s_{2n}(x),$$

$$s_n^2(x) - 4\left(x^2 - 1\right)t_n^2(x) = 4,$$

$$t_{n+1}^2(x) - t_n^2(x) = t_{2n+1}(x),$$

$$s_{n+1}^2(x) - s_n^2(x) = 4\left(x^2 - 1\right)t_{2n+1}(x),$$

$$s_{n+1}^2(x) + s_n^2(x) = 2xs_{2n+1}(x) + 4,$$

$$t_{n+1}^2(x) - t_{n-1}^2(x) = 2xt_{2n}(x),$$

$$s_n(x)s_{n+1}(x) - 4\left(x^2 - 1\right)t_n(x)t_{n+1}(x) = 4x,$$

$$s_n(x)s_{n+1}(x) + 4\left(x^2 - 1\right)t_n(x)t_{n+1}(x) = 2s_{2n+1}(x),$$

$$s_{4n}(x) - 2 = 4\left(x^2 - 1\right)t_{2n}^2(x),$$

$$t_{n+1}(x) - xt_n(x) = \frac{1}{2}s_n(x),$$

$$2s_{n+1}(x) - 2xs_n(x) = 4\left(x^2 - 1\right)t_n(x).$$

Proposition 1 $s_n(2x^2 - 1) - s_n^2(x) = -2$.

Proof Consider the expression $s_n(2x^2 - 1)$. Then α, β, Δ are replaced by α^*, β^*, Δ^*, respectively. So, $\alpha^* = \alpha^2$, $\beta^* = \beta^2$, $\Delta^* = 2x\Delta$ and by using the Binet form, the proof is completed. □

2.2 Associated sequences

Definition 1 The kth associated sequences $\{t_n^{(k)}(x)\}$ and $\{s_n^{(k)}(x)\}$ of $\{t_n(x)\}$ and $\{s_n(x)\}$ are defined by, respectively ($k \geq 1$)

$$t_n^{(k)}(x) = t_{n+1}^{(k-1)}(x) - t_{n-1}^{(k-1)}(x), \tag{6}$$

$$s_n^{(k)}(x) = s_{n+1}^{(k-1)}(x) - s_{n-1}^{(k-1)}(x), \tag{7}$$

where $t_n^{(0)}(x) = t_n(x)$ and $s_n^{(0)}(x) = s_n(x)$.

Presently,

$$t_n^{(1)}(x) = s_n(x) \quad \bigl(\text{by (4)}\bigr),$$
$$s_n^{(1)}(x) = \Delta^2 t_n(x) \quad \bigl(\text{by (5)}\bigr)$$

are the members of the first associated sequences $\{t_n^{(1)}(x)\}$ and $\{s_n^{(1)}(x)\}$. If (6) and (7) are applied repeatedly, the results emerge

$$t_n^{2j}(x) = s_n^{(2j-1)}(x) = \Delta^{2j} t_n(x),$$
$$t_n^{(2j-1)}(x) = s_n^{(2j-2)}(x) = \Delta^{2j-2} s_n(x).$$

Some special values of $t_n(x)$ and $s_n(x)$

$$\begin{cases} t_n(-x) = (-1)^{n+1} t_n(x), \\ t_n(1) = n, \\ t_n(-1) = (-1)^{n+1} n, \\ t_{2n}(0) = 0, \\ t_{2n-1}(0) = (-1)^{n+1}, \end{cases}$$

$$\begin{cases} s_n(1) = 2, \\ s_n(-1) = 2(-1)^n, \\ s_{2n}(0) = 2(-1)^n, \\ s_{2n-1}(0) = 0. \end{cases}$$

2.3 Differentiation formulas

$$\frac{ds_n(x)}{dx} = 2nt_n(x),$$
$$\frac{dt_n(x)}{dx} = \frac{ns_n(x) - 2xt_n(x)}{2(x^2-1)},$$
$$\frac{d^2 s_n(x)}{dx^2} = n\left(\frac{ns_n(x) - 2xt_n(x)}{x^2-1}\right).$$

Since the derivation function of $s_n(x)$ is a polynomial, all of the derivatives must exist for all real numbers. Thus, we can give the following formulas.

Proposition 2

$$\left.\frac{d^2 s_n(x)}{dx^2}\right|_{x=1} = \frac{2}{3}\left(n^4 - n^2\right),$$
$$\left.\frac{d^2 s_n(x)}{dx^2}\right|_{x=-1} = \frac{2}{3}(-1)^{n-1}\left(n^2 - n^4\right).$$

Proof If we take the limit on $s_n''(x) = n(\frac{ns_n(x)-2xt_n(x)}{x^2-1})$, we have the numerical value of s_n'' at $x = 1$ and $x = -1$.

$$s_n''(1) = \frac{n}{2}\lim_{x\to 1}\frac{ns_n(x) - 2xt_n(x)}{(x-1)}.$$

Since $s_n(1) = 2$, $t_n(1) = n$, apply L'Hôpital's rule:

$$\begin{aligned}
s_n''(1) &= \frac{n}{2}\lim_{x\to 1}\frac{\frac{d}{dx}(ns_n(x) - 2xt_n(x))}{\frac{d}{dx}(x-1)} \\
&= \frac{n}{2}\lim_{x\to 1}\frac{d}{dx}\left(ns_n(x) - 2xt_n(x)\right) \\
&= \frac{n}{2}\lim_{x\to 1}\left(2n^2 t_n(x) - 2t_n(x) - 2x\frac{d}{dx}t_n(x)\right) \\
&= \frac{n}{2}\left(2n^2 t_n(1) - 2t_n(1) - 2\lim_{x\to 1}\frac{d}{dx}t_n(x)\right) \\
&= \frac{n}{2}\left(2n^3 - 2n\right) - \frac{1}{2}\lim_{x\to 1}\frac{d}{dx}\left(2nt_n(x)\right) \\
&= n^4 - n^2 - \frac{1}{2}s_n''(1).
\end{aligned}$$

