On Delannoy numbers and Schröder numbers

Zhi-Wei Sun

Department of Mathematics, Nanjing University, Nanjing 210093, People’s Republic of China

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The nth Delannoy number and the nth Schröder number given by

\[ D_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}, \quad \text{and} \quad S_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{1}{k+1} \]

respectively arise naturally from enumerative combinatorics. Let \( p \) be an odd prime. We mainly show that

\[ \sum_{k=1}^{p-1} \frac{D_k}{k^2} \equiv 2 \left( \frac{-1}{p} \right) E_{p-3} \quad (\text{mod } p) \]

and

\[ \sum_{k=1}^{p-1} \frac{S_k}{m^k} \equiv \frac{m^2 - 6m + 1}{2m} \left( 1 - \left( \frac{m^2 - 6m + 1}{p} \right) \right) \quad (\text{mod } p), \]

where \((-\cdot)\) is the Legendre symbol, \( E_0, E_1, E_2, \ldots \) are Euler numbers, and \( m \) is any integer not divisible by \( p \). We also conjecture that

\[ \sum_{k=1}^{p-1} \frac{D_k^2}{k^2} \equiv -2q_p(2)^2 \quad (\text{mod } p) \]

where \( q_p(2) \) denotes the Fermat quotient \( (2^{p-1} - 1)/p \).

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1. Introduction

For \( n \in \mathbb{N} = \{0, 1, 2, \ldots \} \), the (central) Delannoy number \( D_n \) denotes the number of lattice paths from the point \((0,0)\) to \((n,n)\) with steps \((1,0)\), \((0,1)\) and \((1,1)\), while the Schröder number \( S_n \) represents the number of such paths that never rise above the line \( y = x \). It is known that

\[
D_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} = \sum_{k=0}^{n} \binom{n+k}{2k} (2k)
\]

and

\[
S_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{1}{k+1} = \sum_{k=0}^{n} \binom{n+k}{2k} C_k,
\]

where \( C_k \) stands for the Catalan number \( \frac{(2k)!}{k!(k+1)!} \). For information on \( D_n \) and \( S_n \), the reader may consult [CHV,S], and p. 178 and p. 185 of [St].

Despite their combinatorial backgrounds, surprisingly Delannoy numbers and Schröder numbers have some nice number-theoretic properties. As usual, for an odd prime \( p \) we let \( \left( \frac{\cdot}{p} \right) \) denote the Legendre symbol. Recall that Euler numbers \( E_0, E_1, \ldots \) are integers defined by \( E_0 = 1 \) and the recursion:

\[
\sum_{k=0}^{n} \binom{n}{k} E_{n-k} = 0 \quad \text{for } n = 1, 2, 3, \ldots .
\]

Our first theorem is concerned with Delannoy numbers and their generalization.

**Theorem 1.1.** Let \( p \) be an odd prime. Then

\[
\sum_{k=1}^{p-1} \frac{D_k}{k^2} \equiv 2 \left( \frac{-1}{p} \right) E_{p-3} \pmod{p}
\]  

and

\[
\sum_{k=1}^{p-1} \frac{D_k}{k} \equiv -q_p(2) \pmod{p},
\]

where \( q_p(2) \) denotes the Fermat quotient \( (2^{p-1} - 1)/p \). If we set

\[
D_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^k \quad (n \in \mathbb{N}),
\]

then for any \( p \)-adic integer \( x \) we have

\[
\sum_{k=1}^{p-1} \frac{D_k(x)}{k} \equiv \frac{(-1 + \sqrt{-x})^p + (-1 - \sqrt{-x})^p + 2}{p} \pmod{p}.
\]
Corollary 1.1. Let $p$ be an odd prime. We have

$$
\sum_{k=1}^{p-1} \frac{D_k(3)}{k} \equiv -2q_p(2) \pmod{p} \quad \text{provided} \quad p \neq 3,
$$

