A Survey of the Riordan Group

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This survey is based on a series of lectures given by Louis Shapiro at the Center for Combinatorics in Nankai University in the spring of 2005. The audience was an enthusiastic group of graduate students and faculty. The material here is mainly expository though a few of the results are new.

1 Included are many examples, 20 exercises with solutions, a good number of open problems.

We hope that this is an engaging up to date, but not encyclopedic, presentation of both a useful tool and a fresh perspective on some core combinatorics.

We start with some results that will turn out to be connected to the Riordan group and are of independent interest.

Question 1.1 *: How many Dyck paths (or mountain ranges) are there?*

The Answer is given by the Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$. The number of Dyck paths of length $0, 2, 4, 6, 8, 10, \cdots$ are $1, 1, 2, 5, 14, 42, \cdots$, respectively. See Figure 1 as examples:

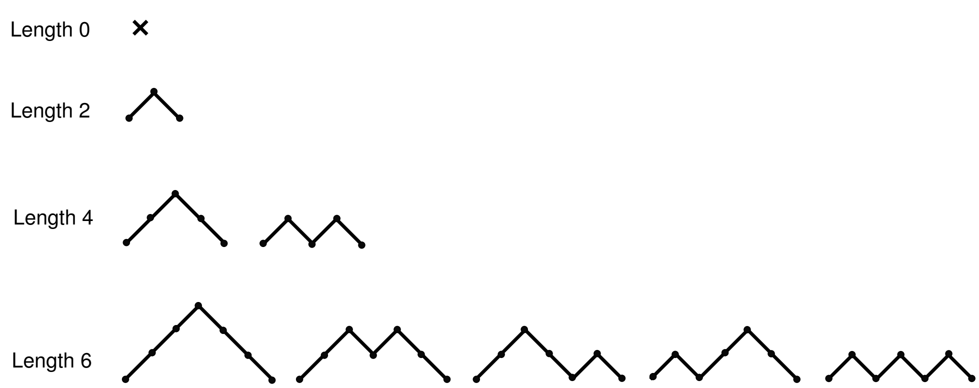


Figure 1: Dyck paths for $n = 0, 1, 2, 3$.

Question 1.2 *: How many points lie on the x-axis on these Dyck paths?*

Indeed, there are C_{n+1} total points on the x-axis. We will return to this example later.

Why are the Catalan numbers and their relatives so important? Statistics and probability are built around the normal curve $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. The normal curve comes from the binomial distribution. Flipping a coin, $Probability(Heads) = Pr(H) = Pr(T) = \frac{1}{2}$ is the simplest case. For a picture we let $H = (1,1) =$ Up step, $T = (1,-1) =$ Down step, see Figure 2. If we analyze this by the first crossing of the x-axis we are back to Dyck paths again.

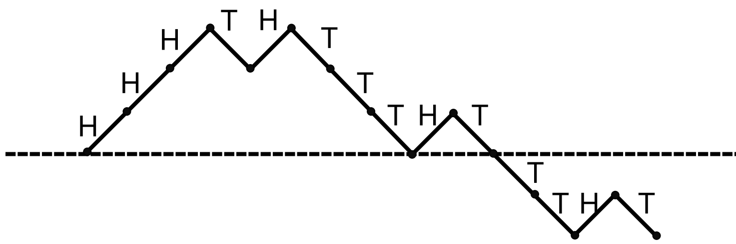


Figure 2: The example

Question 1.3 *: How many ideals are in the ring of upper triangular* 2×2 *matrices over a field* F*?*

Solution. Let $T_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F \right\}$. There are 5 ideals satisfying the condition. For T_n the answer is C_{n+1}, and the number of commutative ideals is 2^n, see [11], [14] (Ch6.24(a)). An interesting project is to look at the varied settings where the Catalan numbers arise and see which 2^n subobjects are counted by 2^n.

Percolation

Start with an $n \times n$ permutation matrix. Put one 1 in each row and column with 0's elsewhere. The transition rule is: any square next to two diagonal 1's becomes a 1. Repeat until no further change is possible, see the Figure 3.

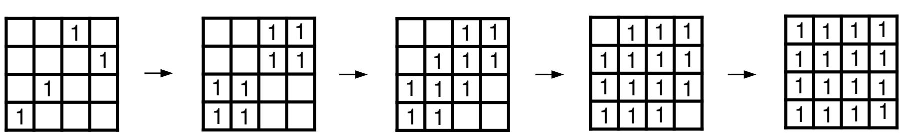


Figure 3: The process

How many of $n!$ possible starting positions fill up entirely? We tabulate the first six terms

n!	$1, 2, 6, 24, 120, 720, \cdots$
Fill up entirely	$1, 2, 6, 22,\ 90\ , 394, \cdots$

.

It turns out that the number of matrices that fill up entirely is given by the large Schröder numbers. Since the rate of growth of these Schröder numbers approaches $3+2\sqrt{2}$, we have

$$\lim_{n\to\infty}\frac{|\text{fill up entirely}|}{n!}=0.$$

Also the following property holds (Kaplansky 1945)

$$\lim_{n\to\infty}\frac{|\text{no growth at all}|}{n!}=e^{-2}.$$

For more information, see [14] (Ch. 6.39(k)).

Some Open Questions and Projects

Let

$$C(z)=\frac{1-\sqrt{1-4z}}{2z}=\sum_{n=0}^{\infty}C_nz^n=\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}z^n=1+z+2z^2+5z^3+14z^4+\cdots$$

and

$$B(z)=\frac{1}{\sqrt{1-4z}}=\sum_{n=0}^{\infty}\binom{2n}{n}z^n=1+2z+6z^2+20z^3+\cdots.$$

We often abbreviate $C(x)$ to C, this is the Catalan number's generating function. Similarly, $B(x)=B$ is the generating function for the Central Binomial Coefficients. We list some open problems and projects as follows:

(A) Give a combinatorial proof (without cross multiplying) that

$$\frac{BC}{2C-B}=\frac{1}{1-3zC}.$$

(B) Count the Dyck paths (or Motzkin paths) with a unique highest point.

(C) q-count the seventeen Catalan identities.

(D) In the Catalan settings find the 2^n subthings that are bijectively equivalent to the commutative ideals in T_n.

See [14] *Vol.* 2, Ch. 6.19 and *Catalan addendum* at the website:

http://www-math.mit.edu/~rstan/ec/.

(E) Give a combinatorial proof of the following matrix identity

$$\begin{bmatrix}1&&&&&\\3&1&&&&\\11&6&1&&&\\45&31&9&1&&\\197&156&60&12&1&\\&&\cdots&&&\ddots\end{bmatrix}\begin{bmatrix}1\\3\\7\\15\\31\\\vdots\end{bmatrix}=\begin{bmatrix}1\\6\\6^2\\6^3\\6^4\\\vdots\end{bmatrix}.$$

Where the first column of the matrix is the little Schröder numbers and $a_{i,j}$ can be obtained by $a_{i-1,j-1} + 3a_{i-1,j} + 2a_{i-1,j+1}$ in the matrix. The number 6 in the righthand side (RHS for short) of the identity could mean unit steps up, down, right, left, forward, or backward.

2 The Fundamental Theorem of Riordan Arrays

We can easily see the following results for Dyck paths of length $2n$:

(1) The minimal number of points on the x-axis is 2;

(2) The maximal number of points on the x-axis is $n+1$.

Here, we have a question: what is the average number of such points?

For $n = 2 \cdot 10^6$, we could propose arguments for $\frac{10^6}{2}$, $\sqrt{10^6}$, and $\log_2 10^6$ as reasonable. There are approximately 5 000 000, 1 000 and 20. You may have a guess of your own. We will return to this question after introducing the fundamental theorem.

2.1 Introduction

We can obtain the identity

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1} \tag{2.1}$$

by the following combinatorial interpretation.

There are n students. We could select a group of k students to serve in the council of our school and then choose the president from the council :

(i) Choose k students from the n students as the council, where $1 \leq k \leq n$. This can be done in $\binom{n}{k}$ ways;

(ii) Select a president from the k students;

(iii) Sum over all the possible numbers k, and we will get the lefthand side (briefly LHS) of the identity.

There is also another way as follows:

(i) Choose one of the n students as the president. There are n possible choices;

(ii) Then there are 2^{n-1} different choices to select the rest of the council from the other $n-1$ students;

(iii) In all, there are $n2^{n-1}$ distinct ways to do this, which is exactly the RHS of the identity.

Obviously, these two ways are equivalent to each other. Thus, we get the above identity.

The book [2], *Proofs that Really Count: The Art of Combinatorial Proof*, by Arthur T. Benjamin, Jennifer J. Quinn and William Watkins is fun to read and we will introduce the fundamental theorem by way of this example. It also has many similar proofs.

Step.1 Try some small cases.

Suppose we ignore the proof we have just done. Assume you were handed a new identity and asked to prove it. You might start by checking the identity when $n = 3$ and $n = 4$:

$$\binom{3}{0} \times 0 + \binom{3}{1} \times 1 + \binom{3}{2} \times 2 + \binom{3}{3} \times 3 = 3 \times 2^2$$

$$\binom{4}{0} \times 0 + \binom{4}{1} \times 1 + \binom{4}{2} \times 2 + \binom{4}{3} \times 3 + \binom{4}{4} \times 4 = 4 \times 2^3.$$

Step.2 Write these and some other small cases out as a lower triangular matrix times a column vector equals a column vector.

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \times 2^0 \\ 2 \times 2^1 \\ 3 \times 2^2 \\ 4 \times 2^3 \\ \vdots \end{bmatrix} \tag{2.2}$$

Step.3 Write down the generating functions (*GF*s for short) for each column.

