A FURTHER NOTE ON LUCASIAN NUMBERS

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1. INTRODUCTION

This paper will extend and unify the results in [4] by completely determining all Lucasian numbers which are terms in certain Lucas sequences. Our specification of all Lucasian numbers will be based on results obtained in [1] in which all terms in particular Lucas sequences which do not have any primitive prime divisors are found.

Before proceeding, we will recall some definitions and known results. Our notation will be the same as that in [4]. Let \( u(r, s) \) and \( v(r, s) \) be Lucas sequences which satisfy the same second-order recursion relation

\[
u_{n+2} = r u_{n+1} + s u_n
\]

and have initial terms \( u_0 = 0, u_1 = 1, v_0 = 2, v_1 = r \) respectively, where \( r \) and \( s \) are integers. Associated with the sequences \( u(r, s) \) and \( v(r, s) \) is the characteristic polynomial

\[
f(x) = x^2 - rx - s
\]

with characteristic roots \( \alpha \) and \( \beta \). Let \( D = r^2 + 4s = (\alpha - \beta)^2 \) be the discriminant of both \( u(r, s) \) and \( v(r, s) \). By the Binet formulas

\[
v_n = (\alpha^n - \beta^n)/(\alpha - \beta)
\]

and

\[v_n = \alpha^n + \beta^n.
\]

The recurrences \( u(r, s) \) and \( v(r, s) \) are said to be degenerate if \( \alpha \beta = -s = 0 \) or \( \alpha/\beta \) is a root of unity. It follows from (3) and (4) that \( u_n \) or \( v_n \) can be equal to zero for \( n \geq 1 \) only if both \( u(r, s) \) and \( v(r, s) \) are degenerate.

This paper is in final form and no version of it will be submitted for publication elsewhere.

The integer \( m \) is a divisor of the recurrence \( u(r, s) \) satisfying the relation (1) if \( m \mid w_n \) for some \( n \geq 1 \). The prime \( p \) is a primitive prime divisor of \( w_n \), \( n \geq 1 \), if \( p \mid w_n \) but \( p \nmid w_i \) for \( 1 \leq i < n \). Given the Lucas sequence \( u(r, s) \), the integer \( m \) is called Lucasian if \( m \) is a divisor of \( u(r, s) \). In our main theorem, Theorem 2.6, we will show that if \( u(r, s) \) and \( v(r, s) \) are nondegenerate and \( gcd(r, s) = 1 \), then \( u_n \) is not Lucasian if \( n \geq 27 \). Theorem 2.6 will also find all terms \( u_n \) and \( v_n \) such that \( a \geq 1 \), \( b \geq 1 \), and \( u_n \mid u_b \).

A related question is to determine all \( a \) and \( b \) such that \( a \geq 1 \), \( b \geq 1 \), and \( v_a \mid v_b \). We will answer this question completely in Theorem 2.8 when both \( v(r, s) \) and \( u(r, s) \) are nondegenerate and \( gcd(r, s) = 1 \).

2. THE MAIN RESULTS

For reference, we will give the main results of [4] in Theorems 2.1-2.4. Theorems 2.5 and 2.8 will then generalize Theorems 2.1, 2.2 and 2.4.

**Theorem 2.1**: Let \( u(r, s) \) and \( v(r, s) \) be nondegenerate Lucas sequences for which \( gcd(r, s) = 1 \) and \( D > 0 \). Let \( a \) and \( b \) be positive integers. Then \( u_a \mid v_b \) if and only if one of the following conditions also holds. For convenience, the value of \( u_a \) is given in these conditions. The expressions \( r = s \) given below means that \( r \) can be any integer.

(i) \( a = 1 \), \( r = s \), \( b = s \), \( u_1 = 1 \);

(ii) \( a = 2 \), \( r = 1 \) or \( 2 \), \( s = s \), \( b = s \), \( u_2 = r = v_1 \);

(iii) \( a = 2 \), \( r \geq 3 \), \( s = s \), \( b \equiv 1 \) (mod 2), \( u_2 = v_1 = v_1 \);

(iv) \( a = 3 \), \( r \geq 3 \), \( s = 1 \), \( b \equiv 0 \) (mod 3), \( u_3 = 2 \);

(v) \( a = 4 \), \( r \geq 1 \), \( s = s \), \( b \equiv 2 \) (mod 4), \( u_4 = v_4 = r^2 + 2s \).

In particular, \( u_n \) is not Lucasian if \( n \geq 5 \).

**Theorem 2.2**: Let \( u(r, s) \) and \( v(r, s) \) be nondegenerate Lucas sequences for which \( gcd(r, s) = 1 \) and \( D < 0 \). Then \( u_n \) is not Lucasian for \( n > e^{452} \).

**Theorem 2.3**: Let \( u(r, s) \) and \( v(r, s) \) be nondegenerate Lucas sequences for which \( gcd(r, s) > 1 \). Then there exists a constant \( N(r, s) \) dependent on \( r \) and \( s \) such that \( u_n \) is not Lucasian for \( n \geq N(r, s) \).

**Theorem 2.4**: Let \( u(r, s) \) and \( v(r, s) \) be nondegenerate Lucas sequences for which \( gcd(r, s) = 1 \) and \( D > 0 \). Let \( a \) and \( b \) be positive integers. If \( |v_a| \geq 3 \), then \( u_a | v_b \) if and only if \( 2a | b \). If \( |v_a| \leq 2 \), then \( v_a | u_b \) if and only if one of the following two conditions also holds (the value of \( v_a \) is given for convenience):

(i) \( a = 1 \), \( |r| = 1 \), \( s = s \), \( b = s \), \( v_1 = r \);

(ii) \( a = 1 \), \( |r| = 2 \), \( s = 1 \) (mod 2), \( b = 0 \) (mod 2), \( v_2 = r = u_2 \).