So, the proof for $x = -1$ is similar. □

2.4 Some summation formulas

In this section we deal with the matrix

$$\mathbf{V} = \begin{bmatrix} 2x & -1 \\ 1 & 0 \end{bmatrix}.$$

By induction, we have

$$\mathbf{V}^m = \begin{bmatrix} t_{m+1}(x) & -t_m(x) \\ t_m(x) & -t_{m-1}(x) \end{bmatrix}. \tag{8}$$

So, the matrix **V** generates Vieta-Pell and Vieta-Pell-Lucas polynomials. Hence,

$$\begin{bmatrix} t_{m+1}(x) \\ t_m(x) \end{bmatrix} = \mathbf{V}^m \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{9}$$

and from (1.10) in [8], we get

$$t_m(x) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{V}^{m-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{10}$$

It is known that

$$\mathbf{V}^{m+n} = \mathbf{V}^{m}\mathbf{V}^{n}. \tag{11}$$

From (8) and (11), the elementary formulas for $t_n(x)$ are obvious

$$t_{m+n+1}(x) = t_{m+1}(x)t_{n+1}(x) - t_m(x)t_n(x),$$

$$t_{m+n}(x) = t_{m+1}(x)t_n(x) - t_m(x)t_{n-1}(x),$$

$$t_{m+n-1}(x) = t_m(x)t_n(x) - t_{m-1}(x)t_{n-1}(x).$$

If we use the matrix technique for summation in [8], we get the first finite summation as follows.

Proposition 3
(i) $\sum_{n=1}^{m} t_n(x) = \frac{t_{m+1}(x)-t_m(x)-1}{2(x-1)}$,
(ii) $\sum_{n=1}^{m} s_n(x) = \frac{s_{m+1}(x)-s_m(x)+2-2x}{2(x-1)}$.

Proof (i) Let the matrix **A**,

$$\mathbf{A} = \mathbf{I} + \mathbf{V} + \mathbf{V}^2 + \cdots + \mathbf{V}^{n-2} + \mathbf{V}^{n-1}$$

be the series of matrices. Then we have

$$\mathbf{VA} = \mathbf{V} + \mathbf{V}^2 + \cdots + \mathbf{V}^{n-1} + \mathbf{V}^{n}.$$

Hence,

$$\begin{aligned}
&(\mathbf{V}-\mathbf{I})\mathbf{A} = \mathbf{V}^n - \mathbf{I},\\
&\mathbf{A} = (\mathbf{V}-\mathbf{I})^{-1}\big(\mathbf{V}^n - \mathbf{I}\big)\\
&\quad = \frac{1}{2(x-1)}\begin{bmatrix} t_{n+1}(x)-t_n(x)-1 & t_{n-1}(x)-t_n(x)+1 \\ t_n(x)-t_{n-1}(x)-1 & t_{n-2}(x)-t_{n-1}(x)+2x-1 \end{bmatrix},\\
&\sum_{n=1}^{m} t_n(x) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{A}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \big(\text{by (10)}\big)\\
&\qquad = \frac{1}{2(x-1)}\begin{bmatrix} t_{m+1}(x)-t_m(x)-1 & t_{m-1}(x)-t_m(x)+1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\\
&\qquad = \frac{1}{2(x-1)}\big[t_{m+1}(x)-t_m(x)-1\big].
\end{aligned}$$

Thus, the proof is completed.
(ii)

$$\begin{aligned}
\sum_{n=1}^{m} s_n(x) &= \sum_{n=1}^{m}\big(\alpha^n + \beta^n\big) \quad \big(\text{by (3)}\big)\\
&= \alpha\left(\frac{1-\alpha^m}{1-\alpha}\right) + \beta\left(\frac{1-\beta^m}{1-\beta}\right)
\end{aligned}$$

$$= \frac{(\alpha+\beta) - 2\alpha\beta - (\alpha^{m+1} + \beta^{m+1}) + \alpha\beta(\alpha^m + \beta^m)}{2-2x}$$

$$= \frac{2x - 2 - (\alpha^{m+1} + \beta^{m+1}) + \alpha^m + \beta^m}{2(1-x)} \quad \text{(by (1))}$$

$$= \frac{s_{m+1}(x) - s_m(x) + 2 - 2x}{2(x-1)} \quad \text{(by (3))}.$$

This completes the proof. □

Theorem 1 *Let* $\mathbf{V}$ *be a square matrix such that* $\mathbf{V}^2 = 2x\mathbf{V} - \mathbf{I}$. *Then, for all* $n \in \mathbb{Z}^+$,

$$\mathbf{V}^n = t_n(x)\mathbf{V} - t_{n-1}(x)\mathbf{I},$$

where $t_n(x)$ *is the nth Vieta-Pell polynomial and* $\mathbf{I}$ *is a unit matrix.*

Proof The proof is obvious from induction. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the research was realized in collaboration with the same responsibility and contributions. Both authors read and approved the final manuscript.

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