As we know, by the *Binomial Theorem*, the GF of the n^{th} row is $(1+x)^n$. But we are interested in the columns instead, and the GFs of first three columns are

$$\frac{1}{1-z}, \quad \frac{z}{(1-z)^2}, \quad \text{and} \quad \frac{z^2}{(1-z)^3}.$$

The GFs of the two column vectors on the LHS and RHS of (2.2) are

$$A(z) = \frac{z}{(1-z)^2} \quad \text{and} \quad B(z) = \frac{z}{(1-2z)^2},$$

respectively.

This is just a special case of the following situation.

Definition 2.1 *A lower triangular infinite matrix, L, is a Riordan Array, if the GF of the* k^{th} *column is*

$$g(z)f(z)^k,$$

for $k = 0, 1, 2, 3, \cdots,$ *where*

$$g(z) = 1 + g_1 z + g_2 z^2 + g_3 z^3 + \cdots,$$

$$f(z) = \qquad z + f_2 z^2 + f_3 z^3 + \cdots.$$

Notation 2.2 *Let $L = (g(z), f(z))$ denote the corresponding Riordan array. [Note: we only need $f(0) = 0$ at this point.]*

For example,

$$P\text{=}Pascal\text{'}s\ triangle = (\frac{1}{1-z}, \frac{z}{1-z}).$$

2.1.1 Fundamental Theorem of Riordan Arrays (FTRA).

Theorem 2.3 *The Fundamental Theorem of Riordan Arrays*

$$(g(z), f(z)) \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix} \tag{2.3}$$

where the generating functions of the two column vectors are $A(z)$ and $B(z)$, respectively. Then this identity is true if and only if the following equation holds

$$g(z)A(f(z)) = B(z). \tag{2.4}$$

Proof. We look at the Riordan Array $(g(z), f(z))$ column by column, and multiply it by the column vector on the LHS

$$\begin{bmatrix} | & & & & & \\ | & | & & & & \\ g & gf & gf^2 & & & \\ | & | & | & | & & \\ | & | & | & | & | & \\ & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{bmatrix} \tag{2.5}$$

This yields

$$a_0 g + a_1 gf + a_2 gf^2 + \cdots = g(a_0 + a_1 f + a_2 f^2 + \cdots) = g(z) \cdot A(f(z)) = B(z),$$

and we have our result. ∎

For example, returning to the student council and (2.1)

$$(\frac{1}{1-z}, \quad \frac{z}{1-z})\frac{z}{(1-z)^2} = \frac{1}{1-z} \cdot \frac{\frac{z}{1-z}}{(1-\frac{z}{1-z})^2} = \frac{z}{(1-2z)^2}.$$

Since

$$B(z) = \frac{z}{(1-2z)^2} = z + 2z(2z) + 3z(2z)^2 + \cdots + n2^{n-1}z^n + \cdots = \sum n2^{n-1}z^n,$$

we have the GF for the column vector on the RHS.

We now return to our first example.

2.2 The average number of points on the x-axis of Dyck paths

Let k denote the number of points on the x-axis of Dyck paths.

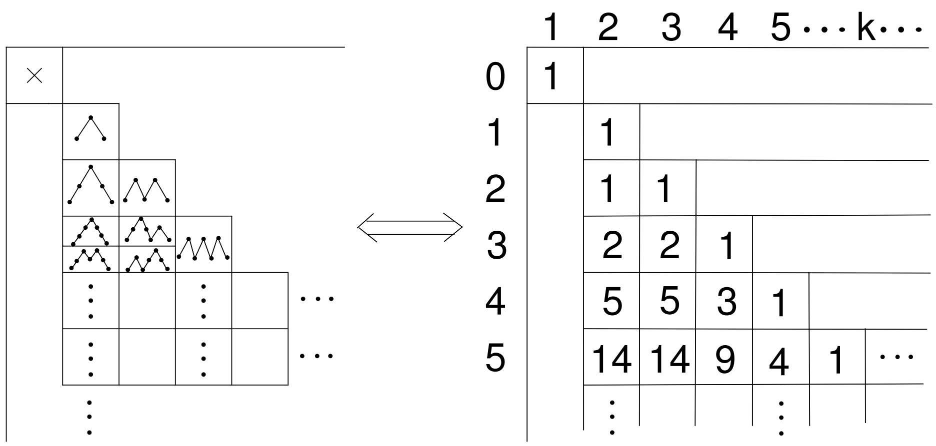


Figure 4: Dyck paths and the number of paths with points on the x-axis

In the table on the right side, the numbers in the leftmost column denote n; the numbers in the topmost row are the number of points on the x-axis corresponding to such paths. After we fill in the blanks with 0, we get a matrix which we denote $\mathcal{C}$, since the row sums as well as 2 of the columns are the Catalan Numbers.

Question: Is $\mathcal{C}$ a Riordan Array?

The GFs for the column in $\mathcal{C}$ are 1, zC, z^2C^2, $z^3C^3, \cdots$. Thus we have that $g(z)$ is 1, and $f(z)$ is zC, where C is the short for the GF of the Catalan Numbers. Thus $(g(z), f(z)) = (1, zC)$ is a Riordan array in notation. Now we want to show that $(1, zC)A(z) = C$, where $A(z)$ is the GF of some number sequence.

From the matrix $\mathcal{C}$, it is easy to verify that this is possible , since

$$\begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ 0 & 2 & 2 & 1 & & & \\ 0 & 5 & 5 & 3 & 1 & & \\ 0 & 14 & 14 & 9 & 4 & 1 & \\ & & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 14 \\ 42 \\ \vdots \end{bmatrix}. \tag{2.6}$$

Denote the GFs of the two column vectors by $A(z)$ and $B(z)$, then

$$A(z) = \frac{1}{1-z},$$

$$B(z) = C(z).$$

Thus,

$$g(z) \cdot A(f(z)) = 1 \cdot \frac{1}{1 - zC} = C = B(z),$$

since $C = 1 + zC^2$.

By FTRA, we have proven that the left hand column has C as its generating function as we wanted.

Let us consider the average number of the points on the x-axis of Dyck paths of length $2n$. From the table above, we know that

$$\begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 2 & 2 & 1 & & \\ 0 & 5 & 5 & 3 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 14 \\ 42 \\ \vdots \end{bmatrix}. \tag{2.7}$$

The entries of the vector in the RHS of this equation are the number of points on the x-axis, which look like the Catalan Numbers, C_{n+1}.

The proof of this is similar to the last proof except now $A(z) = \frac{1}{(1-z)^2}$.

By the $FTRA$,

$$(1, zC)\frac{1}{(1-z)^2} = 1 \cdot \frac{1}{(1-zC)^2} = C^2 = \frac{C-1}{z},$$

which is the generating function for $\sum_{n=0}^{\infty} C_{n+1} z^n$.

The average number of points on the x-axis is

$$\frac{C_{n+1}}{C_n} = \frac{\frac{1}{n+2}\binom{2n+2}{n+1}}{\frac{1}{n+1}\binom{2n}{n}} = \frac{4n+2}{n+2}.$$

As $n \to \infty$, the limit is 4.

Note that by the bijection between Dyck paths and trees, which is similar to preorder traversal, we find that the number of points on the x-axis equals $deg(r) - 1$, where $deg(r)$ denotes the degree of the root. This is a classic result of [6].

3 Riordan Group–The plot continues to be unfold

3.1 Recall

The average number of returns to the x-axis of Dyck paths approaches 4.

Euler discovered that the number of ways to triangulate an $(n+2)$-gon is C_n. We pick such an $(n+2)$-gon and consider each triangle to be a room in a museum. Figure 5 illustrates a typical case. A young man enters the museum through the bottom door. His first plan not loving the art museum experience is to always take the door on the left. His second plan applies when he is walking with a good friend who likes museums. It is the same unless the left hand door leads to the outside when the right hand door is chosen instead. In Figure 5, the first plan yields 2 rooms while the second yields 4. Now assume that n is large and that we are choosing our Euler museum at random uniformly. Then under the first plan the average number of rooms visited approaches 3 while with plan 2 the number approaches 8 [10].

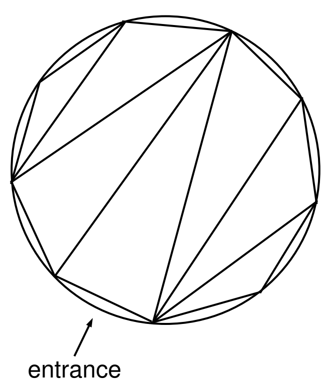


Figure 5: Euler's art museum

Recall also the following from Section 2. A matrix L is a Riordan Array if the k^{th} column of L has gf^k as is GF with $g = 1+g_1z+g_2z^2+\cdots$ and $f(0) = 0$. The notation is $(g(z), f(z))$ or (g, f) .

The fundamental theorem of Riordan array (denoted by FTRA) is

$$(g(z), f(z)) \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{bmatrix} \quad \text{iff} \quad g(z)A(f(z)) = B(z).$$

Let's try another example using the FTRA. We let $b_n = \sum_{k\geq 0} \binom{n-k}{k} 6^k$, and we want to find a closed form expression for b_n.

