BILATERAL GENERATING FUNCTIONS FOR A NEW CLASS OF GENERALIZED LEGENDRE POLYNOMIALS

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ABSTRACT. Recently Chatterjea (1) has proved a theorem to deduce a bilateral generating function for the Ultraspherical polynomials. In the present paper an attempt has been made to give a general version of Chatterjea's theorem. Finally, the theorem has been specialized to obtain a bilateral generating function for a class of polynomials \( \{P_n(x; \alpha, \beta)\} \) introduced by Bhattacharjya (2).

KEY WORDS AND PHRASES. Bilateral generating function, Ultraspherical polynomials, Legendre polynomials, Orthogonal polynomials, Weight function, Rodrigue's formula.

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1. **INTRODUCTION.**

Using the following differential formula for the Ultraspherical polynomials $P_n^\lambda(x)$ due to Tricomi,

$$P_n^\lambda [x(x^2-1)^{-1/2}] = \frac{(-1)^n}{n!} (x^2-1)^{\lambda+n/2} D^n (x^2-1)^{-\lambda}, \quad (1.1)$$

Chatterjea (1) has recently obtained a bilateral generating function for the Ultraspherical polynomials in the form of following theorem.

**THEOREM 1.** If

$$F(x, t) = \sum_{m=0}^{\infty} a^m t^m p_n^\lambda(x),$$

then

$$\rho^{-2\lambda} F\left(\frac{x-t}{\rho}, \frac{\rho y}{\rho}\right) = \sum_{r=0}^{\infty} t^r b_r(y) p_r^\lambda(x), \quad (1.2)$$

where

$$b_r(y) = \sum_{m=0}^{\infty} \binom{r}{m} a_m y^m,$$

and $\rho = (1-2xt+t^2)^{1/2}$.

A closer look at the above relation (1.2) suggests the following interesting general version of Chatterjea's theorem:

2. Let $F \circ G$ be used to denote the composition $F \circ G(x) = F(G(x))$. In terms of this notation, we state

**THEOREM 2.** Suppose that there exist functions $f, g, h$ and $X$ and a sequence of constants $\{c_n\}$ such that the sequence of functions $\{Q_n\}$ is generated by the formula
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\[ c_n f^n Q_n o X = D^n h, \quad n = 0, 1, 2, \ldots, \]  

(2.1)

where \( D \equiv d/dx \). Define the generating function

\[ F(x, t) = \sum_{n=0}^{\infty} a_n t^n Q_n(x). \]  

(2.2)

Then

\[ fF(X, gtz) = f \sum_{n=0}^{\infty} c_n (gt)^n Q_n o X b_n(z), \]

where

\[ b_n(z) = \sum_{k=0}^{\infty} \frac{a_k}{c_k (n-k)!} z^k. \]

PROOF. By Taylor's theorem,

\[ fF(X, gtz) = e^{tD} fF(X, gtz). \]  

(2.3)

To evaluate the right hand side of (2.3), we shall use as our starting point the relations (2.1) and (2.2), and the series expansion for \( e^{tD} \). Thus

\[ e^{tD} fF(X, gtz) = e^{tD} f \sum_{n=0}^{\infty} a_n (gt)^n Q_n o X \]

\[ = e^{tD} \sum_{n=0}^{\infty} \frac{a_n}{c_n} (tz)^n D^n h \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{a_n}{c_n} t^{n+m} \frac{z^n D^{n+m} h}{m!} \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{a_n}{c_n} (gt)^{n+m} c_{n+m} f Q_{n+m} o X/m! \]

\[ = f \sum_{n=0}^{\infty} c_n (gt)^n Q_n o X b_n(z), \]

where

\[ b_n(z) = \sum_{k=0}^{\infty} \frac{a_k}{c_k (n-k)!} z^k. \]

It is worthwhile to remark here that if we choose \( Q_n(x) = p_n^\lambda(x) \),

\[ f(x) = (x^2-1)^{-\lambda} \]

\[ g(x) = (x^2-1)^{-1/2} \]

\[ X(x) = x(x^2-1)^{-1/2} \]

\[ h(x) = (x^2-1)^{-\lambda} \]

and

\[ c_n = n! /(-1)^n \]

then Theorem ? would correspond to Chatterjea's theorem.