**Remark 2.5**: The proofs of Theorem 2.1 and 2.4 given in [4] depend partly on the fact that if \( D > 0 \), then \( u(r, s) \) is strictly increasing for \( n \geq 2 \) and \( v(r, s) \) is strictly increasing for \( n \geq 1 \).

**Theorem 2.6**: Let \( u(r, s) \) and \( v(r, s) \) be nondegenerate Lucas sequences for which \( gcd(r, s) = 1 \). Let \( a \) and \( b \) be positive integers. Then \( u_a \) is Lucasian and \( u_a | v_b \) if and only if one of the following conditions also holds. For convenience, the value of \( u_a \) is given in these conditions.

(i) \( a = 1 \), \( r = s \), \( s = s \), \( b = s \), \( u_1 = 1 \);

(ii) \( a = 2 \), \( |r| = 1 \) or \( 2 \), \( s = s \), \( b = s \), \( u_2 = v_1 = r \);

(iii) \( a = 2 \), \( |r| \geq 3 \), \( s = s \), \( b \equiv 1 \) (mod 2), \( u_2 = v_1 = r \);

(iv) \( a = 3 \), \( r = s \), \( s = 1 - r^2 \), \( b = s \), \( u_3 = u_3 \);

(v) \( a = 3 \), \( r \equiv 1 \) (mod 2), \( s = 1 - r^2 \), \( b = 0 \) (mod 3), \( u_3 = u_3 \);
(vi) $a = 4, \ b = 2 \pmod{4}, \ u_1 = \pm u_2$;
(vii) $a = 4, \ r \equiv 1 \pmod{2}, \ s = (\pm 1 - r^2)/2, \ b = 1 \pmod{2}, \ u_4 = \pm u_2 = \pm u_1 = \pm r$;
(viii) $a = 5, \ |r| = 1, \ s = -2, \ b = *, \ u_5 = -1$;
(ix) $a = 5, \ |r| = 1, \ s = -3, \ b = *, \ u_5 = 1$;
(x) $a = 6, \ |r| = 12, \ s = -55, \ b = *, \ u_6 = 1$;
(xi) $a = 5, \ |r| = 12, \ s = -377, \ b = *, \ u_6 = 1$;
(xii) $a = 6, \ r = *, \ s = \pm 1 - r^2, \ b \equiv 3 \pmod{6}, \ u_6 = \pm u_3$;
(xiii) $a = 7, \ |r| = 1, \ s = -5, \ b = *, \ u_7 = 1$;
(xiv) $a = 8, \ |r| = 1, \ s = -2, \ b = 2 \pmod{4}, \ u_8 = \pm u_2 = \pm 3$;
(xv) $a = 10, \ |r| = 1, \ s = -2, \ b \equiv 5 \pmod{10}, \ u_{10} = -u_5 = \pm 11$;
(xvi) $a = 10, \ |r| = 1, \ s = -3, \ b \equiv 5 \pmod{10}, \ u_{10} = u_5 = \pm 11$;
(xvii) $a = 10, \ |r| = 12, \ s = -55, \ b \equiv 5 \pmod{10}, \ u_{10} = v_5 = \pm 44868$;
(xviii) $a = 10, \ |r| = 12, \ s = -377, \ b \equiv 5 \pmod{10}, \ u_{10} = v_5 = \pm 5319292$;
(xix) $a = 13, \ |r| = 1, \ s = -2, \ b = *, \ u_{13} = 1$;
(xx) $a = 14, \ |r| = 1, \ s = -5, \ b \equiv 7 \pmod{14}, \ u_{14} = v_7 = \pm 559$;
(xxi) $a = 26, \ |r| = 1, \ s = -2, \ b \equiv 13 \pmod{26}, \ u_{26} = -u_{13} = \pm 181$.

Remark 2.7: By Theorem 2.6, if $u(r, s)$ is nondegenerate and $\gcd(r, s) = 1$, then there exist only 12 possible indices $n$ for which $u_n$ can be Lucasian. It is noteworthy that for the Lucas sequences $u(\pm 1, -2)$, $u_n$ is Lucasian for 10 of these indices, namely $n = 1, 2, 3, 4, 5, 6, 8, 10, 12, 26$. The only other nondegenerate Lucas sequences $u(r, s)$ for which $\gcd(r, s) = 1$ and $u_n$ is Lucasian for 5 or more indices $n$ are $u(\pm 1, -3)$ and $u(\pm 1, -5)$. For $u(\pm 1, -3)$, $u_n$ is Lucasian for the 6 indices $n = 1, 2, 3, 4, 5, 10$, while for $u(\pm 1, -5)$, $u_n$ is Lucasian for the 5 indices $n = 1, 2, 4, 7, 14$.

Theorem 2.8: Let $u(r, s)$ and $v(r, s)$ be nondegenerate Lucas sequences for which $\gcd(r, s) = 1$. Let $a$ and $b$ be positive integers. Then $v_n/u_n$ if and only if one of the following conditions also holds. For convenience, the value of $u_n$ is given in condition (ii)-(vii).