Step 1

$$\begin{aligned}
&\binom{0}{0} && = 1\\
&\binom{1}{0} && = 1\\
&\binom{2}{0} + \binom{1}{1}6 && = 7\\
&\binom{3}{0} + \binom{2}{1}6 && = 13\\
&\binom{4}{0} + \binom{3}{1}6 + \binom{2}{2}6^2 && = 55;
\end{aligned}$$

Step 2

$$\begin{bmatrix} 1 & & & & \\ 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & & & \\ 1 & 3 & 1 & & \\ 1 & 4 & 3 & & \\ 1 & 5 & 6 & 1 & \\ & & \cdots & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 6^2 \\ 6^3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 7 \\ 13 \\ 55 \\ \vdots \end{bmatrix};$$

Step 3 Easily, we see that $g(z) = \frac{1}{1-z}$, $f(z) = \frac{z^2}{1-z}$ and $A(z) = \frac{1}{1-6z}$;

Step 4 By FTRA, we have

$$\begin{aligned}
(\frac{1}{1-z}, \frac{z^2}{1-z})(\frac{1}{1-6z}) &= \frac{1}{1-z} \cdot \frac{1}{1-6\frac{z^2}{1-z}} = \frac{1}{1-z-6z^2} \\
&= \frac{1}{(1-3z)(1+2z)} = \frac{1}{5}(\frac{3}{1-3z} + \frac{2}{1+2z}).
\end{aligned}$$

Then $b_n = [z^n]\frac{1}{5}(\frac{3}{1-3z} + \frac{2}{1+2z}) = \frac{1}{5}(3^{n+1} + (-1)^n 2^{n+1})$.

3.2 Riordan Group

Definition 3.1 *The Riordan group*

$$R = \{(g(z), f(z)) | (g(z), f(z)) \textit{ is a Riordan array and } f(z) = f_0 + f_1 z + f_2 z^2 + f_3 z^3 + \cdots,$$

where $f_0 = 0,\ f_1 = 1\}$*, i.e., each member of* R *is a lower triangular matrix with* 1*'s on the main diagonal. The multiplication in* R *is* $(g(z), f(z))(G(z), F(z)) = (g(z)G(f(z)), F(f(z)))$*. The identity is* $I = (1, z)$*. The inverse of* $(g(z), f(z))$ *is* $(\frac{1}{g(\bar{f}(z))}, \bar{f}(z))$*, where* $\bar{f}(z)$ *is the compositional inverse of* $f(z)$*, i.e.,* $f(\bar{f}(z)) = \bar{f}(f(z)) = z$*.*

To check the inverse property, we compute

$$(\frac{1}{g(\bar{f}(z))}, \bar{f}(z))(g(z), f(z)) = (\frac{1}{g(\bar{f}(z))} g(\bar{f}(z)), f(\bar{f}(z))) = (1, z).$$

3.3 Subgroups

Some of the important subgroups of R are the Appel, Associated, Checkerboard, Bell, Stochastic and Commutator subgroups.

(I) The Appel subgroup is $\{(g(z), z)\}$.

One example is

$$\begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 2 & 1 & 1 & & & & \\ 4 & 2 & 1 & 1 & & & \\ 8 & 4 & 2 & 1 & 1 & & \\ 16 & 8 & 4 & 2 & 1 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix}.$$

(II) The associated subgroup is $\{(1, f(z))\}$.

One example is

$$\begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ 0 & 2 & 2 & 1 & & & \\ 0 & 5 & 5 & 3 & 1 & & \\ 0 & 14 & 14 & 9 & 4 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix}.$$

(III) The Checkerboard subgroup is all $\{(g(z), f(z)) \mid g(z)$ is an even function and $f(z)$ is an odd function$\}$. For instance,

$$\begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 2 & 0 & 1 & & & & \\ 0 & 3 & 0 & 1 & & & \\ 6 & 0 & 4 & 0 & 1 & & \\ 0 & 10 & 0 & 5 & 0 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix} = (B(z^2), zC(z^2)).$$

(IV) The Bell subgroup is all $\{(g(z), zg(z))\}$.

Pascal's triangle $= (\frac{1}{1-z}, \frac{z}{1-z})$ is an example.

(V) The Stochastic subgroup is

$$\left\{ (g(z),f(z)) \quad | \quad (g(z),f(z)) \begin{bmatrix} 1\\1\\1\\ \vdots \end{bmatrix} = \begin{bmatrix} 1\\1\\1\\ \vdots \end{bmatrix} \right\} = \text{stabilizer of} \begin{bmatrix} 1\\1\\1\\ \vdots \end{bmatrix}.$$

(VI) The Commutator subgroup is (almost certainly)

$$H := R \cap \left\{ \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ \times & 0 & 1 & & & \\ \times & \times & 0 & 1 & & \\ \times & \times & \times & 0 & 1 & \\ & & \dots & & & \ddots \end{bmatrix} \right\}.$$

$H \subseteq$ Commutator and probably H=Commutator (This last is unsolved but work is currently in progress).

(VII) The Derivative subgroup is $\{(g(z),f(z)) \mid f'(z) = g(z)\}$.

Of all these subgroups only the Appel and the Commutator are normal. We also can decompose R as the semidirect product of the Appel subgroup and the Associated subgroup. To see this, note first that

$$\text{Appel} \cap \text{Associated} = \{(1,z)\}$$

and then

$$(g(z),f(z)) = (g(z),z)(1,f(z)).$$

Note $(1,-z)$ is an element of order 2, where we allow $f(z) = \pm z + f_2 z^2 + \cdots$. Each element of order 2 generates a subgroup of order 2. Two other examples are

$$\begin{bmatrix} 1 & & & & & \\ 1 & -1 & & & & \\ 1 & -2 & 1 & & & \\ 1 & -3 & 3 & -1 & & \\ 4 & -4 & 6 & -4 & 1 & \\ & & \dots & & & \ddots \end{bmatrix} = \left(\frac{1}{1-z}, \frac{-z}{1-z} \right)$$

and somewhat surprisingly

$$\begin{bmatrix} 1 & & & & & & & \\ 1 & -1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ 2 & -3 & 3 & -1 & & & & \\ 4 & -4 & 6 & -4 & 1 & & & \\ 8 & -13 & 13 & -10 & 5 & -1 & & \\ 17 & -28 & 30 & -24 & 15 & -6 & 1 & \\ & & \dots & & & & & \ddots \end{bmatrix},$$

where the first column of the above matrix is the GF for RNA secondary structures [14].

3.4 Open Question

Is there a systematic way to classify elements of order two?

4 Riordan Group

4.1 How to find $f = f(z)$?

Having been given a matrix L in which we suspect is Riordan and $g(z)$, how can we find $f(z)$?

Method 1 Comparing the coefficients of gf and g.

For example,

$$L = \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 1 & 0 & 1 & & & & & \\ 0 & 2 & 0 & 1 & & & & \\ 2 & 0 & 3 & 0 & 1 & & & \\ 0 & 5 & 0 & 4 & 0 & 1 & & \\ 5 & 0 & 9 & 0 & 5 & 0 & 1 & \\ & & & & \cdots & & & \ddots \end{bmatrix}.$$

In this matrix, the nonzero entries of the first two columns are Catalan numbers. The GFs of the first two columns are g and gf. As we know, $g(z) = \sum_{n=0}^{\infty} C_n z^{2n}$. Then, we find $f = f(z)$ by comparing the coefficients of $gf(z)$ and $g(z)$ as follows:

$$\begin{array}{lll} & g = 1 + 0z + z^2 + 0z^3 + 2z^4 + 0z^5 + 5z^6 + 0z^7 + 14z^8 + \cdots & (1) \\ (1) \text{ and } (3) \Longrightarrow (2) & f = 0 + z + 0z^2 + z^3 + 0z^4 + 2z^5 + 0z^6 + 5z^7 + 0z^8 + 14z^9 + \cdots & (2) \\ \hline & gf = 0 + z + 0z^2 + 2z^3 + 0z^4 + 5z^5 + 0z^6 + 14z^7 + \cdots & (3). \end{array}$$

Therefore, we can guess that $f = f(z) = zC(z^2)$. We can then compute $g(z)f(z)^2$ to see if we are getting the coefficients of the next column.

Method 2 Recursions or "dot diagrams".

For another example,

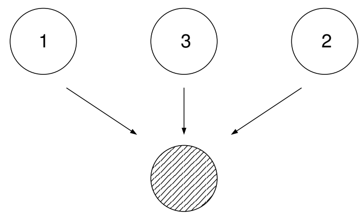
$$L = \begin{bmatrix} 1 & & & & & \\ 3 & 1 & & & & \\ 11 & 6 & 1 & & & \\ 45 & 31 & 9 & 1 & & \\ 197 & 156 & 60 & 12 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix}.$$

We might be interested since $1, 3, 11, 45, 197, \ldots$ are the little Schröder numbers.

There is a recursion between four entries of any two adjacent rows as follows:

$$a_{n,k-1} + 3a_{n,k} + 2a_{n,k+1} = a_{n+1,k}.$$

The picture for this is



Thus, in the Riordan group, we look at the k^{th} column and we find that

$$gf^k = z(gf^{k-1} + 3gf^k + 2gf^{k+1}).$$

Because we need to move down one row in the matrix, the equation contains the term z. Dividing by gf^{k-1}, we obtain

$$\begin{aligned} f &= z(1 + 3f + 2f^2) \\ &= \frac{1 - 3z - \sqrt{1 - 6z + z^2}}{4z} \\ &= z(1 + 3z + 11z^2 + 45z^3 + \cdots). \end{aligned}$$

If we denote $1 + 3z + 11z^2 + 45z^3 + \cdots$ by $g(z)$, then $f(z) = zg(z)$ and

$$g(z) = 1 + 3f + 2f^2 = 1 + 3zg + 2zg \cdot f = 1 + z(3g + 2gf).$$

According to this, we get the following Riordan Array

$$\begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 11 & 6 & 1 & & \\ 45 & 31 & 9 & 1 & \\ & & \cdots & & \ddots \end{bmatrix} = (g(z), zg(z)),$$

where $g = \frac{1-3z-\sqrt{1-6z+z^2}}{4z^2}$. Thus, we are in the Bell subgroup.