(1.4)

$$
\sum_{k=1}^{p-1} \frac{D_k(-4)}{k} \equiv \frac{3 - 3^p}{p} \pmod{p},
$$

(1.5)

$$
\sum_{k=1}^{p-1} \frac{D_k(-9)}{k} \equiv -6q_p(2) \pmod{p},
$$

(1.6)

and also

$$
\sum_{k=1}^{p-1} \frac{D_k(-2)}{k} \equiv -4 P_{p-(\frac{2}{p})} \pmod{p},
$$

(1.7)

where the Pell sequence $\{P_n\}_{n \geq 0}$ is given by

$$
P_0 = 0, \quad P_1 = 1, \quad \text{and} \quad P_{n+1} = 2P_n + P_{n-1} \quad (n = 1, 2, 3, \ldots).
$$

If $p \neq 5$, then

$$
\sum_{k=1}^{p-1} \frac{D_k(-5)}{k} \equiv -2q_p(2) - \frac{5}{p} F_{p-(\frac{5}{p})} \pmod{p},
$$

(1.8)

where the Fibonacci sequence $\{F_n\}_{n \geq 0}$ is defined by

$$
F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad (n = 1, 2, 3, \ldots).
$$

Now we propose two conjectures which seem challenging in the author's opinion.

Conjecture 1.1. Let $p > 3$ be a prime. We have

$$
\sum_{k=1}^{p-1} \frac{D_k^2}{k^2} \equiv -2q_p(2)^2 \pmod{p},
$$

(1.9)

$$
\sum_{k=1}^{p-1} \frac{D_k}{k} \equiv -q_p(2) + pq_p(2)^2 \pmod{p^2},
$$

(1.10)

$$
\sum_{k=1}^{p-1} D_k S_k \equiv -2p \sum_{k=1}^{p-1} \frac{(-1)^k + 3}{k} \pmod{p^4},
$$

and

$$
\sum_{k=1}^{(p-1)/2} D_k S_k \equiv \begin{cases} 
4x^2 \pmod{p} & \text{if } p \equiv 1 \pmod{4} \text{ and } p = x^2 + y^2 \ (2 \mid x, \ 2 \mid y), \\
0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
$$
Also, \( \sum_{n=1}^{p-1} s_n^2/n \equiv -6 \pmod{p} \), where

\[
s_n := \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k+1} = D_n - S_n.
\]

**Remark 1.1.** Let \( p \) be an odd prime. Though there are many congruences for \( q_p(2) \pmod{p} \), (1.9) is curious since its left-hand side is a sum of squares. It is known that \( \sum_{k=1}^{p-1} 1/k \equiv -p^2B_{p-3}/3 \pmod{p^3} \) if \( p > 3 \), where \( B_0, B_1, B_2, \ldots \) are Bernoulli numbers. If \( p > 3 \). In addition, we can prove that \( \sum_{k=1}^{p-1} D_k \equiv (-1/p) - p^2E_{p-3} \pmod{p^3} \) and \( \sum_{k=0}^{p-1} D_k^2 \equiv (\frac{2}{p}) \pmod{p} \).

**Conjecture 1.2.** Let \( p > 3 \) be a prime. Then

\[
\sum_{k=0}^{p-1} (-1)^k D_k(2)^3 \equiv \sum_{k=0}^{p-1} (-1)^k D_k \left( -\frac{1}{4} \right)^3 \equiv \left( -\frac{2}{p} \right) \sum_{k=0}^{p-1} (-1)^k D_k \left( \frac{1}{8} \right)^3
\]

\[
\equiv \begin{cases} 
(-1/p)(4x^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\
0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}.
\end{cases}
\]

Also,

\[
\left( -\frac{1}{p} \right) \sum_{k=0}^{p-1} (-1)^k D_k \left( \frac{1}{2} \right)^3 \equiv \begin{cases} 
4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ and } p = x^2 + 6y^2 \ (x, y \in \mathbb{Z}), \\
8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ and } p = 2x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\
0 \pmod{p^2} & \text{if } (-6/p) = -1.
\end{cases}
\]