(i) $2ab, \ r = *, \ s = *$;
(ii) $a = 1, \ |r| = 1, \ s = *, \ b = *, \ u_1 = r$;
(iii) $a = 2, \ r \equiv 1 \pmod{2}, \ s = (\pm 1 - r^2)/2, \ b = *, \ v_2 = \pm 1$;
(iv) $a = 2, \ r \equiv 0 \pmod{2}, \ s = (\pm 2 - r^2)/2, \ b \equiv 0 \pmod{2}, \ v_2 = \pm 2$;
(v) $a = 4, \ |r| = 1, \ s = -2, \ b = *, \ u_4 = 1$;
(vi) $a = 4, \ |r| = 2, \ s = -7, \ b \equiv 0 \pmod{2}, \ v_4 = 2$;
(vii) $a = 5, \ |r| = 2, \ s = -3, \ b \equiv 0 \pmod{2}, \ v_5 = \pm 2$.

Remark 2.9: Theorem 2.6 generalizes Theorems 2.1 and 2.2, which were proven in [4]. Theorem 2.8 generalizes Theorem 2.4 which was also proved in [4]. As contrasted to the proofs of Theorems 2.1, 2.2, and 2.4, the proofs of Theorems 2.6 and 2.7, which will be given in Section 4, do not treat the cases $D > 0$ and $D < 0$ separately.

The key result in proving Theorems 2.6 and 2.8 is Theorem 2.10 given below which is the main theorem of [1].

Theorem 2.10: Let $u(r, s)$ be a nondegenerate Lucas sequence for which $\gcd(r, s) = 1$. Then $u_n$ has a primitive prime divisor if $n > 30$. Moreover $u_n$ has no primitive prime divisor only if $n = 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 18, 30$.

Remark 2.11: Consider all nondegenerate Lucas sequences $u(r, s)$ for which $\gcd(r, s) = 1$. Tables 1 and 3 of [1] list all terms $u_n, n \geq 1$, which have no primitive prime divisors. We note that in [1], the authors define a prime $p$ to be a primitive prime divisor of $u_n$ if $p | u_n$ but
In contrast to this definition, we do not include \( p \) as a primitive prime divisor of \( u_n \) if \( p \mid D \), but \( p \nmid u_k \) for \( 1 \leq k < n \).

For reference, Theorem 2.12 lists all degenerate Lucas sequences \( u(r, s) \) and \( v(r, s) \).

**Theorem 2.12:** The Lucas sequences \( u(r, s) \) and \( v(r, s) \) are degenerate if and only if one of the following conditions holds:

(i) \( s = 0 \).

(ii) \( \alpha/\beta = 1 \) and \( D = r^2 + 4s = 0 \).

(iii) \( \alpha/\beta = -1 \), \( r = 0 \) and \( s = N \) for some non-zero integer \( N \).

(iv) \( \alpha/\beta \) is a primitive cube root of unity, \( r = N \), and \( s = -N^2 \) for some non-zero integer \( N \).

(v) \( \alpha/\beta \) is a primitive fourth root of unity, \( r = 2N \), and \( s = -2N^2 \) for some non-zero integer \( N \).

(vi) \( \alpha/\beta \) is a primitive sixth root of unity, \( r = 3N \), and \( s = -3N^2 \) for some non-zero integer \( N \).

**Proof:** This is proved in [3, p. 613].

3. NECESSARY LEMMAS AND DEFINITIONS

The following lemmas and definition will be needed for the proofs of Theorems 2.6 and 2.8. Lemmas 3.1, 3.2, 3.3, and 3.5 are well-known (and follow readily from (1), (3), and (4)).

**Lemma 3.1:** \( u_{2n} = u_n u_n \).

**Lemma 3.2:**

\[
\begin{align*}
u_n(-r, s) &= (-1)^n u_n(r, s), \\
v_n(-r, s) &= (-1)^n v_n(r, s).
\end{align*}
\]

It follows from Lemma 3.2 that, \( u_n(-r, s) \) is Lucasian if and only if \( u_n(r, s) \) is Lucasian.

**Lemma 3.3:** Consider the Lucas sequences \( u(r, s) \) and \( v(r, s) \). Then \( u_n \mid u_{in} \) for all \( i \geq 1 \) and \( v_n \mid v_{jn} \) for all \( j \geq 0 \).

**Lemma 3.4:** If \( u_n \) is not Lucasian, and \( \alpha \mid c \), then \( u_n \) is not Lucasian.

**Proof:** By Lemma 3.3, \( u_n \mid u_{in} \). It is now evident that \( u_n \) is not Lucasian if \( u_n \) is not Lucasian.

**Lemma 3.5:** Let \( u(r, s) \) and \( v(r, s) \) be Lucas sequences for which \( 2 \nmid gcd(r, s) \).

(i) Suppose \( r \) is odd and \( s \) is even. Then \( 2 \nmid u_n \) and \( 2 \nmid v_n \) for \( n \geq 1 \).

(ii) Suppose \( r \) and \( s \) are both odd. Then \( 2 \mid u_n \) if and only if \( 3 \mid n \), and \( 2 \mid v_n \) if and only if \( 3 \mid n \).

(iii) Suppose \( r \) is even and \( s \) is odd. Then \( 2 \mid u_n \) if and only if \( 2 \mid n \), and \( 2 \mid v_n \) for all \( n \geq 0 \).