Next we want to look at several approaches to obtaining the inverse in the Riordan group.

Method 1 By inspection.

Proceeding one diagonal at a time, the computations are easy.

For example, let

$$L = \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 1 & 0 & 1 & & & & & \\ 0 & 2 & 0 & 1 & & & & \\ 2 & 0 & 3 & 0 & 1 & & & \\ 0 & 5 & 0 & 4 & 0 & 1 & & \\ 5 & 0 & 9 & 0 & 5 & 0 & 1 & \\ & & & & \cdots & & & \ddots \end{bmatrix} = (C(z^2), zC(z^2)).$$

By inspection, we first work down to

$$\begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ -1 & 0 & 1 & & & & & \\ & -2 & 0 & 1 & & & & \\ & & -3 & 0 & 1 & & & \\ & & & -4 & 0 & 1 & & \\ & & & & -5 & 0 & 1 & \\ & & & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 1 & 0 & 1 & & & & & \\ 0 & 2 & 0 & 1 & & & & \\ 2 & 0 & 3 & 0 & 1 & & & \\ 0 & 5 & 0 & 4 & 0 & 1 & & \\ 5 & 0 & 9 & 0 & 5 & 0 & 1 & \\ & & & & \cdots & & & \ddots \end{bmatrix} = I$$

and after a few more diagonal we see that

$$(C(z^2), zC^2(z))^{-1} \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ -1 & 0 & 1 & & & & & \\ 0 & -2 & 0 & 1 & & & & \\ 1 & 0 & -3 & 0 & 1 & & & \\ 0 & 3 & 0 & -4 & 0 & 1 & & \\ -1 & 0 & 6 & 0 & -5 & 0 & 1 & \\ & & & & \cdots & & & \ddots \end{bmatrix}.$$

This yields the inverse of the matrix. In this inverse, if we ignore the signs, we find that the row sums are the Fibonacci Numbers. This illustrates a loose inverse relation between the two famous series, the Catalan numbers and Fibonacci numbers.

Method 2   We can do this rigorously using the Riordan group inverse.

In the above example, we recall that $C(z) = 1 + z(C(z))^2$.

Since

$$\begin{aligned} f(z) &= zC(z^2) \\ &= z[1 + z^2[C(z^2)]^2] \\ &= z + z \cdot [zC(z^2)]^2, \end{aligned}$$

we have

$$f(z) = z + z \cdot f(z)^2.$$

Replace z by $\overline{f}(z)$ and we get

$$\begin{aligned} z &= \overline{f}(z) + \overline{f}(z) \cdot z^2 \\ &= \overline{f}(z)(1 + z^2) \\ &\Longrightarrow \overline{f}(z) = \frac{z}{1 + z^2}, \end{aligned}$$

since $\overline{f}(f(z)) = f(\overline{f}(z)) = z$.

$$\begin{aligned} g(\overline{f}(z)) &= C(\overline{f}(z)^2) \\ &= C(\frac{z^2}{(1+z^2)^2}) \\ &= 1+z^2. \end{aligned}$$

It follows that

$$(\frac{1}{g(\overline{f}(z))}, \overline{f}(z)) = (\frac{1}{1+z^2}, \frac{z}{1+z^2}) = L^{-1}.$$

4.2 Exponential Generating Functions and the Riordan Group.

We now can double our list of examples by making one simple change in our definition.

The exponential generating function of the k^{th} column of L is

$$g(z)\frac{(f(z))^k}{k!} \qquad k = 0, 1, 2, 3, \cdots.$$

For instance,

(1)

$$P = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ & & & \cdots & & & \ddots \end{bmatrix} = (e^z, z).$$

To verify this note that

$$g(z) = e^z = 1 + 1 \cdot z + 1 \cdot \frac{z^2}{2!} + \cdots$$

and

$$e^z \cdot \frac{z^k}{k!} = \sum \frac{z^{n+k}}{n!k!} = \sum_{n \geq k} \binom{n}{k} \frac{z^n}{n!}.$$

(2)

$$P = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ 0 & 2 & 3 & 1 & & & \\ 0 & 6 & 11 & 6 & 1 & & \\ 0 & 24 & 52 & 35 & 10 & 1 & \\ & & & \cdots & & & \ddots \end{bmatrix} = (1, ln\frac{1}{1-z}) =: St(1).$$

Recall that

$$\begin{aligned}\ln\frac{1}{1-x} &= \int \frac{1}{1-x}dx \\ &= \int (1+x+x^2+\cdots)dx \\ &= x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}+\cdots \\ &= \sum_{n=0}(n-1)!\frac{x^{n+1}}{n!} \\ &= x+\frac{x^2}{2!}+2!\frac{x^3}{3!}+\cdots .\end{aligned}$$

(3)

$$P=\begin{bmatrix}1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ 0 & 1 & 3 & 1 & & & \\ 0 & 1 & 7 & 6 & 1 & & \\ 0 & 1 & 15 & 25 & 10 & 1 & \\ & & & \cdots & & & \ddots\end{bmatrix} = (1, e^z-1) =: St(2),$$

which are the Stirling numbers.

The Stirling numbers of the first kind are denoted $s(n,k)$. Recall that $s(n,k)$ is the number of permutations on n letters that can be written as a product of k disjoint cycles.

The Stirling numbers of the second kind are denoted $S(n,k)$. Recall that $S(n,k)$ is the number of ways to partition a set with n elements into k disjoint blocks.

The Bell numbers count all ways of partitioning a set with n elements into blocks. In other words, $B_n = \sum_{k\geq 0} S(n,k)$. From the relation between Stirling numbers and Bell numbers, we have immediately that

$$\begin{bmatrix}1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ 0 & 1 & 3 & 1 & & & \\ 0 & 1 & 7 & 6 & 1 & & \\ 0 & 1 & 15 & 25 & 10 & 1 & \\ & & & \cdots & & & \ddots\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\\1\\1\\ \vdots\end{bmatrix} = \begin{bmatrix}1\\1\\2\\5\\15\\52\\ \vdots\end{bmatrix}.$$

By applying the fundamental theorem of Riordan Arrays, we now have a one line derivation of the generating function for the Bell Numbers

$$(1, e^z-1)\cdot e^z = 1\cdot e^{e^z-1} = B(z).$$

4.3 The Riordan Group meets Hankel matrices and the Stieltjes transform.

First we look at the telephone exchange problem. If a telephone exchange has n subscribers, in how many ways can people be talking? We call this number T_n and we have no conference calls. See the following Figure 6 as an example.

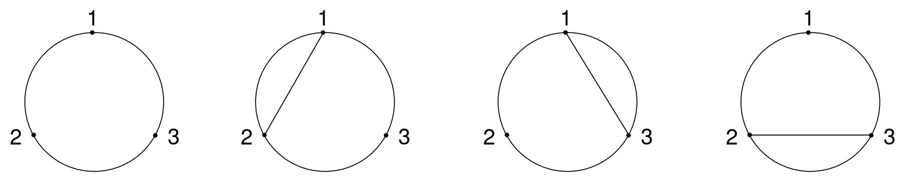


Figure 6: The example showing that $T_3 = 4$

Suppose by dividing into cases and recording our results for small n that we have

n	0	1	2	3	4	5	6	7	8	$\cdots$
T_n	1	1	2	4	10	26	76	232	764	$\cdots$

We now form a Hankel matrix with constant terms from our sequence on the off diagonal.

Step 1. Let

$$H_T = \begin{bmatrix} 1 & 1 & 2 & 4 & 10 & \\ 1 & 2 & 4 & 10 & 26 & \\ 2 & 4 & 10 & 26 & 76 & \cdots \\ 4 & 10 & 26 & 76 & 232 & \\ 10 & 26 & 76 & 232 & 764 & \\ & & & \cdots & & \ddots \end{bmatrix}$$

and do row reduction (Gauss elimination). This yields

$$\begin{bmatrix} 1 & 1 & 2 & 4 & 10 & \\ & 1 & 2 & 6 & 16 & \\ & & 2 & 6 & 24 & \cdots \\ & & & 6 & 24 & \\ & & & & 24 & \\ & & \cdots & & & \ddots \end{bmatrix}.$$

Step 2. By the Fundamental Theorem of Applied Linear Algebra,

$$H_T = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 4 & 6 & 3 & 1 & & \\ 10 & 16 & 12 & 4 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 0! & & & & & \\ & 1! & & & & \\ & & 2! & & & \\ & & & 3! & & \\ & & & & 4! & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 4 & 10 & \\ & 1 & 2 & 6 & 16 & \\ & & 1 & 3 & 12 & \\ & & & 1 & 4 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix} = LDU.$$

This yields

$$\begin{bmatrix} 1 & 1 & 2 & 4 & 10 & \\ & 1 & 2 & 6 & 16 & \\ & & 2 & 6 & 24 & \cdots \\ & & & 6 & 24 & \\ & & & & 24 & \\ & & \cdots & & & \ddots \end{bmatrix}.$$

Here L is a lower triangular with 1's on the main diagonal, U is a upper triangular with 1's on the main diagonal and D is a diagonal matrix. Since the Hankel matrix is symmetric, $L = U^T$.

Remarks

(1.) The principal submatrices have determinant $d_0 d_1 d_2 \cdots d_n$, which is the product of diagonal elements.

(2.) Often, L is a Riordan matrix.

(3.) Project: Which sequences yield a given determinant sequence?

(4.) We can examine L to learn more.