And

\[
\sum_{k=0}^{p-1} (-1)^k D_k(-4)^3 \equiv \left( -\frac{5}{p} \right) \sum_{k=0}^{p-1} (-1)^k D_k \left( -\frac{1}{16} \right)^3
\]

\[
\equiv \begin{cases} 
4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ and } p = x^2 + 15y^2 \ (x, y \in \mathbb{Z}), \\
12x^2 - 2p \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ and } p = 3x^2 + 5y^2 \ (x, y \in \mathbb{Z}), \\
0 \pmod{p^2} & \text{if } (-15/p) = -1.
\end{cases}
\]

**Remark 1.2.** Note that \((-1)^n D_n(x) = D_n(-x - 1)\) for any \( n \in \mathbb{N} \), since

\[
D_n(-x - 1) = \sum_{k=0}^{n} \binom{n}{k} \binom{-n-1}{k} \sum_{j=0}^{n} \binom{k}{j} x^j
\]

\[
= \sum_{j=0}^{n} \binom{n}{j} x^j \sum_{k=0}^{n} \binom{-n-1}{k} \binom{n-j}{n-k}
\]

\[
= \sum_{j=0}^{n} \binom{n}{j} x^j (-j - 1) = (-1)^n D_n(x).
\]
Concerning Schröder numbers we establish the following result.

**Theorem 1.2.** Let $p$ be an odd prime and let $m$ be an integer not divisible by $p$. Then

\[
\sum_{k=1}^{p-1} \frac{S_k}{m^k} \equiv \frac{m^2 - 6m + 1}{2m} \left( 1 - \left( \frac{m^2 - 6m + 1}{p} \right) \right) \pmod{p}.
\]  

(1.11)

**Example 1.1.** Theorem 1.2 in the case $m = 6$ gives that

\[
\sum_{k=1}^{p-1} \frac{S_k}{6^k} \equiv 0 \pmod{p}
\]

for any prime $p > 3$.

(1.12)

For technical reasons, we will prove Theorem 1.2 in the next section and show Theorem 1.1 and Corollary 1.1 in Section 3.

2. Proof of Theorem 1.2

**Lemma 2.1.** Let $p$ be an odd prime and let $m$ be any integer not divisible by $p$. Then

\[
\sum_{k=1}^{p-1} \frac{C_k}{m^k} \equiv \frac{m-4}{2} \left( 1 - \left( \frac{m(m-4)}{p} \right) \right) \pmod{p}.
\]  

(2.1)

**Proof.** This follows from [Su10, Theorem 1.1] in which the author even determined \( \sum_{k=1}^{p-1} \frac{C_k}{m^k} \pmod{p^2} \). However, we will give here a simple proof of (2.1).

For each $k = 1, \ldots, p-1$, we clearly have

\[
\binom{(p-1)/2}{k} \equiv \binom{-1/2}{k} = \frac{(2k)}{(-4)^k} \pmod{p}.
\]

Note also that

\[
C_{p-1} = \frac{1}{2(p-1)} \prod_{k=1}^{p-1} \frac{p+k}{k} \equiv -1 \pmod{p}.
\]

