The Moment Problem

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1 Motivation

2 What the moment problem is?

3 Existence and uniqueness of the solution - operator approach

4 Jacobi matrix and Orthogonal Polynomials

5 Sufficient conditions for determinacy

6 The set of solutions of indeterminate moment problem
Chebychev’s question: *If for some positive function* $f$, 

\[
\int_{\mathbb{R}} x^n f(x) dx = \int_{\mathbb{R}} x^n e^{-x^2} dx, \quad n = 0, 1, \ldots
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*can we then conclude that* $f(x) = e^{-x^2}$?
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That is: *Is the normal density uniquely determined by its moment sequence?*
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- Chebychev’s question: If for some positive function $f$, 
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  can we then conclude that $f(x) = e^{-x^2}$?

- That is: Is the normal density uniquely determined by its moment sequence?

- Answer: yes in the sense that $f(x) = e^{-x^2}$ a.e. wrt Lebesque measure on $\mathbb{R}$. 
  
  What happens if one replaces the normal density by something else? 
  The general answer to the Chebychev’s question is no. Suppose, e.g., $X \sim N(0, \sigma^2)$ and consider densities of exp($X$) (lognormal distribution) or sinh($X$) then we lost the uniqueness.

  A tough problem: What can be said when there is no longer uniqueness?
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\textit{can we then conclude that }$f(x) = e^{-x^2}$\textit{?}

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Answer: \textit{yes} in the sense that $f(x) = e^{-x^2}$ \textit{a.e. wrt Lebesque measure on }$\mathbb{R}$.

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A tough problem: *What can be said when there is no longer uniqueness?*
Let $I \subset \mathbb{R}$ be an open interval. For a positive measure $\mu$ on $I$ the $n$th moment is defined as

$$\int_I x^n d\mu(x), \quad \text{(provided the integral exists)}.$$ 

Suppose a real sequence $\{s_n\}_{n \geq 0}$ is given. The moment problem on $I$ consists of solving the following three problems:

1. Does there exist a positive measure on $I$ with moments $\{s_n\}_{n \geq 0}$? If so,
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1. Does there exist a positive measure on \( I \) with moments \( \{s_n\}_{n \geq 0} \)?
   If so,

2. is this positive measure uniquely determined by moments \( \{s_n\}_{n \geq 0} \)? (\textit{determinate case})
   If this is not the case,
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   If this is not the case,
3. how one can describe all positive measures on $I$ with moments $\{s_n\}_{n \geq 0}$? (indeterminate case)
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- uniqueness $\sim$ determinate case vs. non-uniqueness $\sim$ indeterminate case
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One can restrict oneself to cases:

- \( I = \mathbb{R} \) - Hamburger moment problem \( (\mathcal{M}_H = \text{set of solutions}) \)
- \( I = [0, +\infty) \) - Stieltjes moment problem \( (\mathcal{M}_S = \text{set of solutions}) \)
- \( I = [0, 1] \) - Hausdorff moment problem

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The moment problem has a solution on \([0, 1]\) iff sequence \(\{s_n\}_{n \geq 0}\) is completely monotonic, i.e.,

\[ (-1)^k (\Delta^k s)_n \geq 0 \]

for all \(k, n \in \mathbb{Z}_+\), where \((\Delta s)_n = s_{n+1} - s_n\).
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- measure with finite support is uniquely determined by its moments (Vandermonde matrix),
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Consequently, we will further discuss the Stieltjes and Hamburger moment problem only.
Existence of the solution