**Lemma 3.6:** Let \( v(r, s) \) be a Lucas sequence for which \( 2 \nmid gcd(r, s) \).

(i) If \( r \) and \( s \) are both odd and \( 2^k \parallel v_3 \) for some positive integer \( k \), then \( 2 \parallel v_n \) for \( n \equiv 0 \pmod{6} \) and \( 2^k \parallel v_n \) for \( n \equiv 3 \pmod{6} \). Recall that \( 2^k \parallel a \) if \( 2^k \mid a \), but \( 2^{k+1} \nmid a \).

(ii) If \( r \) is even and \( s \) is odd and \( 2^k \parallel v_1 = r \), then \( 2 \parallel v_{2n} \) and \( 2^k \parallel v_{2n+1} \) for all \( n \geq 0 \).

**Proof:** This is proved in [8].

**Definition 3.7:** For the Lucas sequence \( u(r, s) \) the rank of appearance of the positive integer \( m \) in \( u(r, s) \), denoted by \( \omega(m) \), is the least positive integer \( n \), if it exists, such that \( m \mid u_n \). The rank of appearance of \( m \) in \( v(r, s) \), denoted by \( \overline{\omega}(m) \), is defined similarly.

**Lemma 3.8:** Let \( u(r, s) \) and \( v(r, s) \) be Lucas sequences for which \( gcd(r, s) = 1 \). Let \( p \) be an odd prime. If \( \omega(p) \) is odd, then \( \overline{\omega}(p) \) does not exist and \( u_{\omega(p)} \) is not Lucasian.
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**Proof:** This was proved by Carmichael [2, p. 47].

**Lemma 3.9:** Let \( u(r, s) \) and \( v(r, s) \) be Lucas sequences for which \( \gcd(r, s) = 1 \). Suppose that \( p \) is an odd prime and \( \omega(p) = 2n \). Then \( \omega(p) = n \).

**Proof:** This is proved in Proposition 2(iv) of [7].

**Definition 3.10:** The 2-valuation of the integer \( n \), denoted by \([n]_2\) is the largest integer \( k \) such that \( 2^k | n \).

**Lemma 3.11:** Let \( v(r, s) \) be a Lucas sequence for which \( \gcd(r, s) = 1 \). Suppose that \( u_a \) is Lucasian and that \( p \) and \( q \) are distinct odd prime divisors of \( u_a \). Then \( \omega(p) = 2 \) if \( \omega(q) = 2 \).

**Proof:** This is proved in Proposition 2(ix) of [7].

**Lemma 3.12:** Let \( u(r, s) \) and \( v(r, s) \) be nondegenerate Lucas sequences for which \( \gcd(r, s) = 1 \). Let \( a \) and \( b \) be positive integers and let \( d = \gcd(a, b) \).

(i) \( \gcd(u_a, u_b) = u_d \).

(ii) \( \gcd(v_a, v_b) = \begin{cases} v_d & \text{if } [a]_2 = [b]_2, \\ 1 \text{ or } 2 & \text{otherwise}; \end{cases} \)

(iii) \( \gcd(u_a, v_b) = \begin{cases} u_d & \text{if } [a]_2 > [b]_2, \\ 1 \text{ or } 2 & \text{otherwise}. \end{cases} \)

**Proof:** This is proved in [6] and [3, section 5].

**Lemma 3.13:** Let \( u(r, s) \) and \( v(r, s) \) be nondegenerate Lucas sequences for which \( \gcd(r, s) = 1 \). Then \( m | u_a \) if and only if \( \omega(m) | n \). Moreover, if \( m \geq 3 \), then \( m | u_a \) if and only if \( \omega(m) | n \) and \( \omega(m) = [n]_2 \).

**Proof:** The results follow from Lemmas 3.3 and 3.12.

**Lemma 3.14:** Let \( u(r, s) \) be nondegenerate Lucas sequence for which \( 2 \nmid \gcd(r, s) \). If \( r \) is even, \( s \) is odd, and \( 4 | a \), then \( u_a \) is not Lucasian. If \( r \) and \( s \) are both odd and \( 6 | a \), then \( u_a \) is not Lucasian.

**Proof:** First suppose that \( r \) is even and \( s \) is odd. By Lemma 3.5 (iii), \( 2 | v_n \) for all \( n \geq 0 \). Moreover, \( u_2 = v_1 = r \). Suppose that \( 2^k | r \). By Lemma 3.6 (ii), \( 2^{k+1} \nmid v_n \) for any \( n \geq 0 \). However, \( 2 | v_2 \), and hence by Lemma 3.1, \( 2^{k+1} | u_4 = u_2 v_2 \). Thus, \( u_4 \) is not Lucasian, and consequently by Lemma 3.4, \( u_4 \) is not Lucasian if \( 4 | a \).

Now suppose that both \( r \) and \( s \) are odd. By Lemma 3.5(ii), \( 2 | u_a \) and \( 2 \nmid v_a \). Suppose that \( 2^k | v_3 \). By Lemma 3.6 (i), \( 2^{k+1} \nmid v_n \) for any \( n \geq 0 \). However, \( 2^{k+1} | u_6 = u_3 v_3 \). Therefore, \( u_6 \) is not Lucasian, and hence by Lemma 3.4, \( u_6 \) is not Lucasian if \( 6 | a \).