Let's continue with the same example as an case where more can learned.

$$L = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 4 & 6 & 3 & 1 & & \\ 10 & 16 & 12 & 4 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} = (g(z), f(z)).$$

By solving the first few terms in

$$g(z)f(z) = z + 2\frac{z^2}{2!} + 6\frac{z^3}{3!} + 16\frac{z^4}{4!} + \cdots = (1 + z + 2\frac{z^2}{2!} + 4\frac{z^3}{3!} + 10\frac{z^4}{4!} + \cdots)f(z),$$

we see that $f(z)$ seems to be just $f(z) = z$. This indicates that we are in the Appel subgroup.

We next note that $l_{n+1,0} = l_{n,0} + l_{n,1}$. In terms of generating functions this means

$$g(z) = 1 + \int (g(z) + g(z)z)dz.$$

Why the integral sign? We want to move down one row, we are using exponential generating functions, and $\int \frac{z^n}{n!}dz = \frac{z^{n+1}}{(n+1)!}$ does the job. Hence,

$$\begin{aligned} g' &= g + zg = g(1+z) \\ &\Longrightarrow \frac{g'}{g} = 1 + z \\ &\Longrightarrow g = e^{z+\frac{z^2}{2}}. \end{aligned}$$

We have been looking at the telephone exchange setting, but we are also counting n elements in the symmetric group S_n where $x^2 = I$. Also we are counting symmetric $n \times n$ permutation matrices and standard Young tableaux. The generating function for the Hermite polynomials is closely related to $e^{-t^2+2tx} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$. At this point the Encyclopedia of Integer Sequences leads to supply much more valuable information.

The web site is *http://www.research.att.com/~njas/sequences/index.html*.

Let $l(n, k)$=the number of arrangements where k people are sitting anxiously by the phone. Set $n = m + q$, then $T_n = \sum_{k=0}^{\infty} l(m, k)k!l(q, k)$, see Figure 7 for example. We break

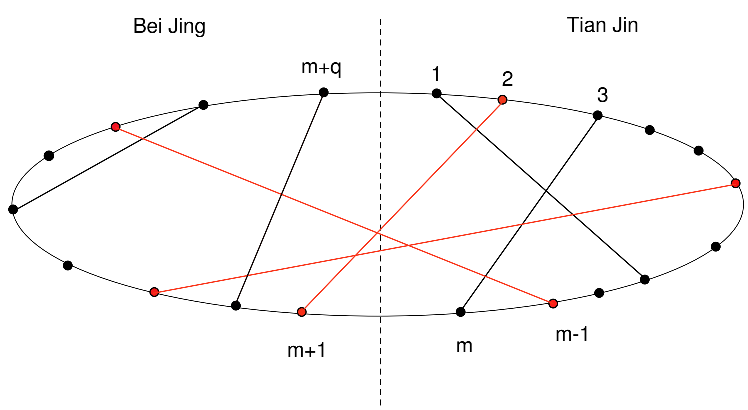


Figure 7: The telephone exchange

the n subscribers into m subscribers in one group and q in the other, let k be the number of calls between the two groups. Note that T_n depends on $n = m + q$ but not on m or q individually. Thus we can get constant terms on the off diagonals. This proves that our Hankel matrices LDU decomposition is indeed valid.

Production matrices

We start with a lower triangular matrix L. Remove the top row and move the other rows up, call this matrix $\bar{L}$. Solve $LP = \bar{L}$ and P is the production matrix we want, i.e., $P = L^{-1}\bar{L}$.

Following the same example

$$\begin{bmatrix} 1 &&&&& \\ 1 & 1 &&&& \\ 2 & 2 & 1 &&& \\ 4 & 6 & 3 & 1 && \\ 10 & 16 & 12 & 4 & 1 & \\ && \cdots &&& \ddots \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \\ 1 & 1 & 1 & 0 & 0 & \\ 0 & 2 & 1 & 1 & 0 & \\ 0 & 0 & 3 & 1 & 1 & \\ 0 & 0 & 0 & 4 & 1 & \\ && \cdots &&& \ddots \end{bmatrix} = \begin{bmatrix} 1 & 1 &&&&& \\ 2 & 2 & 1 &&&& \\ 4 & 6 & 3 & 1 &&& \\ 10 & 16 & 12 & 4 & 1 && \\ 26 & 50 & 40 & 20 & 5 & 1 & \\ && \cdots &&&& \ddots \end{bmatrix}.$$

Thus

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \\ 1 & 1 & 1 & 0 & 0 & \\ 0 & 2 & 1 & 1 & 0 & \\ 0 & 0 & 3 & 1 & 1 & \\ 0 & 0 & 0 & 4 & 1 & \\ && \cdots &&& \ddots \end{bmatrix}.$$

Note that it is easy to find the first few rows of P even without computing L^{-1}.

We have indicated that $T(z) = g(z) = 1 + z + 2\frac{z^2}{2!} + 4\frac{z^3}{3!} + \cdots$, but going over to ordinary generating functions we can use the columns of P to write down the continued fraction expansion as follows:

$$\hat{T}(z) = 1 + z + 2z^2 + 4z^3 + 10z^4 + \cdots = \cfrac{1}{1 - z - \cfrac{z^2}{1 - z - \cfrac{2z^2}{1 - z - \cfrac{3z^2}{\cdots}}}}.$$

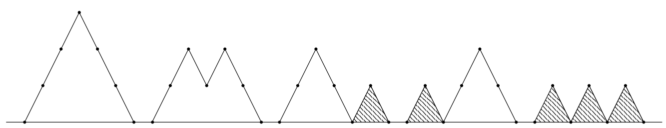
The continued fraction expansion can be terminated after k steps to give the k^{th} *Páde* approximation. Other applications of Production matrices will be discussed in later lectures. The book, *Combinatorial Enumeration*, by Goulden and Jackson, particularly Section 5.2, discusses many examples of tridiagonal production matrices using the language of continued fractions.

5 The average number of hills in Dyck paths

We return to our first example of Dyck path.

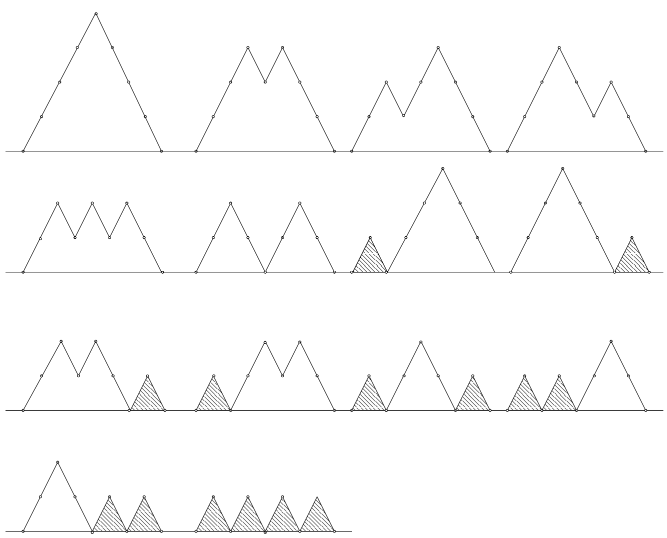
Definition 5.1 *A hill is a UD pair of steps that forms a peak of height* 1*, and we denote the number of hills in Dyck paths of length* $2n$ *by* H_n*.*

When $n = 3$, $C_3 = 5$, see the following figure.



Thus, $H_3 = 5$ and the average number of hills is $5/5 = 1$.

When $n = 4$, $C_4 = 14$, we have



Thus, $H_3 = 14$ and the average number of hills is still $14/14 = 1$.

Can we prove that $H_n = C_n$ for all $n \geq 1$? The answer is YES! We will prove this conclusion using the fundamental theorem, the FTRA.

Let

$$\mathcal{F} = \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ 2 & 2 & 0 & 1 & & \\ 6 & 4 & 3 & 0 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} = (F, zF).$$

The numbers in the k^{th} column are the numbers of Dyck paths of length $2n$ with k hills, for $k = 0, 1, 2, \cdots$.

Denote the GF of the number of Dyck paths with no hills by $F(z)$, then $F(z) = 1 + 0 \cdot z + 1 \cdot z^2 + 2z^3 + 6z^4 + \cdots$. We can decompose Dyck paths by occurrences of hills, and we get the following relation. As before we abbreviate to F and C. We obtain

$$\begin{aligned} C &= F + FzF + FzFzF + \cdots \\ &= F + zF^2 + z^2F^2 + \cdots \\ &= \frac{F}{1 - zF}, \end{aligned}$$

i.e.,

$$C = GF(\textit{no hills}) + GF(\textit{one hill}) + GF(\textit{two hills}) + \cdots .$$

Obviously,

$$\mathcal{F} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 14 \\ \vdots \end{bmatrix} = C,$$

since every Dyck path has some number of hills. By FTRA, we have

$$(F, zF)\frac{1}{1 - z} = F \cdot \frac{1}{1 - zF} = \frac{F}{1 - zF} = C(z).$$

We want to know the total number of hills, so

$$\mathcal{F} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ 2 & 2 & 0 & 1 & & \\ 6 & 4 & 3 & 0 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \\ 14 \\ \vdots \end{bmatrix}.$$

The entries of the last column vector are the coefficients of the generating function $H(z)$ for the total number of hills. Hence,

$$H(z) = (F, zF) \cdot \frac{z}{(1 - z)^2} = F \cdot \frac{zF}{(1 - zF)^2} = z \cdot (\frac{F}{1 - zF})^2 = zC(z)^2 = C(z) - 1,$$

since $C(z) = 1 + zC(z)^2$.

Thus, the total number of hills is given by the Catalan Numbers except when $n = 0$. Thus the average number of hills is exactly 1 for $n \geq 1$.

Can we generalize this result? The answer is also YES!

