Orthogonal polynomials and the moment problem

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Master of Science.

Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

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Abstract

The classical moment problem concerns distribution functions on the real line. The central feature is the connection between distribution functions and the moment sequences which they generate via a Stieltjes integral. The solution of the classical moment problem leads to the well known theorem of Favard which connects orthogonal polynomial sequences with distribution functions on the real line. Orthogonal polynomials in their turn arise in the computation of measures via continued fractions and the Nevanlinna parametrisation. In this dissertation classical orthogonal polynomials are investigated first and their connection with hypergeometric series is exhibited. Results from the moment problem allow the study of a more general class of orthogonal polynomials. \(q\)-Hypergeometric series are presented in analogy with the ordinary hypergeometric series and some results on \(q\)-Laguerre polynomials are given. Finally recent research will be discussed.
Acknowledgements

I acknowledge the support of the National Research Foundation of South Africa in funding my MSc.
I am grateful for the help and support of Dr Sarah Jane Johnston while researching the topic and compiling my dissertation.
Lastly I thank my family for their companionship.
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