Finally, suppose that in addition to \( r \) and \( s \) both being odd, \( 4 | u_3 \). Since \( u_3 = r^2 + s \), this can occur only if \( s \equiv 3 \pmod{4} \). However, then \( v_3 = r(r^2 + 3s) \equiv 2 \pmod{4} \), and \( 2 \nmid v_3 \). By Lemma 3.6 (i), it follows that \( 4 \) is not a divisor of \( v(r, s) \). Since \( u_4 | u_3 \) for all \( t \geq 1 \) by Lemma 3.3, we see that \( 4 | u_3 \) for all \( t \geq 1 \). Hence, \( u_a \) is not Lucasian if \( 3 | a \).

**Lemma 3.15:** Let \( u(r, s) \) and \( v(r, s) \) be nondegenerate Lucas sequences for which \( \gcd(r, s) = 1 \). Then \( u_a \) is Lucasian if \( a \geq 1 \) if and only if at least one of the following four conditions holds:

(i) \( a \) is odd and \( u_a = \pm 1 \);

(ii) \( a \) is even and \( u_{a/2} = \pm 1 \);

(iii) \( a \) is even, \( u_{a/2} \) is Lucasian, and \( v_{a/2} = \pm 1 \);
(iv) $a = 3$ and $u_a = \pm 2$.

**Proof:** We first show sufficiency. The sufficiency of (i) is obvious. The sufficiency of (ii) and (iii) follow from the fact that $u_a = u_{a/2}v_{a/2}$ by Lemma 3.1. The sufficiency of (iv) follows from the fact that by Lemma 3.5, $2|u_a$ for $a \geq 1$ only if $2|v_a$.

We now show necessity. Suppose first that $a$ is odd, $u_a \neq \pm 1$, and $u_a$ is Lucasian. If $u_a$ has an odd prime divisor $p$, then by Lemma 3.13, $\omega(p)|a$ and hence $\omega(p)$ is odd. It now follows from Lemmas 3.4 and 3.8 that $u_a$ is not Lucasian.

Now assume that $u_a \geq 2$ and $u_a$ is a power of 2. By Lemma 3.5, we must have that $r$ and $s$ are both odd and $3|a$. Since $2|u_a = r^s + s$, we see that $u_a$ has no primitive prime divisor if $a > 3$. It now follows from Theorem 2.10 that $a$ must equal 3. By Lemma 3.14, $u_a$ is not Lucasian if $4|u_a$. Hence, $a = 3$ and $u_a = \pm 2$. By our above discussion, we see that if $a$ is odd and $u_a$ is Lucasian, then either $u_a = \pm 1$ or $a = 3$ and $u_a = \pm 2$.

At this point, we assume that $a$ is even, $u_{a/2} \neq \pm 1$, and $u_{a/2} \neq \pm 1$. Suppose that $2|u_{a/2}$.

By Lemma 3.5, either $r$ and $s$ are both odd, or $r$ is even and $s$ is odd. If $r$ and $s$ are both odd, then $3|(a/2)$ and $2|v_{a/2}$ by Lemma 3.5. Suppose that $2^k || v_3$. Then $2^{k+1}|u_a = u_{a/2}v_3$, and hence $2^{k+1}|u_a$ by Lemma 3.3. Thus, $u_a$ is not Lucasian by Lemma 3.6 (i). If $r$ is even and $s$ is odd, then $a/2$ is even by Lemma 3.5. Moreover, by Lemma 3.5 (iii), $2|u_a$ for all $n \geq 0$. Suppose that $2^k || v_3$. Then $2^k|u_2 = r = v_1$, and hence $2^{k+1}|u_{a/2}$ by Lemma 3.3. Then $2^{k+1}|u_a = u_{a/2}v_{a/2}$, and $u_a$ is not Lucasian by Lemma 3.6 (ii).

Next suppose that $2|u_{a/2}$. Since $2 \not| u_{a/2}$ by our above arguments, we see by Lemma 3.5 that $a$ is even, $s$ is odd, and $a/2$ is odd. Then $u_{a/2}$ has an odd prime divisor. By our earlier discussion and Lemma 3.4, it follows that both $u_{a/2}$ and $u_a$ are not Lucasian.

The only remaining case to consider is the one in which $u_{a/2}$ has an odd prime divisor $p$ and $u_{a/2}$ has an odd prime divisor $q$. Since $gcd(u_{a/2}, v_{a/2}) = 1$ or 2 by Lemma 3.12 (iii), $p \neq q$.

Suppose that $2^1 || a$. If $t = 1$, then $\omega(p)|(a/2)$ and $\omega(p)$ is odd by Lemma 3.13. Thus, both $u_{a/2}$ and $u_a$ are not Lucasian by Lemmas 3.4 and 3.8. Consequently we must have that $t \geq 2$.

By Lemmas 3.13 and 3.9, it follows that

$$[\omega(p)]_2 \leq t - 2. \tag{7}$$

Since $q|u_{a/2}$, we see by Lemma 3.13 that

$$[\omega(q)]_2 = t - 1. \tag{8}$$

Since $pq|u_a$ it now follows from Lemma 3.11 and (7) and (8) that $u_a$ is not Lucasian. Hence, if $a$ is even and $u_a$ is Lucasian, then $u_{a/2} = \pm 1$ or $u_{a/2} = \pm 1$. We note further that if $u_{a/2} = \pm 1$ and $u_a$ is Lucasian, then $u_{a/2}$ is also Lucasian since $u_a = u_{a/2}v_{a/2}$. Necessity is thus shown and our result follows.