For \( \{s_n\}_{n \geq 0} \), we denote \( H_N(s) \) the \( N \times N \) Hankel matrix with entries \( (H_N(s))_{ij} := s_{i+j}, \quad i, j \in \{0, 1, \ldots, N - 1\} \).
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Define two sesquilinear forms \( H_N \) and \( S_N \) on \( \mathbb{C}^N \) by

\[
H_N(x, y) := \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \overline{x_i} y_j s_{i+j} \quad \text{and} \quad S_N(x, y) := \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \overline{x_i} y_j s_{i+j+1}.
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- Hence \( H_N(x, y) = (x, H_N(s)y) \) and \( S_N(x, y) = (x, H_N(Ts)y) \) ((..,) Euclidean inner product).
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Let \( \mu \in \mathcal{M}_H \) or \( \mu \in \mathcal{M}_S \) with infinite support. By observing that

\[
H_N(y, y) = \int \left| \sum_{i=0}^{N-1} y_i x^i \right|^2 d\mu(x) \quad \text{and} \quad S_N(y, y) = \int x \left| \sum_{i=0}^{N-1} y_i x^i \right|^2 d\mu(x),
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one immediately gets the following.
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one immediately gets the following.

**Necessary condition for the existence**

A necessary condition for the Hamburger moment problem to have a solution (with infinite support) is the sesquilinear form \( H_N \) is PD for all \( N \in \mathbb{Z}_+ \). A necessary condition for the Stieltjes moment problem to have a solution (with infinite support) is both sesquilinear forms \( H_N \) and \( S_N \) are PD for all \( N \in \mathbb{Z}_+ \).
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- Let $H_N$ be PD for all $N \in \mathbb{N}$. 
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- For $P, Q \in \mathbb{C}[x],$

\[
P(x) = \sum_{k=0}^{N-1} a_k x^k, \quad \text{and} \quad Q(x) = \sum_{k=0}^{N-1} b_k x^k,
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define positive definite inner product

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\langle P, Q \rangle := H_N(a, b).
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  $A$ is a symmetric operator.
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- Especially,
  \[
  \langle 1, A^n 1 \rangle = s_n, \quad n \in \mathbb{N}.
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- A has a self-adjoint extension since it commutes with a complex conjugation operator $C$ on $\mathbb{C}[x]$ (von Neumann).
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- If each $S_N$ is PD, then

$$\langle P, A[P] \rangle = S_N(a, a) \geq 0, \quad \text{for all } P \in \mathbb{C}[x],$$

and it follows $A$ has a non-negative self-adjoint extension $A_F$, the Friedrichs extension.
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Let $A'$ be a self-adjoint extension of $A$. By the spectral theorem there is a projection valued spectral measure $E_{A'}$ and positive measure
\[ \mu(.) = \langle 1, E_{A'}(.)1 \rangle. \]
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- Especially, for $f(x) = x^n$, one finds

\[
s_n = \langle 1, A^n1 \rangle = \langle 1, (A')^n1 \rangle = \int_{\mathbb{R}} x^n d\mu(x),
\]

since $\text{Dom}(A^n) \subset \text{Dom}((A')^n)$.
Existence of the solution

- We see a self-adjoint extension of $A$ yields a solution of the Hamburger moment problem.
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- Moreover, a non-negative self-adjoint extension has $\text{supp}(\mu) \subset [0, \infty)$ and so yields a solution of the Stieltjes moment problem.
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**Theorem (Existence)**

i) A necessary and sufficient condition for $\mathcal{M}_H \neq \emptyset$ (with infinite support) is

$$\det H_N(s) > 0 \quad \text{for all } N \in \mathbb{N}. $$

ii) A necessary and sufficient condition for $\mathcal{M}_S \neq \emptyset$ (with infinite support) is

$$\det H_N(s) > 0 \land \det S_N(s) > 0 \quad \text{for all } N \in \mathbb{N}. $$
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- Historically, this result has not been obtained by using the spectral theorem that was invented later.
In view of the connection of the moment problem and self-adjoint extensions, the following result is reasonable.

Theorem (Uniqueness)

i) A necessary and sufficient condition for the Hamburger moment problem to be determinate is that the operator $A$ is essentially self-adjoint (i.e., it has a unique self-adjoint extension).

ii) A necessary and sufficient condition for the Stieltjes moment problem to be determinate is that the operator $A$ has a unique non-negative self-adjoint extension.