Therefore

\[
\sum_{k=1}^{p-1} \frac{C_k}{m^k} \equiv \sum_{0 \leq k < p-1} \binom{(p-1)/2}{k} \frac{1}{k+1} \left( -\frac{4}{m} \right)^k + \frac{C_{p-1}}{m^{p-1}}
\]

\[
\equiv -\frac{m}{4} \times \frac{2}{p+1} \sum_{k=1}^{(p-1)/2} \binom{(p+1)/2}{k+1} \left( -\frac{4}{m} \right)^{k+1} - 1
\]

\[
\equiv -\frac{m}{2} \left( \left( 1 - \frac{4}{m} \right)^{(p+1)/2} - 1 - \frac{p+1}{2} \left( -\frac{4}{m} \right) \right) - 1
\]
\[ \equiv - \frac{m}{2} \left( \frac{m-4}{m} \times \frac{(m(m-4))^{(p-1)/2}}{m^{p-1}} - 1 + \frac{2}{m} \right) - 1 \]

\[ \equiv - \frac{m-4}{2} \left( \frac{m(m-4)}{p} \right) + \frac{m}{2} - 2 \pmod{p} \]

and hence (2.1) follows. \( \square \)

**Lemma 2.2.** For any odd prime \( p \) we have

\[ \sum_{k=1}^{p-1} S_k \equiv 2 \left( \frac{-1}{p} \right) - 2^p \pmod{p^2}. \] (2.2)

**Proof.** Recall the known identity (cf. (5.26) of [GKP, p. 169])

\[ \sum_{n=0}^{m} \binom{n}{k} = \binom{m+1}{k+1} \quad (k, m \in \mathbb{N}). \]

Then

\[ \sum_{n=0}^{p-1} S_n = \sum_{n=0}^{p-1} \sum_{k=0}^{n} \binom{n+k}{2k} C_k = \sum_{k=0}^{p-1} C_k \sum_{n=k}^{p-1} \binom{n+k}{2k} \]

\[ = \sum_{k=0}^{p-1} C_k \binom{p+k}{2k+1} = \sum_{k=0}^{p-1} \frac{p}{k!(k+1)!(2k+1)} \prod_{0<j\leq k} (p^2 - j^2) \]

\[ \equiv \sum_{k=0}^{p-1} \frac{p(-1)^k(k!)^2}{k!(k+1)!(2k+1)} = p \sum_{k=0}^{p-1} (-1)^k \left( \frac{2}{2k+1} - \frac{1}{k+1} \right) \pmod{p^2}. \]

Observe that

\[ 2p \sum_{k=0}^{p-1} \frac{(-1)^k}{2k+1} = p \sum_{k=0}^{p-1} \left( \frac{(-1)^k}{2k+1} + \frac{(-1)^{p-1-k}}{2(p-1-k)+1} \right) \]

\[ = p \sum_{k=0}^{p-1} (-1)^k \left( \frac{1}{2k+1} + \frac{1}{2p - (2k+1)} \right) \]

\[ \equiv p(-1)^{(p-1)/2} \left( \frac{1}{p} + \frac{1}{2p - p} \right) = 2 \left( \frac{-1}{p} \right) \pmod{p^2}. \]

Also,

\[ -p \sum_{k=0}^{p-1} \frac{(-1)^k}{k+1} = p \sum_{k=1}^{p} \frac{(-1)^k}{k} \]

\[ \equiv - \sum_{k=1}^{p-1} \frac{p}{k} \left( \frac{p-1}{k-1} \right) - 1 = - \sum_{k=0}^{p-1} \frac{p}{k} = 1 - 2^p \pmod{p^2}. \]
Combining the above, we obtain

\[ \sum_{n=0}^{p-1} S_n \equiv 2 \left( -\frac{1}{p} \right) + 1 - 2^p \pmod{p^2} \]

and hence (2.2) holds. \( \square \)

**Proof of Theorem 1.2.** In the case \( m \equiv 1 \pmod{p} \), (1.11) reduces to the congruence

\[ \sum_{k=1}^{p-1} S_k \equiv -2 \left( 1 - \left( -\frac{1}{p} \right) \right) \pmod{p} \]

which follows from (2.2) in view of Fermat’s little theorem.