By FTRA, if we use ordinary GFs and let g, f be two generalized GFs, then we have

$$(g,f)\cdot\begin{bmatrix}1\\1\\1\\1\\\vdots\end{bmatrix}=(g,f)\frac{1}{1-z}=\frac{g}{1-f}$$

and

$$(g,f)\cdot\begin{bmatrix}0\\1\\2\\3\\\vdots\end{bmatrix}=(g,f)\frac{z}{(1-z)^2}=\frac{gf}{(1-f)^2}.$$

Let

$$\frac{g}{1-f}=\frac{gf}{(1-f)^2}+1,$$

then we obtain

$$g=\frac{(1-f)^2}{1-2f}.$$

For a second example, we let $f(z)=f=z$ so that

$$g=\frac{(1-z)^2}{1-2z}=1+\frac{z^2}{1-2z}$$

and

$$\begin{bmatrix}1&&&&&&\\0&1&&&&&\\1&0&1&&&&\\2&1&0&1&&&\\4&2&1&0&1&&\\8&4&2&1&0&1&\\&&&\cdots&&&\ddots\end{bmatrix}.$$

The row sums are essentially 2^{n-1} which suggests compositions. The numbers in the k^{th} column are the number of compositions of n of the form

$$\underbrace{1+1+\ldots+1}_{k}+\underset{\substack{\uparrow\\ \text{not } 1}}{a_1}+\ldots+a_m$$

.

If we use exponential GFs, we have

$$(g,f)\cdot\begin{bmatrix}1\\1\\1\\1\\\vdots\end{bmatrix}=(g,f)e^z=ge^f$$

and

$$(g,f)\cdot\begin{bmatrix}0\\1\\2\\3\\\vdots\end{bmatrix}=(g,f)(ze^z)=gfe^f,$$

since

$$1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots=e^z,$$

$$0+z+2\cdot\frac{z^2}{2!}+3\cdot\frac{z^3}{3!}+\cdots=z(1+\frac{z}{1!}+\frac{z^2}{2!}+\cdots)=ze^z.$$

Let

$$ge^f=gfe^f+1,$$

then

$$g(1-f)e^f=1,$$

$$g=\frac{e^{-f}}{1-f}.$$

The easiest example is $f(z)=z$, $g(z)=\frac{e^{-z}}{1-z}$, and its Riordan array is

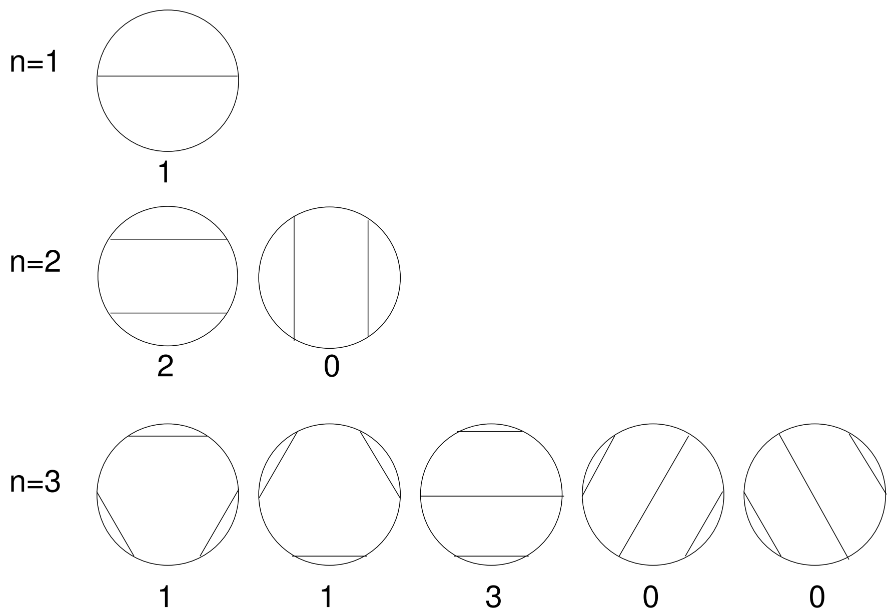
$$\begin{bmatrix}1&&&&&&\\0&1&&&&&\\1&0&1&&&&\\2&3&0&1&&&\\9&8&6&0&1&&\\44&45&20&10&0&1&\\&&\cdots&&&&\ddots\end{bmatrix}=\begin{bmatrix}1&&&&&&\\0\cdot1&1&&&&&\\1\cdot1&0\cdot2&1&&&&\\2\cdot1&1\cdot3&0\cdot3&1&&&\\9\cdot1&2\cdot4&1\cdot6&0\cdot4&1&&\\44\cdot1&9\cdot5&2\cdot10&1\cdot10&0\cdot5&1&\\&&\cdots&&&&\ddots\end{bmatrix}=(\frac{e^{-z}}{1-z},z).$$

This $g(z)$ is the generating function of the number of *derangements* of n ordered objects. *Derangements* are permutations without fixed points (i.e., having no cycles of length one). The number of *derangements* of length n, say $d(n)$, satisfies the recurrence relations:

$$d(n)=(n-1)[d(n-1)+d(n-2)]\quad\text{and}\quad d(n)=nd(n-1)+(-1)^n$$

with $d(1)=0$ and $d(2)=1$.

Let us return to Dyck paths. Here is another appearance of the average number being one phenomenon.



The number of noncrossing matchings is C_n as is the number of horizontal lines. We leave finding a bijection proof to readers and move on to our next example.

Question 5.2 : *How many permutations in S_n have their local maxima in ascending order?*

For example, in S_9, let us consider the following two permutation:

(1) In 132465987, the three local maxima numbers are $3, 6, 9$, and $3 < 6 < 9$ are in ascending order;

(2) In 132465978, the four local maxima numbers are $3, 6, 9, 8$, but $3 < 6 < 9 > 8$ are not in ascending order.

The first few terms for this sequence are $1, 2, 5, 16, 290, 1511, \cdots$. For instance, when $n = 3$, only 312 does not satisfy the ascending condition. Thus, the third term is 5. The corresponding Hankel matrix is

$$\begin{bmatrix} 1 & 2 & 5 & 16 & \\ 2 & 5 & 16 & 63 & \\ 5 & 16 & 63 & 290 & \\ 16 & 63 & 290 & 1511 & \\ & & \cdots & & \ddots \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 5 & 6 & 1 & & & \\ 16 & 31 & 12 & 1 & & \\ 63 & 264 & 106 & 20 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 2 & & & \\ & & & 6 & & \\ & & & & 24 & \\ & & & & & \ddots \end{bmatrix}.$$

Solve $g(z) \cdot f(z) = (gf)(z)$.

$$
\begin{array}{rl}
 & g = 1 + 2z + 5\frac{z^2}{2!} + 16\frac{z^3}{3!} + 63\frac{z^4}{4!} + \cdots (1) \\
(1) \text{ and } (3) \Longrightarrow (2) & f = 0 + z + 2\frac{z^2}{2!} + 4\frac{z^3}{3!} + 8\frac{z^4}{4!} + \cdots (2) = (e^{2z}-1)/2 \text{ (the educated guess)} \\
\hline
 & gf = 0 + z + 6\frac{z^2}{2!} + 31\frac{z^3}{3!} + 264\frac{z^4}{4!} + \cdots (3).
\end{array}
$$

From the first two columns, we can see the following recurrence relation.

$$l_{n+1,0} = 2l_{n,0} + l_{n,1}$$

$$\Longrightarrow g = 1 + \int (2g + gf)dz$$

$$g' = (2+f)g$$

$$\frac{g'}{g} = 2 + f = 2 + \frac{e^{2x}-1}{2} = \frac{1}{2}e^{2z} + \frac{3}{2}$$

$$\ln g = \frac{3}{2}z + \frac{1}{4}e^{2z} + C,$$

where C is a constant. Let $z = 0$, then $g(0) = 1$. Thus, we find $C = -\frac{1}{4}$ and

$$g = \exp(\frac{6z - 1 + e^{2z}}{4}).$$

6 Overview of the Riordan Group.

Now we give an overview of the Riordan group as illustrated in Figure 8.

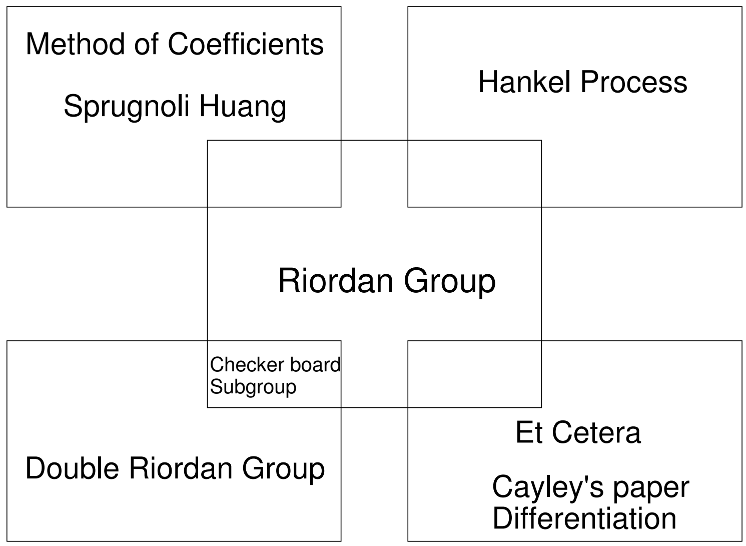


Figure 8: The overwiew

6.1 Ordinary generating functions

Theorem 6.1 *(Merlin, Rogers, Sprngnoli, Verri) If* $L = (g, f)$ *is Riordan, there exists a function* $A(z)$ *such that* $f(z) = zA(f(z))$*. The A-sequence is* $(a_0, a_1, a_2, \cdots)$ *and a z-sequence* $l_{n+1,0} = z_0 l_{n,0} + z_1 l_{n,1} + \cdots$.