It is not easy to prove the theorem. In one direction, it is not clear that distinct self-adjoint extensions $A_1'$ and $A_2'$ give rise to distinct measures $\mu_1$ and $\mu_2$.

The other direction is even less clear. For not only is it not obvious, it is false that every solution of the moment problem arises from some measure given by spectral measure of some self-adjoint extension.

A solution of the moment problem which comes from a self-adjoint extension of $A$ is called $N$-extremal solution (von Neumann [Simon], extremal [Shohat-Tamarkin]).
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In one direction, it is not clear that distinct self-adjoint extensions $A'_1$ and $A'_2$ give rise to distinct measures $\mu_1$ and $\mu_2$. 
In view of the connection of the moment problem and self-adjoint extensions, the following result is reasonable.

**Theorem (Uniqueness)**

i) A necessary and sufficient condition for the Hamburger moment problem to be determinate is that the operator $A$ is essentially self-adjoint (i.e., it has a unique self-adjoint extension).

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The other direction is even less clear. For not only is it not obvious, it is **false** that every solution of the moment problem arise from some measure given by spectral measure of some self-adjoint extension.
In view of the connection of the moment problem and self-adjoint extensions, the following result is reasonable.

**Theorem (Uniqueness)**

i) A necessary and sufficient condition for the Hamburger moment problem to be determinate is that the operator $A$ is essentially self-adjoint (i.e., it has a unique self-adjoint extension).

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It is not easy to prove the theorem.

- In one direction, it is not clear that distinct self-adjoint extensions $A'_1$ and $A'_2$ give rise to distinct measures $\mu_1$ and $\mu_2$.
- The other direction is even less clear. For not only is it not obvious, it is **false** that every solution of the moment problem arise from some measure given by spectral measure of some self-adjoint extension.
- A solution of the moment problem which comes from a self-adjoint extension of $A$ is called *N-extremal* solution (von Neumann [Simon], extremal [Shohat-Tamarkin]).
Consider set \( \{1, x, x^2, \ldots \} \subset \mathcal{H}^{(s)} \) which is linearly independent \((H_N \text{ PD})\) and span \( \mathcal{H}^{(s)} \).
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By applying the Gramm-Schmidt procedure, we obtain an orthonormal basis \( \{P_n\}_{n=0}^{\infty} \) for \( \mathcal{H}(s) \).
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By construction, \( P_n \) is a polynomial of degree \( n \) with real coefficients and

\[
\langle P_m, P_n \rangle = \delta_{mn}
\]

for all \( m, n \in \mathbb{Z}_+ \). These are well-known *Orthogonal Polynomials*.
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\( \{P_n\}_{n=0}^{\infty} \) are determined by moment sequence \( \{s_n\}_{s=0}^{\infty} \),

\[
P_n(x) = \frac{1}{\sqrt{\det[H_{n+1}(s)H_n(s)]}} \begin{vmatrix}
  s_0 & s_1 & \ldots & s_n \\
  s_1 & s_2 & \ldots & s_{n+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  s_{n-1} & s_n & \ldots & s_{2n-1} \\
  1 & x & \ldots & x^n 
\end{vmatrix}
\]
Since \( \text{span}(1, x, \ldots, x^n) = \text{span}(P_0, P_1, \ldots, P_n) \), \( xP_n(x) \) has an expansion in \( P_0, P_1, \ldots, P_{n+1} \).
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Moreover, if \( 0 \leq j < n - 1 \), one has

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There are sequences $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$, and $\{c_n\}_{n=0}^{\infty}$ such that

$$xP_n(x) = c_nP_{n+1}(x) + b_nP_n(x) + a_{n-1}P_{n-1}(x), \quad (P_{-1}(x) := 0),$$

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for \( n \in \mathbb{Z}_+ \).

Furthermore, by the Gramm-Schmidt procedure, \( c_n > 0 \), and

\[
c_n = \langle P_{n+1}, xP_n \rangle = \langle P_n, xP_{n+1} \rangle = a_n.
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where \( a_n > 0 \) and \( b_n \in \mathbb{R} \).