**Remark 3.16:** Suppose that $u(r, s)$ and $v(r, s)$ are nondegenerate, $gcd(r, s) = 1$, and $u_a$ is Lucasian. We note that if $v_{a/2} = \pm 1$, then $u_a = u_a / v_{a/2} = \pm u_{a/2}$. It now follows from Lemma 3.15 that if it is not the case that $a = 3$ and $u_a = \pm 2$, then either $u_a$ or $u_{a/2}$ has no primitive prime divisor. In what follows, we will apply the results of [1], which determine all $n$ such that $w_n$ has no primitive prime divisor to find all $a$ for which $w_n$ is not Lucasian and it is not the case that $a = 3$ and $u_a = \pm 2$. 
Lemma 3.17: Let \( u(r, s) \) be a nondegenerate Lucas sequence for which \( \gcd(r, s) = 1 \). If \( a \) is odd, then \( u_a \) is not Lucasian if \( a \geq 12 \), \( a \neq 14 \), and \( a \neq 26 \).

Proof: Suppose that \( a \) is odd, \( a \geq 9 \) and \( a \neq 13 \). By Lemma 3.15, if \( u_a \) were to be Lucasian, then \( u_a \) must equal \( \pm 1 \), and consequently \( u_a \) would have no primitive prime divisor. It now follows from Theorem 2.10 that \( u_a \) is not Lucasian.

We now assume that \( a = 18 \), \( a = 30 \), or \( a = 60 \). By our earlier discussion, \( u_9 \) and \( u_{15} \) are both not Lucasian. Thus, \( u_a \) is not Lucasian by Lemma 3.4. Now assume that \( a \) is even and either \( a = 22 \) or \( a \geq 28 \). By our previous argument, we can assume that \( a \neq 30 \), \( a \neq 36 \), and \( a \neq 60 \). If \( u_a \) were to be Lucasian, then by Remark 3.16, either \( u_{a/2} \) or \( u_a \) would have no primitive prime divisors. Therefore, by Theorem 2.10, \( u_a \) is not Lucasian in this case also.

We next assume that \( a = 20 \) and \( u_a \) is Lucasian. By Theorem 2.10, \( u_{20} \) has a primitive prime divisor. Thus by Lemma 3.15 and Remark 3.16, \( u_{10} = \pm 1 \) and \( u_{10} \) has no primitive prime divisor. By Table 1 of [1], the only recurrences \( u(r, s) \) for which \( u_{10} \) has no primitive prime divisor are \( u(\pm 2, -3) \), \( u(\pm 5, -7) \), and \( u(\pm 5, -18) \). We see by inspection that in each case, \( u_{10} \neq \pm 1 \). Hence, \( u_{20} \) is not Lucasian.

Now assume that \( a = 16 \) and \( u_a \) is Lucasian. Since \( u_{16} \) has a primitive prime divisor by Theorem 2.10, we see by Lemma 3.15 and Remark 3.16 that \( u_8 = \pm 1 \). By Table 1 of [1], the only recurrences \( u(r, s) \) for which \( u_8 \) has no primitive prime divisor are \( u(\pm 2, -7) \) and \( u(\pm 1, -2) \). In each case, we see by inspection that \( u_8 \neq \pm 1 \). Thus \( u_{16} \) is not Lucasian.

We finally suppose that \( a = 12 \) and \( u_a \) is Lucasian. By Lemma 3.15 either \( u_6 = \pm 1 \) or \( v_6 = \pm 1 \). First assume that \( u_6 = \pm 1 \). Since \( u_4 = u_3 \), \( v_3 = r(r^2 = 3s) = \pm 1 \). Thus, \( r = \pm 1 \). Then \( r^2 + 3s = 3s + 1 = \pm 1 \). Hence, \( 3s = -2 \), which is impossible, or \( 3s = 0 \) and \( s = 0 \), which is excluded by the assumption that \( u(r, s) \) is nondegenerate. Therefore, by Lemma 3.15 and Remark 3.16, \( v_6 = \pm 1 \) and \( u_{12} \) has no primitive prime divisor. Since \( v_6 = \pm 1 \), it follows from Lemma 3.5 that \( r \) is odd and \( s \) is even. By Table 1 of [1], the only recurrences \( u(r, s) \) for which \( r \) is odd and \( s \) is even and \( u_{12} \) has no primitive prime divisor are \( u(\pm 1, -2) \) and \( u(\pm 1, -4) \). In both cases, we see that for the corresponding Lucas sequence, \( u(r, s) \), \( v_6 \neq \pm 1 \). Thus, \( u_{12} \) is not Lucasian. By Lemma 3.4, \( u_{24} \) is also not Lucasian.

Lemma 3.18: Suppose that \( u(r, s) \) and \( v(r, s) \) are nondegenerate Lucas sequences for which \( \gcd(r, s) = 1 \). Suppose that \( a \) is even, \( u_{a/2} = \pm 1 \), \( |u_a| \geq 3 \), and \( \overline{\omega}(v_{a/2}) = a/2 \). Then \( u_a \) is if and only if \( b \equiv a/2 \) (mod \( a \)).

Proof: Since \( u_a = u_{a/2} v_{a/2} \) by Lemma 3.1, \( u_a = \pm v_{a/2} \). Note that \( b \equiv a/2 \) (mod \( a \)) if and only if \( b = (2k + 1)(a/2) \) for some \( k \geq 0 \). The result now follows from Lemma 3.12 (ii).

4. PROOFS OF THE MAIN THEOREMS

We are now ready for the proofs of Theorems 2.6 and 2.8.