Below we assume that \( m \not\equiv 1 \pmod{p} \). Then

\[ p - 1 \sum_{n=1}^{m^n} \equiv p - 1 \sum_{n=1}^{m^{p-1-n}} = \frac{m^{p-1} - 1}{m - 1} \equiv 0 \pmod{p} \]

and hence

\[ \sum_{n=1}^{p-1} S_n \equiv \sum_{n=1}^{p-1} S_n - 1 = \sum_{n=1}^{p-1} \sum_{k=1}^{n} \frac{(n+k)}{2k} C_k \equiv \sum_{k=1}^{p-1} \sum_{n=k}^{p-1} \frac{(n+k)}{2k} \frac{m^n}{m^{n-k}} \pmod{p}. \]

Given \( k \in \{1, \ldots, p-1\} \), we have

\[ \sum_{n=k}^{p-1} \frac{(n+k)}{2k} \frac{m^n}{m^{n-k}} = \sum_{r=0}^{p-1-k} \frac{(2k+r)}{m^r} = \sum_{r=0}^{p-1-k} \frac{(-2k-1)}{(-m)^r} \equiv \sum_{r=0}^{p-1-k} \frac{(p-1-2k)}{(-m)^r} \pmod{p}. \]

If \( (p - 1)/2 < k < p - 1 \), then

\[ C_k = \frac{(2k)!}{k! (k+1)!} \equiv 0 \pmod{p}. \]

Therefore

\[ \sum_{n=1}^{p-1} S_n \equiv \sum_{k=1}^{(p-1)/2} C_k \frac{1 - 1}{m} \pmod{p^{p-1}} \]

\[ \equiv \sum_{k=1}^{p-1} C_k \frac{m}{m-1} \pmod{p} \]

where \( m_0 \) is an integer with \( m_0 \equiv (m - 1)^2/m \pmod{p} \). By Lemma 2.1,
\[
\sum_{k=1}^{p-1} \frac{C_k}{m_0^{k}} \equiv \frac{m_0 - 4}{2} \left( 1 - \left( \frac{m_0(m_0 - 4)}{p} \right) \right) \\
= \frac{mm_0 - 4m}{2} \left( 1 - \left( \frac{mm_0(mm_0 - 4m)}{p} \right) \right) \\
= \frac{(m - 1)^2 - 4m}{2} \left( 1 - \left( \frac{(m - 1)^2 - 4m}{p} \right) \right) \pmod{p}.
\]

So (1.11) follows. We are done. \(\Box\)

3. Proofs of Theorem 1.1 and Corollary 1.1

We need some combinatorial identities.

**Lemma 3.1.** For any \(n \in \mathbb{N}\), we have

\[
\sum_{r=0}^{2n} \frac{(-1)^r(2n)}{r} = \frac{(-16)^n}{(2n + 1)(2n)}
\]

(3.1)

and

\[
\sum_{r=0}^{2n} \frac{(-1)^r(2n)}{(2n + 1 - 2r)^2} = \frac{(-16)^n}{(2n + 1)^2(2n)}.
\]

(3.2)

that is,

\[
\sum_{k=-n}^{n} \frac{(-1)^k(2n)}{(2k + 1)^2(n - k)} = \frac{16^n}{(2n + 1)^2(2n)} \quad \text{for } s = 1, 2.
\]

(3.3)

**Proof.** If we denote by \(a_n\) the left-hand side of (3.1), then the well-known Zeilberger algorithm (cf. [PWZ]) yields the recursion

\[
a_{n+1} = -\frac{8(n + 1)}{2n + 3} a_n \quad (n = 0, 1, 2, \ldots).
\]

So (3.1) can be easily proved by induction. (3.2) is equivalent to [Su11, (2.5)] which was shown by a similar method. Clearly (3.3) is just a combination of (3.1) and (3.2). We are done. \(\Box\)

**Proof of Theorem 1.1.** Let \(s \in \{1, 2\}\) and let \(x\) be any \(p\)-adic integer. We claim that

\[
\delta_{s,2} \delta_{p,3} + \sum_{n=1}^{p-1} \frac{D_n(x)}{n^s} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-x)^k}{k^s} \pmod{p}.
\]