Hence, \( A \) has, in the basis \( \{P_n\}_{n=0}^{\infty} \), has tridiagonal matrix representation and \( \text{Dom}(A) \) is the set of sequences of finite support.
The realization of elements of $\mathcal{H}^{(s)}$ as $\sum_{n=0}^{\infty} \lambda_n P_n$, with $\sum_{n=0}^{\infty} |\lambda_n|^2 < \infty$ gives a different realization of $\mathcal{H}^{(s)}$ as a set of sequences $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ with the usual $\ell^2(\mathbb{Z}_+)$ inner product.
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• $\mathbb{C}[x]$ corresponds to finitely supported sequences $\lambda$. 
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Thus, given a set of moments $\{s_n\}_{n=0}^{\infty}$, we can find real $\{b_n\}_{n=0}^{\infty}$ and positive $\{a_n\}_{n=0}^{\infty}$ so that the moment problem is associated to self-adjoint extensions of the Jacobi matrix,

$$A = \begin{pmatrix}
     b_0 & a_0 & a_1 & \cdots \\
     a_1 & b_1 & a_1 & \cdots \\
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     & & & \ddots
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There are explicit formulae for the $b_n$'s and $a_n$'s in terms of the determinants of the $s_n$'s.
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There are explicit formulae for the $b_n$’s and $a_n$’s in terms of the determinants of the $s_n$’s.

The set of moments $\{s_n\}_{n=0}^{\infty}$ is associated to the Jacobi matrix $A$ through identity

$$ s_n = (e_0, A^n e_0). $$
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The set of moments $\{s_n\}_{n=0}^{\infty}$ is associated to the Jacobi matrix $A$ through identity

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Consequently, we reveal following correspondences:

- moment sequence $\leftrightarrow$ Jacobi matrix
- Orthogonal Polynomials $\leftrightarrow$ three-term recurrence
It is desirable to be able to tell whether the moment problem is determinate (or indeterminate) just by looking at the moment sequence \( \{s_n\}_{n=0}^{\infty} \), or the Jacobi matrix (seq. \( \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \)), or orthogonal polynomials \( \{P_n\}_{n=1}^{\infty} \).
Sufficient conditions for determinacy - moment sequence

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**Carleman, 1922, 1926**

If

1) \( \sum_{n=1}^{\infty} \frac{1}{2^n \sqrt{|s_{2n}|}} = \infty \) \quad \text{or} \quad 2) \( \sum_{n=1}^{\infty} \frac{1}{a_n} = \infty \)

then the Hamburger moment problem is determinate.

If

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then both Hamburger and Stieltjes moment problems are determinate.
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then both Hamburger and Stieltjes moment problems are determinate.

- Hence, e.g., if \( \{a_n\}_{n=0}^{\infty} \) is bounded or there are \( R, C > 0 \) such that
  \[
  |s_n| \leq CR^n n!,
  \]
  for all \( n \) sufficiently large, we have determinate Hamburger m.p. If
  \[
  |s_n| \leq CR^n (2n)!,
  \]
  for all \( n \) sufficiently large, we have determinate Stieltjes m.p.
Let 
\[
\lim_{n \to \infty} b_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{a_n^2}{b_n b_{n+1}} = L < \frac{1}{4}.
\]
then the Hamburger moment problem is determinate if 
\[
\liminf_{n \to \infty} \sqrt[n]{b_n} < \frac{1 + \sqrt{1 - 4L}}{1 - \sqrt{1 - 4L}}
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and indeterminate if the opposite (strict) inequality holds.
Sufficient conditions for determinacy - Jacobi matrix

Chihara, 1989

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- Chihara uses totally different approach to the problem - concept of chain sequences.
Recall \( \{P_n\}_{n=0}^{\infty} \) are determined by the three-term recurrence

\[
xP_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x)
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with initial settings \( P_0(x) = 1 \) and \( P_1(x) = \frac{1}{b_0} (x - a_0) \).
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Let us denote by \( \{Q_n\}_{n=0}^{\infty} \) a polynomial sequence that solve the same recurrence as \( \{P_n\}_{n=0}^{\infty} \) with initial conditions \( Q_0(x) = 0 \) and \( Q_1(x) = \frac{1}{b_0} \).