APPLICATIONS: Earlier, Bhattacharjya (2) introduced a new class of generalized Legendre polynomials \( \{p_n(x; \alpha, \beta)\} \) which are orthogonal with the
The Rodrigue's formulae for these polynomials are (2, 6.6) and (6.8):

\[ P_{2m} \left( x^{-1/2}; \alpha, \beta \right) = \frac{x^{m+(\alpha+1)/2} (1-x)^{(\beta-\alpha)/2}}{(1-x^2)^{(\beta-\alpha)/2} x^{-(\alpha+1)/2} x^{-m-(\alpha+1)/2}}, \quad (2.4) \]

and

\[ P_{2m+1} \left( x^{-1/2}; \alpha, \beta \right) = \frac{x^{m+1+\alpha/2} (1-x)^{(\beta-\alpha)/2}}{(1-x^2)^{(\beta-\alpha)/2} x^{-(\alpha+3)/2} x^{-m-(\alpha+3)/2}}, \quad (2.5) \]

Here we note that the sequences \{P_{2n} \left( x^{-1/2}; \alpha-2n, \beta \right) \} and \{ P_{2n+1} \left( x^{-1/2}; \alpha-2n, \beta \right) \} are amenable to a method of Theorem 2 for finding bilateral generating functions.

Let \( Q_n (x) = P_{2n} (x; \alpha-2n, \beta) \equiv P_{2n} (x) \). For simplicity of notation, set \( y = -(\alpha+1)/2 \) and \( \delta = (\alpha-\beta)/2 \). Then (2.1) holds with \( f(x) = x^y (1-x)^\delta \), \( g(x) = (1-x)^{-1} \), \( X(x) = x^{-1/2} \) and \( c_n = \frac{\phi(n)}{n} = (-n-(\alpha-1)/2)_n \). Upon replacing \( t \) by \(-t\) and \( z \) by \(-y\), we get

\[ \frac{x-t}{x} \left( \frac{1-(x-t)^\delta}{x-1-x} \right) F \left( \frac{1}{1/2}, \frac{-t}{1-x}, -\frac{yt}{1-(1-x)} \right) = \sum_{r=0}^{\infty} \left( \frac{t}{1-x} \right)^r \phi(r) * P_{2r} \left( x^{-1/2} \right) b_r (-y), \quad (2.6) \]

where

\[ F \left( \frac{1}{x^{1/2}}, \frac{t}{1-x} \right) = \sum_{m=0}^{\infty} a_m \left( \frac{-t}{1-x} \right)^m P_{2m} \left( x^{-1/2} \right) \]

and

\[ b_r (-y) = \sum_{m=0}^{\infty} a_m \left( \frac{-y}{\phi(m)(r-m)!} \right)^m. \quad (2.7) \]

Now replacing \( x^{-1/2} \) by \( s \) and \( t/(1-x) \) by \( t \) in (2.6), we are led to the following bilateral generating function for generalized even Legendre polynomials:

**COROLLARY.** 1: If

\[ F(x,t) = \sum_{m=0}^{\infty} a_m t^m P_{2m} (x), \]
then
\[
\left[1-(x^2-1)t\right]^y (1+t)^\delta \sum_{r=0}^{\infty} \frac{(-t)^r \phi(r)}{r!} \cdot P_{2r}(x) b_r(-y),
\]
where \(b_r(-y)\) is given by (2.7).

In the same way, let \(Q_n(x) = P_{2n+1}(x; \alpha-2n, \beta) = P_{2n+1}(x)\), and set
\[
y = -(\alpha+2)/2, \quad \delta = (\alpha-\beta)/2.
\]
Then (2.1) holds with \(f(x) = x^y (1-x)^\delta\),
\[
g(x) = (1-x)^{-1}, \quad X(x) = x^{-1/2} \quad \text{and} \quad c_n = \psi(n) = (-n-(\alpha+1)/2)\gamma(n).
\]
Replacing \(t\) by \(-t\) and \(z\) by \(-y\) and making the same substitution as before in (2.7), we are led to the following bilateral generating function for generalized odd Legendre polynomials.

**COROLLARY 2:** If
\[
F(x,t) = \sum_{m=0}^{\infty} a_m t^m P_{2m+1}(x),
\]
then
\[
\left[1-(x^2-1)t\right]^y (1-t)^\delta \sum_{r=0}^{\infty} \frac{(-t)^r \psi(r)}{r!} \cdot P_{2r+1}(x) c_r(-y),
\]
where
\[
c_r(-y) = \sum_{m=0}^{\infty} \frac{a_m (-y)^m \gamma(m)}{m! (r-m)!}.
\]
Taking \(\alpha = \beta\) in Corollary 1 and 2, we can obtain bilateral generating functions for generalized Legendre polynomials due to Dutta and More (3).

Next, we note that (2),
\[
P_{2m}(x; 0, 0) = \frac{(-1)^m m! P_{2m}(x)}{(-2m + 1/2)_m},
\]
and
\[
P_{2m}(x; 0, 0) = \frac{(-1)^m m! P_{2m+1}(x)}{(-2m - 1/2)_m},
\]
where \(P_{2m}(x)\) and \(P_{2m+1}(x)\) are even and odd Legendre polynomials. Therefore, by (2.8), (2.9) and the above two corollories we can obtain bilateral generating functions for Legendre polynomials.
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