(3.4)

Clearly,

\[
\sum_{n=1}^{p-1} \frac{D_n(x)}{n^s} = \sum_{n=1}^{p-1} \sum_{k=1}^{n} \left( \frac{n+k}{2k} \right) x^k \left( \frac{n+k}{2k} \right) \sum_{n=k}^{p-1} \left( \frac{2k}{k} \right) x^k \sum_{n=k}^{p-1} \left( \frac{n+k}{2k} \right) x^k.
\]
Note that \( \sum_{n=1}^{p-1} 1/n^s \equiv -\delta_{s,2} \delta_{p,3} \mod p \) since
\[
\sum_{k=1}^{p-1} \frac{1}{(2k)^s} \equiv \sum_{n=1}^{p-1} \frac{1}{n^s} \mod p.
\]
As \( p \mid (2k) \) for \( k = (p+1)/2, \ldots, p-1 \), and
\[
\sum_{n=k}^{p-1} \frac{(n+k)2k}{n^s} = \sum_{r=0}^{p-1-k} \frac{(2k+r)^s}{(k+r)^s} \equiv (-2)^s \sum_{r=0}^{p-1-k} \frac{(-1)^r (p-1-2k)^s}{(p-2k-2r)^s} \mod p
\]
for \( k = 1, \ldots, (p-1)/2 \), by applying Lemma 3.1 we obtain from the above that
\[
\delta_{s,2} \delta_{p,3} + \sum_{n=1}^{p-1} \frac{D_n(x)}{n^s} \equiv (-2)^s \sum_{k=1}^{(p-1)/2} \frac{2k}{k^s} \sum_{r=0}^{p-1-k} \frac{(-16)(p-1)/2-k}{(p-2k)^s(p-1)/2-k} \equiv \sum_{k=1}^{(p-1)/2} \frac{2k}{k^s} \sum_{r=0}^{k} \frac{(-1/2)^r}{(p-1/2-k)^r} \equiv \sum_{k=1}^{(p-1)/2} \frac{2k}{k^s} \frac{(-1/2)^k}{k^k} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-x)^k}{k^k} \mod p.
\]
In the case \( s = 2 \) and \( x = 1 \), (3.4) yields the congruence
\[
\delta_{p,3} + \sum_{n=1}^{p-1} \frac{D_n(x)}{n^2} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^2} \mod p.
\]
By Lehmer [L, (20)],
\[
\sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \equiv \delta_{p,3} + \left( \frac{-1}{p} \right) E_{p-3} \mod p
\]
and hence
\[
\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^2} = 2 \sum_{k=1}^{(p-1)/2} \frac{1}{k^2} - \sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \equiv \delta_{p,3} + 2 \left( \frac{-1}{p} \right) E_{p-3} \mod p
\]
since \( \sum_{k=1}^{(p-1)/2} (1/k^2 + 1/(p-k)^2) = \sum_{k=1}^{p-1} 1/k^2 \equiv 0 \mod p \) if \( p > 3 \). So (1.1) follows.
With the help of (3.4) in the case \( s = x = 1 \), we have
\[
\sum_{n=1}^{p-1} \frac{D_n}{n} = \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k} \equiv \frac{1}{2} \sum_{k=1}^{(p-1)/2} \left( \frac{(-1)^k}{k} + \frac{(-1)^{p-k}}{p-k} \right) \\
\equiv -\frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{k} (p-1) = -\frac{1}{2p} \sum_{k=1}^{p-1} \binom{p}{k} = -q_p(2) \pmod{p}.
\]
This proves (1.2).