The Hamburger moment problem is determinate if and only if

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\sum_{n=0}^{\infty} \left( P_n(0) + Q_n(0) \right) = \infty.
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These two polynomial sequences are linearly independent and any solution of the three-term recurrence is a linear combination of them.

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Actually, one can write some \( x \in \mathbb{R} \) instead of zero in the condition. It is even necessary and sufficient that there exists a \( z \in \mathbb{C} \setminus \mathbb{R} \) such that both \( \{P_n(z)\}_{n=0}^{\infty} \) and \( \{Q_n(z)\}_{n=0}^{\infty} \) do not belong to \( \ell^2(\mathbb{Z} +) \).
Sufficient conditions for determinacy - Orthogonal Polynomials

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Sometimes the natural starting point is not orthogonal polynomials of Jacobi matrix but a density $w$ with moments $\{s_n\}_{n=0}^\infty$.

Krein, 1945

Let $w$ be a density of $\mu$ (i.e., $d\mu(x) = w(x)\,dx$) where either

1) $\text{supp}(w) = \mathbb{R}$ and $\int_{-\infty}^{\infty} \ln(w(x)) \,dx > -\infty$,

2) $\text{supp}(w) = [0, \infty)$ and $\int_{0}^{\infty} \ln(w(x)) \sqrt{x}(1+x) \,dx > -\infty$.

Suppose that for all $n \in \mathbb{Z}^+$:

$\int_{\mathbb{R}} |x|^n w(x) \,dx < \infty$.

Then the moment problem (Hamburger in case (1), Stieltjes in case (2)) with moments $s_n = \int x^n w(x) \,dx$ is indeterminate.
Sufficient conditions for indeterminacy - density of measure

Sometimes the natural starting point is not orthogonal polynomials of Jacobi matrix but a density \( w \) with moments \( \{s_n\}_{n=0}^{\infty} \).

**Krein, 1945**

Let \( w \) be a density of \( \mu \) (i.e., \( d\mu(x) = w(x)dx \)) where either

1) \( \text{supp}(w) = \mathbb{R} \) and

\[
\int_{\mathbb{R}} \frac{\ln(w(x))}{1 + x^2} \, dx > -\infty,
\]

or

2) \( \text{supp}(w) = [0, \infty) \) and

\[
\int_{0}^{\infty} \frac{\ln(w(x))}{\sqrt{x}(1 + x)} \, dx > -\infty.
\]

Suppose that for all \( n \in \mathbb{Z}_+ \):

\[
\int_{\mathbb{R}} |x|^n w(x) \, dx < \infty.
\]

Then the moment problem (Hamburger in case (1), Stieltjes in case(2)) with moments

\[
s_n = \frac{\int x^n w(x) \, dx}{\int w(x) \, dx}
\]

is indeterminate.
The set of solutions of indeterminate moment problem

- The problem about describing $\mathcal{M}_H$ was solved by Nevanlinna in 1922 using complex function theory.

A function $\phi$ is called a Pick function (beware Herglotz) if it is holomorphic in $\mathbb{C}^+ := \{z \in \mathbb{C} | \text{Im}(z) > 0\}$ and $\text{Im}(\phi(z)) \geq 0$ for $z \in \mathbb{C}^+$. Denote the set of Pick functions by $P$. $P \cup \{\infty\}$ denotes the one-point compactification of $P$. ($P$ inherits the topology of holomorphic functions on $\mathbb{C} \setminus \mathbb{R}$.)