Proof of Theorem 2.6: A substantial part of this proof deals with examining nondegenerate Lucas sequences \( u(r, s) \) for which \( \gcd(r, s) = 1 \) and either \( u_{a/2} \) or \( u_a \) has no primitive prime divisor. Since there exist infinitely many such Lucas sequences when \( a/2 \) or \( a = 2, 3, 4 \) or \( 6 \) by Table 3 of [1], we will treat these cases separately.

If \( u_a = 1 \), then clearly \( u_a | u_n \) for all \( n \geq 1 \). The result for the case in which \( a = 1 \) now follows since \( u_a = 1 \). Suppose that \( a = 2 \). Then \( u_2 = r = v_1 \). If \( |r| = 1 \) or 2, then \( u_a | u_n \) for all \( n \) by our above observation and by Lemma 3.5 (iii). If \( |r| \geq 3 \), then condition (iii) holds for Lemma 3.18.
Now assume that $a = 3$. By Lemma 3.15, $u_3$ is Lucasian only if $u_3 = \pm 1$ or $u_3 = \pm 2$. We note that $u_3 = r^2 + s = \pm 1$ if and only if $s = \pm 1 - r^2$, and condition (iv) is satisfied. If $u_3 = r^2 + s = \pm 2$, then $s = \pm 2 - r^2$. Hence, $s \equiv r \pmod{2}$. Consequently, $r$ and $s$ are both odd since $\gcd(r, s) = 1$. By Lemma 3.5 (ii), it follows that condition (v) holds.

Next suppose that $a = 4$. By Lemma 3.15 either $u_2 = r = \pm 1$ or $v_2 = r^2 + 2s = \pm 1$. If $u_2 = \pm 1$, then
\[ u_4 = u_2v_2 = \pm uv_2 = \pm (r^2 + 2s) = \pm (2s + 1). \]  
(9)

We claim that $|v_3| \geq 3$. Suppose that $v_3 = \pm 1$. By (9), either $2s + 1 = 1$ or $2s + 1 = -1$. If $2s + 1 = 1$, then $s = 0$, which is excluded since $u(r, s)$ and $v(r, s)$ are both nondegenerate. If $2s + 1 = -1$, then $u(r, s)$ and $v(r, s)$ would be degenerate by Theorem 2.12 (iv), contrary to assumption. Noting that $v_2 = 2s + 1$ is odd, we see that $|u_4| = |v_2| \geq 3$. Since $v_1 = r = \pm 1$, it follows from Lemma 3.18 that condition (vi) holds. If $v_2 = r^2 + 2s = \pm 1$, then
\[ u_4 = u_2v_2 = \pm uv_2 = \pm r = \pm v_1. \]  
(10)

Moreover, $v_2 = \pm 1$ if and only if $s = (\pm 1 - r^2)/2$ and $r \equiv 1 \pmod{2}$. If $r = \pm 1$, then $s = 0$, which contradicts the fact that both $u(r, s)$ and $v(r, s)$ are nondegenerate. Hence, $|u_4| = |v_2| = |r| = |v_1| \geq 3$. We now see by Lemma 3.18 that condition (vii) holds.

We now assume that $a = 6$. By Lemma 3.15, either $u_3 = \pm 1$ or $v_3 = \pm 1$. However, by the treatment of the case $a = 12$ in the latter part of the proof of Lemma 3.17, we see that $v_3$ cannot equal $\pm 1$. Thus $v_3 = \pm 1$. Then $u_6 = u_3v_3 = \pm v_3$. We will show that $|v_3| \geq 3$ and $\bar{u}(v_3) = 3$. Since $u_6 = r^2 + s$, it will then follow from Lemma 3.18 that condition (xiii) holds.

We note that $v_3 \neq 0$ since $v(r, s)$ is nondegenerate. Suppose that
\[ 0 < |v_3| = |r(r^2 + 3s)| \leq 2. \]

Then $r = \pm 1$ or $r = \pm 2$. Substituting these values into $s = 1 - r^2$, and noting that $s \neq 0$, we obtain a value for $|v_3| = |r(r^2 + 3s)| > 3$ in each case.

We now claim that $\bar{u}(v_3) = 3$. If $\bar{u}(v_3) < 3$, then by Lemmas 3.3 and 3.12, $\bar{u}(v_3) = 1$ and $|v_3| = |v_3|$. If $|v_3| = |v_3|$, then, since $r \neq 0$, $r^2 + 3s = \pm 1$. Since, also, $r^2 + s = \pm 1$, we have $s = 0$ or $\pm 1$; but $s \neq 0$ by Theorem 2.12 (i), and $r^2 + s = \pm 1$ is not possible for an integer $r \neq 0$. Hence, $|v_3| \neq |v_3|$, and it follows that condition (xiii) holds.

We next suppose that $a = 8$. By Lemma 3.15, $u_4 = \pm 1$ or $v_4 = \pm 1$. First suppose that
\[ u_4 = u_2v_2 = r(r^2 + 2s) = \pm 1. \]

Then $r = \pm 1$ and $r^2 + 2s = 2s + 1 = \pm 1$. If $2s + 1 = 1$, then $s = 0$, which is excluded since $u(r, s)$ is nondegenerate. If $2s + 1 = -1$, then $r = \pm 1$ and $s = -1$, which is also excluded by Lemma 2.12 (iv) since $u(r, s)$ is nondegenerate. Now assume that $v_4 = \pm 1$. Then $u_8 = u_4v_4 = \pm u_4$, and $u_8$ has no primitive prime divisor. By Table 1 of [1], the only recurrences $u(r, s)$ for which $u_8$ has no primitive prime divisor are $u(\pm 2, -7)$ and $u(\pm 1, -2)$. By inspection, we see that $u_4(\pm 2, -7) = 2$, while $u_4(\pm 1, -2) = 1$. We further observe that if $r = \pm 1$ and $s = -2$, then
\[ u_8 = u_4 = \pm u_2 = \pm 3, \]

while
\[ u_2 = v_1 = \pm 1. \]
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It now follows from Lemma 3.18 that condition (xvi) holds.