Now fix a \( p \)-adic integer \( x \). Observe that
\[
p \sum_{k=1}^{(p-1)/2} \frac{(-x)^k}{k} \equiv -2 \sum_{k=1}^{(p-1)/2} \frac{p}{2k} \binom{p-1}{2k-1} (-x)^k \\
= \sum_{j=1}^{p} \binom{p}{j} (-1)^{p-j} ((\sqrt{-x})^j + (\sqrt{-x})^j) \\
= (-1 + \sqrt{-x})^p + (-1 - \sqrt{-x})^p + 2 \pmod{p^2}.
\]
Combining this with (3.4) in the case \( s = 1 \) we immediately get (1.3).

The proof of Theorem 1.1 is now complete. \( \square \)

**Remark 3.1.** By modifying our proof of (1.2) and using the new identity \( \sum_{r=0}^{2n} \binom{2n}{r} / (2n+1 - 2r) = 2^{2n} / (2n+1) \), we can prove the congruence \( \sum_{k=1}^{p-1} (-1)^k S_k/k \equiv 4((\frac{2}{p}) - 1) \pmod{p} \) for any odd prime \( p \). Combining this with \( \sum_{k=1}^{p-1} (-1)^k D_k/k \equiv -4P_{p-\left(\frac{2}{p}\right)/p} \pmod{p} \) (an equivalent form of (1.7)) we obtain that \( \sum_{n=1}^{p-1} (-1)^k S_k/k \equiv 4(1 - (\frac{2}{p}) - P_{p-\left(\frac{2}{p}\right)/p} \pmod{p} \).

**Proof of Corollary 1.1.** Note that \( \omega = (-1 + \sqrt{-3})/2 \) is a primitive cubic root of unity. If \( p \neq 3 \), then
\[
(-1 + \sqrt{-3})^p + (-1 - \sqrt{-3})^p = (2\omega)^p + (2\omega^2)^p = -2^p
\]
and hence (1.3) with \( x = 3 \) yields the congruence in (1.4).

Clearly (1.5) follows from (1.3) with \( x = -4 \).

Since \( 2^p - 4^p + 2 = (2 - 2^p)(2^p + 1) \equiv 6(1 - 2^{p-1}) \pmod{p^2} \), (1.3) in the case \( x = -9 \) yields (1.6). The companion sequence \( \{Q_n\}_{n \geq 0} \) of the Pell sequence is defined by \( Q_0 = Q_1 = 2 \) and \( Q_{n+1} = 2Q_n + Q_{n-1} \quad (n = 1, 2, 3, \ldots) \). It is well known that
\[
Q_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n \quad \text{for all } n \in \mathbb{N}.
\]
(1.3) with \( x = -2 \) yields the congruence
\[
\sum_{k=1}^{p-1} \frac{D_k(-2)}{k} \equiv 2 - Q_p \pmod{p}.
\]
Since \( Q_p - 2 \equiv 4P_{p-\left(\frac{2}{p}\right)/p} \pmod{p^2} \) by the proof of [ST, Corollary 1.3], (1.7) follows immediately.
Recall that the Lucas sequence \( \{L_n\}_{n \geq 0} \) is given by

\[
L_0 = 2, \quad L_1 = 1, \quad \text{and} \quad L_{n+1} = L_n + L_{n-1} \quad (n = 1, 2, 3, \ldots).
\]

It is well known that

\[
L_n = \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{1 - \sqrt{5}}{2} \right)^n \quad \text{for all } n \in \mathbb{N}.
\]

Putting \( x = -5 \) in (1.3) we get

\[
\frac{2 - 2p L_p}{p} = \frac{2^p (1 - L_p) + 2 - 2^p}{p} \equiv -\frac{2}{p} \left( L_p - 1 \right) - 2q_p (2) \pmod{p}.
\]

It is known that

\[
2(2p - 1) \equiv 5F_{p - \left( \frac{5}{2} \right)} \pmod{p^2}
\]

provided \( p \neq 5 \) (see the proof of [ST, Corollary 1.3]). So (1.8) holds if \( p \neq 5 \). We are done. \( \square \)

References