Nevanlinna, 1922

The solutions of the Hamburger moment problem in the indeterminate case are parametrized via homeomorphism $\phi \mapsto \mu_\phi$ of $P \cup \{\infty\}$ onto $\mathcal{M}_H$ given by

$$\int_{\mathbb{R}} d\mu_\phi(x) = \phi(z) - A(z)\phi(z) - C(z)B(z)\phi(z) - D(z)$$

for $z \in \mathbb{C} \setminus \mathbb{R}$, where $A, B, C, D$ are certain entire functions determined by the problem (i.e., the moment sequence, or orthogonal polynomials, ...).

$A, B, C, D$ are called Nevanlinna functions and $(A \ C \ B \ D)$ the Nevanlinna matrix.

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The solutions of the Hamburger moment problem in the indeterminate case are parametrized via homeomorphism $\phi \mapsto \mu_\phi$ of $\mathcal{P} \cup \{ \infty \}$ onto $\mathcal{M}_H$ given by

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Note first that, for $k \in \mathbb{Z}_+$,

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$$d\mu_{\vartheta}(u) = \frac{1}{\sqrt{\pi}} u^{-\ln u} [1 + \vartheta \sin(2\pi \ln u)] du,$$

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- Hence polynomials are not dense in $L^2(d\mu_\vartheta)$. This is a typical situation for solutions of indeterminate moment problems which are not N-extremal.
In some sense, to solve indeterminate Hamburger moment problem means to find the Nevanlinna functions $A, B, C, \text{and} \ D$ (in particular $B$ and $D$).

$A(z) = \sum_{k=0}^{\infty} Q_k(0) Q_k(z)$,
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They can be computed by using orthogonal polynomials,

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where sums converge locally uniformly in $\mathbb{C}$. 

More on $A, B, C, D$: $A, B, C, D$ are entire functions of order $\leq 1$, if the order is 1, the exponential type is 0 [Riesz, 1923] $A, B, C, D$ have the same order, type and Phragmén-Lindelöf indicator function [Berg and Pedersen, 1994].
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If $\phi(z) = t \in \mathbb{R} \cup \{\infty\}$ then $\phi \in \mathcal{P} \cup \{\infty\}$ and $\mu_t$ is a discrete measure of the form

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N-extremal solutions are indeed extreme points in $\mathcal{M}_H$ - but not the only ones.
If we set

\[ \phi(z) = \begin{cases} \beta + i\gamma, & \Im z > 0, \\ \beta - i\gamma, & \Im z < 0, \end{cases} \]

for \( \beta \in \mathbb{R} \) and \( \gamma > 0 \), then \( \phi \in \mathcal{P} \) and \( \mu_{\beta,\gamma} \) is absolutely continuous with density

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The solution \( \mu_{0,1} \) is the one that maximizes certain entropy integral, see Krein’s condition. More general and additional information are provided in [Gabardo, 1992].
Suppose \( \{s_n\}_{n=0}^\infty \) is a sequence of Stieltjes moments such that the moment problem is indeterminate in the sense of Hamburger.
Nevanlinna parametrization in the case of Stieltjes moment problem

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- To describe $\mathcal{M}_S$ one can still use the Nevanlinna parametrization.

$\alpha \leq \phi(x) \leq 0$ for $x < 0$, [Pedersen, 1997]

The quantity $\alpha \leq 0$ plays an important role and can be obtained as the limit $\alpha = \lim_{n \to \infty} P_n(0) Q_n(0)$.

The moment problem is determinate in the sense of Stieltjes if and only if $\alpha = 0$.

The only N-extremal solutions supported within $[0, \infty)$ are $\mu$ with $\alpha \leq t \leq 0$.

For the indeterminate Stieltjes moment problem there is a slightly more elegant way how to describe $\mathcal{M}_S$ known as Krein parametrization, [Krein, 1967].
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Just restrict oneself to consider only the Pick functions \( \phi \) which have an analytic continuation to \( \mathbb{C} \setminus [0, \infty) \) such that \( \alpha \leq \phi(x) \leq 0 \) for \( x < 0 \), [Pedersen, 1997]
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Thank you, and see you in Beskydy!