Example 6.2 *Take the Schröder triangle for example.* $f(z) = z(1 + 3f(z) + 2(f(z))^2)$*, substitute* $\overline{f(z)}$ *for z, we get* $z = \overline{f(z)}(1 + 3z + 2z^2)$*, so* $\overline{f(z)} = \frac{z}{1+3z+2z^2}$*. The process can be described by the following dot diagram (Figure 9).*

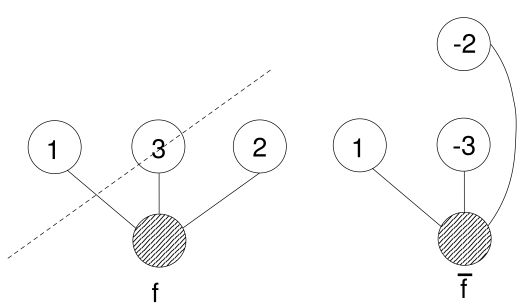


Figure 9: The dot diagram

So

$$Sch = \begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 11 & 6 & 1 & & \\ 45 & 31 & 9 & 1 & \\ & \cdots & & & \ddots \end{bmatrix} \Leftrightarrow P_{Sch} = \begin{bmatrix} 3 & 1 & & & & \\ 2 & 3 & 1 & & & \\ & 2 & 3 & 1 & & \\ & & 2 & 3 & 1 & \\ & \cdots & & & & \ddots \end{bmatrix}.$$

6.2 Exponential generating functions

Theorem 6.3 *(Emeric Deutsch) Let $L = (g(z), f(z))$ be an element of the Riordan group as EGF version. Consider two ordinary generating functions*

$$R(y) = r_0 + r_1 y + r_2 y^2 + r_3 y^3 + \cdots$$

and

$$C(y) = c_0 + c_1 y + c_2 y^2 + c_3 y^3 + \cdots$$

defined by the differential equations

$$R(f(z)) = f'(z) \quad and \quad C(f(z)) = \frac{g'(z)}{g(z)}.$$

Then

$$\begin{bmatrix} c_0 & r_0 & 0 & 0 & \\ 1!c_1 & \frac{1!}{1!}(c_0 + r_1) & r_0 & 0 & \\ 2!c_2 & \frac{2!}{1!}(c_1 + r_2) & \frac{2!}{2!}(c_0 + 2r_1) & r_0 & \\ 3!c_3 & \frac{3!}{1!}(c_2 + r_3) & \frac{3!}{2!}(c_1 + 2r_2) & \frac{3!}{3!}(c_0 + 3r_1) & \\ & & \cdots & & \ddots \end{bmatrix}.$$

If we define $c_{-1} = 0$ then we have simply that $p_{i,j} = \frac{i!}{j!}(c_{i-j} + j r_{i-j+1})$.

Example 6.4 *Pascal's matrix is (e^z, z), so $f(z) = z$, $g(z) = e^z$, $R(z) = 1$ and $C(z) = 1$, then*

$$\begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & 1 & 1 & \\ & & & & \ddots & \ddots \end{bmatrix}.$$

Example 6.5 *Ascending local maxima matrix is $g(z) = exp\left(\frac{6z-1+e^{2z}}{4}\right)$, $f(z) = \frac{1}{2}(e^{2z} - 1)$, $R(z) = 2z + 1$ and $C(z) = 2 + z$, so*

$$\begin{bmatrix} 2 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ 0 & 2 & 6 & 1 & & & \\ 0 & 0 & 3 & 8 & 1 & & \\ 0 & 0 & 0 & 4 & 10 & 1 & \\ & & & & & \ddots & \ddots \end{bmatrix}.$$

Example 6.6 *Stirling-First kind.*

$$S(1) = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 2 & 3 & 1 & & \\ 6 & 11 & 6 & 1 & \\ & \cdots & & & \ddots \end{bmatrix} \Leftrightarrow P_{S(1)} = \begin{bmatrix} 1 & 1 & & & & \\ & 2 & 1 & & & \\ & & 3 & 1 & & \\ & & & 4 & 1 & \\ & & & & \ddots & \ddots \end{bmatrix}.$$

Example 6.7 *Stirling-Second kind.*

$$S(2) = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 3 & 1 & & \\ 1 & 7 & 6 & 1 & \\ & \cdots & & & \ddots \end{bmatrix} \Leftrightarrow P_{S(2)} = \begin{bmatrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ & & & & \cdots & \ddots \end{bmatrix}.$$

Example 6.8 *Involutions by the number of fixed points.*

$$Inv = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ 0 & 3 & 0 & 1 & & & \\ 3 & 0 & 6 & 0 & 1 & & \\ 0 & 15 & 0 & 10 & 0 & 1 & \\ & \cdots & & & & & \ddots \end{bmatrix} = (e^{\frac{z^2}{2}}) \Leftrightarrow P_{Inv} = \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 2 & 0 & 1 & & & \\ & & 3 & 0 & 1 & & \\ & & & 4 & 0 & 1 & \\ & & & & 5 & 0 & \\ & & & & & \ddots & \ddots \end{bmatrix}.$$

Example 6.9 *The Genocchi numbers give an example that is almost Riordan. They can be defined by*

$$z \tan \frac{z}{2} = \sum_{m \geq 1} G_n \frac{z^{2n}}{(2n)!} = \frac{z^2}{2!} + \frac{z^4}{4!} + 3\frac{z^6}{6!} + 17\frac{z^8}{8!} + \cdots .$$

Instead, we use

$$\frac{d^2 z}{dz^2}\big(z \tan \frac{z}{2}\big) = 1 + \frac{z^2}{2!} + 3\frac{z^4}{4!} + 17\frac{z^6}{6!} + \cdots = \sec^2 \frac{z}{2}\big(1 + \frac{z}{2} \tan \frac{z}{2}\big).$$

We use the Hankel matrix and find

$$L = \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 1 & 0 & 1 & & & & & \\ 0 & 3 & 0 & 1 & & & & \\ 3 & 0 & 7 & 0 & 1 & & & \\ 0 & 17 & 0 & 13 & 0 & 1 & & \\ 17 & 0 & 69 & 0 & 22 & 0 & 1 & \\ & \cdots & & & & & & \ddots \end{bmatrix}$$

and

$$P_L = \begin{bmatrix} 0 & 1 & & & & & & \\ 1 & 0 & 1 & & & & & \\ & 2 & 0 & 1 & & & & \\ & & 4 & 0 & 1 & & & \\ & & & 6 & 0 & 1 & & \\ & & & & 9 & 0 & 1 & \\ & & & & & \ddots & \ddots & \ddots \end{bmatrix},$$

although the tridiagonal is not quadratic on the lower subdiagonal.

Let

$$P^+ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ & & P & & \\ & & & & \end{bmatrix}.$$

When is P^+ itself a Riordan matrix? The answer is yes when it is Identity, $S(1)$, $S(2)$, tangent numbers, secant numbers, $(2n+1)!!$ and $(2n)!!$; the answer is no when it is Pascal's matrix, i.e., $(1+z, z)$.

Theorem 6.10 *If $L = (g(z), f(z))$ is an element of the Riordan group (exponential version), then P^+ is also in this Riordan group iff L is in the Derivative Subgroup, i.e., $g'(z) = g(z)$.*

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Exercises and Solutions

1. Use the Riordan array method to evaluate $\sum_{k=0}^{n}\binom{n}{k}(-1)^k(k+1)$.

Solution.

$$\begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \\ 5 \\ -6 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

At this point we have a good idea of what the answer will turn out to be. We now procede to a real proof using the FTRA.

We have $g(z) = \frac{1}{1-z}$, $f(z) = \frac{z}{1-z}$, and $A(z) = \frac{1}{(1+z^2)^2}$. Then

$$(g(z), f(z)) \cdot A(z) = \frac{1}{1-z} \cdot \frac{1}{(1+\frac{z}{1-z})^2} = 1 - z.$$

Hence,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(k+1) = \begin{cases} 1 & \text{if} \quad n = 0 \\ -1 & \text{if} \quad n = 1 \\ 0 & \text{otherwise} \end{cases}$$

∎

2. Let $H_n = n^{th}$ harmonic sum $= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, show that $\sum_{k\geq 1}(-1)^{k-1}\binom{n}{k}\frac{1}{k} = H_n$

Solution. The corresponding Riordan array is:

$$\begin{bmatrix} 1 & & & & \\ 2 & -1 & & & \\ 3 & -3 & 1 & & \\ 4 & -6 & 4 & -1 & \\ & \cdots & & & \ddots \end{bmatrix}\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 + \frac{1}{2} \\ \\ \vdots \end{bmatrix}$$

$$= (\frac{1}{(1-z)^2}, \frac{-z}{1-z}) \cdot \frac{-\ln(1-z)}{z} = \frac{-\ln(1-z)}{z(1-z)} := B(z).$$

The Riordan array corresponding to the nth harmonic sum can be expressed as:

$$\begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ & \cdots & & & \ddots \end{bmatrix}\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 + \frac{1}{2} \\ 1 + \frac{1}{2} + \frac{1}{3} \\ \vdots \end{bmatrix}.$$

Then the GF of the nth harmonic sum, $H(z) = (\frac{1}{1-z}, z) \cdot \frac{-\ln(1-z)}{z} = \frac{-\ln(1-z)}{z(1-z)}$.