We now assume that \( a \geq 5 \) and \( a \) is odd. By Lemmas 3.15 and 3.17, it follows \( a = 5, 7, \) or \( 13 \) and \( u_n \) has no primitive prime divisor. By examining Table 1 of [1] and evaluating the term \( u_n \) in all recurrences \( u(r, s) \) for which \( u_n \) has no primitive prime divisor, we see that \( u_n \) is Lucasian if and only if one of the conditions (viii), (ix), (x), (xi), (xii), or (xiv) holds.

Finally, we suppose that \( a \geq 10 \) and \( a \) is even. By Lemmas 3.15 and 3.17, we see that \( a = 10, 14, \) or \( 26 \) and either \( v_{a/2} = \pm 1 \) or \( u_{a/2} = \pm 1 \). If \( v_{a/2} = \pm 1 \), then \( u_n = u_{a/2} v_{a/2} = \pm u_{a/2} \), and \( u_n \) has no primitive prime divisor. By Theorem 2.10, \( u_n \) has a primitive prime divisor if \( a = 14 \) or \( 26 \). Thus, \( v_{a/2} \) can equal \( \pm 1 \) only if \( a = 10 \). From Table 1 or [1], we see that \( u_{10}(r, s) \) has no primitive prime divisor only if \( r = \pm 2, s = -3, r = \pm 5, s = -7, \) or \( r = \pm 5, s = -18 \). In each case, we observe that \( v_5(r, s) \neq \pm 1 \). Hence, we must have that \( v_{a/2} = \pm 1 \). Since \( a/2 \geq 5 \) and \( a/2 \) is odd, we obtain all the values of \( r \) and \( s \) for which \( u_{a/2} = \pm 1 \) from conditions (viii) - (xi), (xiii), and (xiv). By inspection of these recurrences \( u(r, s) \) and the corresponding recurrences \( v(r, s) \), we obtain the values of \( u_n(r, s) \) and ascertain that \( |u_n(r, s)| \geq 3 \) and \( \Omega(v_{a/2}(r, s)) = a/2 \). It follows from Lemma 3.18 that conditions (xv) - (xviii), (xx), and (xxi) hold. The results now follow.

Proof of Theorem 2.8: By Theorem 2.4, if \( |v_n| \geq 3 \), then \( v_n|u_n \) if and only if \( 2|a|b \). Thus, we can assume that \( |v_n| = 1 \) or \( 2 \). First suppose that \( v_n = \pm 1 \). Then \( u_{2n} = \pm u_n \), and \( u_n \) is Lucasian if \( u_n \) is Lucasian. Using this observation, the proof of Theorem 2.6 determines all instances in which \( v_n = \pm 1 \) in the course of finding all terms \( u_{2n} \) which are Lucasian. It follows from conditions (ii), (vi), and (xvi) of Theorem 2.6 that \( u_n = \pm 1 \) if and only if conditions (ii), (iii), or (v) of Theorem 2.8 hold.

We now assume that \( v_n = \pm 2 \). If \( a = 1 \) and \( v_1 = r = \pm 2 \), it follows from Lemma 3.5 (iii) that \( u_1|u_k \) if and only if \( 2|a|b \). Now suppose that \( a = 2 \) and \( v_2 = r^2 + 2s = \pm 2 \). It follows that \( v_2 = \pm 2 \) if and only if \( s = (\pm 2 - r^2)/2 \) and \( r \) is even. Lemma 3.5 (iii) now implies that \( v_2|u_k \) if and only if condition (iv) of Theorem 2.8 holds.

Next assume that \( a = 3 \) and \( v_3 = r(r^2 + 3s) = \pm 2 \). Then \( |r| = 1 \) or \( 2 \). If \( r = \pm 1 \), then
\[
r^2 + 3s = 3s + 1 = \pm 2.
\]

Then \( 3s = 1 \), which is impossible, or \( 3s = -3 \), which yields \( r = \pm 1, s = -1 \). However, this case is excluded by Theorem 2.12 (iv), since \( v(r, s) \) is nondegenerate.

Finally, assume that \( a \geq 4 \). By Lemma 3.5, we see that \( \omega(2) \leq 3 \). Hence \( u_{2n} = u_n u_n \) has no primitive prime divisor. From Table 1 of [1], we find that there are exactly 10 Lucas sequences \( u(r, s) \) for which some term \( u_{2n} \) has no primitive prime divisor for \( n \geq 4 \). By examining each of these recurrences, we see that \( u_n = \pm 2 \) for \( a \geq 4 \) if and only if \( r \) and \( s \) have the values given in conditions (vi) and (vii) of Theorem 2.8. We note that by Lemma 3.5 (iii), if \( |r| = 2 \) then \( 2|u_n \) if and only if \( 2|n \). The result now follows.

ACKNOWLEDGMENT

The author wishes to thank the referee for his or her careful reading of the manuscript and suggestions which improved this paper.
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AMS Classification Numbers: 11B39