Thus $B(z) = H(z)$ completing the proof of the identity. ∎

3. Consider the matrix

$$F = \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 1 & 0 & 1 & & & & & \\ 0 & 2 & 0 & 1 & & & & \\ 1 & 0 & 3 & 0 & 1 & & & \\ 0 & 3 & 0 & 4 & 0 & 1 & & \\ 1 & 0 & 6 & 0 & 5 & 0 & 1 & \\ & & \cdots & & & & & \ddots \end{bmatrix}$$

and note that

$$F \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 5 \\ \vdots \end{bmatrix}.$$

What is the identity that this is the matrix form of? What happens if you apply F^{-1} to both sides of the equation?

Solution. We see that the skew diagonals of F are just Pascal's triangle. Thus, we get the well known expression,

$$F_n = \sum_{k \geq 0} \binom{n-k}{k}.$$

$$F^{-1} = \begin{bmatrix} 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ -1 & 0 & 1 & & & & & & \\ 0 & -2 & 0 & 1 & & & & & \\ 2 & 0 & -3 & 0 & 1 & & & & \\ 0 & 5 & 0 & -4 & 0 & 1 & & & \\ -5 & 0 & 9 & 0 & -5 & 0 & 1 & & \\ 0 & -14 & 0 & 14 & 0 & -6 & 0 & 1 & \\ & & \cdots & & & & & & \ddots \end{bmatrix} = (C(-z^2), zC(-z^2)).$$

Here $C(z) = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} z^n$ is the GF for the Catalan numbers and

$$F^{-1} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 5 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}.$$

Thus in some loose sense the Fibonacci numbers and the Catalan numbers are inverses of each other. ∎

4. Prove that

$$\sum_{k\geq 0}\binom{n+1}{n-2k}5^k = 2^n F_n,$$

where $(F_n)_{n\geq 0} = 1, 1, 2, 3, 5, 8, 13, \ldots$ are the Fibonaci numbers.

Solution.

To prove $\sum_{k\geq 0}\binom{n+1}{n-2k}5^k = 2^n F_n$, we only need to show

$$\begin{bmatrix} \binom{1}{0} & & & \\ \binom{2}{1} & & & \\ \binom{3}{2} & \binom{3}{0} & & \\ \binom{4}{3} & \binom{4}{1} & & \\ \binom{5}{4} & \binom{5}{2} & \binom{5}{0} & \\ \binom{6}{5} & \binom{6}{3} & \binom{6}{1} & \\ & \cdots & & \ddots \end{bmatrix} \begin{bmatrix} 5^0 \\ 5^1 \\ 5^2 \\ 5^3 \\ 5^4 \\ 5^5 \\ \vdots \end{bmatrix} = \begin{bmatrix} 2^0 F_0 \\ 2^1 F_1 \\ 2^2 F_2 \\ 2^3 F_3 \\ 2^4 F_4 \\ 2^5 F_5 \\ \vdots \end{bmatrix},$$

i.e., we need to show

$$\begin{bmatrix} 1 & & & \\ 2 & & & \\ 3 & 1 & & \\ 4 & 4 & & \\ 5 & 10 & 1 & \\ 6 & 20 & 6 & \\ & \cdots & & \ddots \end{bmatrix} \begin{bmatrix} 5^0 \\ 5^1 \\ 5^2 \\ 5^3 \\ 5^4 \\ 5^5 \\ \vdots \end{bmatrix} = \begin{bmatrix} 2^0 F_0 \\ 2^1 F_1 \\ 2^2 F_2 \\ 2^3 F_3 \\ 2^4 F_4 \\ 2^5 F_5 \\ \vdots \end{bmatrix}.$$

The column generating functions of the matrix on the left side are $\frac{1}{(1-z)^2}$, $\frac{z^2}{(1-z)^4}$, $\frac{z^4}{(1-z)^6}$, $\cdots$ respectively and since we are looking at binomial coefficients we can see that $g(z) = 1/(1-z)^2$ and $f(z) = z^2/(1-z)^2$.

By FTRA, we have

$$\begin{aligned} &\left(\frac{1}{(1-z)^2}, \frac{z^2}{(1-z)^2}\right) \cdot \frac{1}{1-5z} \\ =& \frac{1}{(1-z)^2} \cdot \frac{1}{1-5\frac{z^2}{(1-z)^2}} \\ =& \frac{1}{(1-z)^2 - 5z^2} \\ =& \frac{1}{1-2z-4z^2}, \end{aligned}$$

and the generating function of the right hand side is

$$h(z)=\sum_{n\geq 0}2^nF_nz^n=\sum_{n\geq 0}F_n\,(2z)^n=\frac{1}{1-(2z)-(2z)^2}.$$

Therefore, $\sum_{k\geq 0}\binom{n+1}{n-2k}5^k=2^nF_n$ holds. ∎

5. Discover as much as you can about the infinite matrix.

$$\begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 2 & 1 & 1 & & & & \\ 5 & 3 & 1 & 1 & & & \\ 14 & 8 & 4 & 1 & 1 & & \\ 42 & 24 & 11 & 5 & 1 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix}.$$

Hint. This is an open ended question that will show up in several later questions as well. The idea here is research oriented. What can you do if you have found the first terms of a matrix or sequence and want to learn more? All this is experimental rather than rigorous proof but may be a great help in leading to a rigorous proof. You might look at the row sums and see if you can guess the sequence. You might also find the first five terms of of $f(z)$ by solving $(1+z+2z^2+5z^3+14z^4+\cdots)(f(z))=z+z^2+3z^3+8z^4+24z^5+\cdots$. Then you can look up the terms of $f(z)$ in Sloane's EIS at http://www.research.att.com/~njas/sequences/. ∎

6. Let $D=\{(g(z),f(z))|f'(z)=g(z)\}$, a subset of the Riordan group, show that D is a subgroup.

Solution.

- Closure: For any $(g_1(z),f_1(z)),(g_2(z),f_2(z))\in D$, we have $f_1'(z)=g_1(z)$, $f_2'(z)=g_2(z)$. Because $(g_1(z),f_1(z))(g_2(z),f_2(z))=(g_1(z)g_2(f_1(z)),f_2(f_1(z)))$, and $[f_2(f_1(z))]'$ $=f_2'(f_1(z))f_1'(z)=g_2(f_1(z))g_1(z)$, so $(g_1(z),f_1(z))(g_2(z),f_2(z))\in D$.
- Associativity: Because D is a subset of a group.
- Identity: Since $z'=1$, $I=(1,z)\in D$.
- Inverse: For any $(g(z),f(z))\in D$, the inverse is $(1/g(\bar{f}(z)),\bar{f}(z))$. But $f\left(\overline{f}(z)\right)=z$ yields $\frac{d}{dz}\left(f\left(\overline{f}(z)\right)\right)=f'\left(\overline{f}(z)\right)\cdot\overline{f}'(z)=1$. Thus $\overline{f}'(z)=\frac{1}{f'\left(\overline{f}(z)\right)}=\frac{1}{g\left(\overline{f}(z)\right)}$ and $(g(z),f(z))^{-1}\in D$.

  Then we finish our proof. ∎

7. We continue with exercise 5 and show a second approach to learning about $f(z)$.Start by looking at the row sums.

$$\begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 2 & 1 & 1 & & & & \\ 5 & 3 & 1 & 1 & & & \cdots \\ 14 & 8 & 4 & 1 & 1 & & \\ 42 & 24 & 11 & 5 & 1 & 1 & \\ & & \cdots & & & & \end{bmatrix} = (C(z), f(z)).$$

Solution. By observation, we can see

$$\begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 2 & 1 & 1 & & & & \\ 5 & 3 & 1 & 1 & & & \\ 14 & 8 & 4 & 1 & 1 & & \\ 42 & 24 & 11 & 5 & 1 & 1 & \\ & & \cdots & & & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 10 \\ 28 \\ 84 \\ \vdots \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 14 \\ 42 \\ \vdots \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

A reasonable guess is that $1, 2, 4, 10, 28, 84, \ldots$ is twice the Catalan numbers except at the start. The GF for this sequence would then be $2C - 1$.

By FTRA, we would then have

$$(g(z), f(z)) \cdot A(z) = (C(z), f(z)) \cdot (\frac{1}{1-z}) = 2C(z) - 1,$$

$$\Longrightarrow C(z) \cdot \frac{1}{1 - f(z)} = 2C - 1,$$

$$\Longrightarrow f(z) = 1 - \frac{C}{2C-1} = \frac{C-1}{2C-1}.$$

Our visit with the EIS indicates that $f(z) = zF(z)$ where $F(z)$ is the generating fuction for the Fine numbers. These are a sequence that is closely tied to the Catalan numbers and a detailed survey is given in [Deutsch, Shapiro]. One fact is that $F(z) = F = C/(1 + zC)$ and you can use this to show that $f(z) = \frac{C-1}{2C-1}$. ∎

8. Show that the total number of points on the x-axis on all Dyck paths of length $2n$ is C_{n+1} where $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the m^{th} Catalan number.

Solution. (Using Riordan Arrays).

Classifying Dyck paths of length $2n$ according to the total number of points on the x-axis, one can obtain the following equation:

$$\begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 2 & 2 & 1 & & \\ 0 & 5 & 5 & 3 & 1 & \\ & & \cdots & & & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 14 \\ 42 \\ \vdots \end{bmatrix}.$$

The entries of the vector in the RHS of this equation are the number of points on the x-axis, which form the *Catalan Numbers* C_{n+1}. This looks promising. To complete the proof we use the FTRA. Since the GF for $k+1$ points on the x-axis is $(zC)^k$ we see that the matrix is $(1, zC)$. The GF for the sequence $1, 2, 3, \ldots$ is $1/(1-z)^2$ and the FTRA yields $(1, zC) \cdot \left(1/(1-z)^2\right) = 1 \cdot 1/(1-zC)^2 = C^2 = (C-1)/z = \sum_{n \geq 0} C_{n+1} z^n$ and thus we have C_{n+1} total points. ∎