Permutations Containing and Avoiding 123 and 132 Patterns

Aaron Robertson

Department of Mathematics
Temple University
Philadelphia, PA 19122
aaron@math.temple.edu

Abstract

We prove that the number of permutations which avoid 132-patterns and have exactly one 123-pattern, equals \((n - 2)2^{n-3}\), for \(n \geq 3\). We then give a bijection onto the set of permutations which avoid 123-patterns and have exactly one 132-pattern. Finally, we show that the number of permutations which contain exactly one 123-pattern and exactly one 132-pattern is \((n - 3)(n - 4)2^{n-5}\), for \(n \geq 5\).

Introduction

In 1990, Herb Wilf asked the following: How many permutations of length \(n\) avoid a given pattern, \(p\)? By pattern-avoiding we mean the following: Let \(\pi\) be a permutation of length \(n\) and let \(p = (p_1, p_2, \ldots, p_k)\) be a permutation of length \(k \leq n\) (we will call this a pattern of length \(k\)). Let \(J\) be a set of \(r\) integers, and let \(j \in J\). Define \(\text{place}(j, J)\) to be 1 if \(j\) is the smallest element in \(J\), 2 if it is the second smallest, ..., and \(r\) if it is the largest. The permutation \(\pi\) avoids the pattern \(p\) if and only if there does not exist a set of indices \(I = (i_1, i_2, \ldots, i_k)\), such that \(p = (\text{place}(\pi(i_1), I), \text{place}(\pi(i_2), I), \ldots, \text{place}(\pi(i_k), I))\).

In two beautiful papers ([B] and [N]), the number of subsequences containing exactly one 132-pattern and exactly one 123-pattern are enumerated. Noonan shows in [N] that the number of permutations containing exactly one 123-pattern is the simple formula \(\frac{3}{n}\left(\begin{array}{c}2n \\ n+3\end{array}\right)\). Bóna proves that the even simpler formula \(\left(\begin{array}{c}2n-3 \\ n-3\end{array}\right)\) enumerates the number of permutations containing exactly one 132-pattern. Bóna’s result proved a conjecture first made by Noonan and Zeilberger in [NZ].

Noonan and Zeilberger considered in [NZ] the number of permutations of length \(n\) which contain exactly \(r\) \(p\)-patterns, for \(r \geq 1\). In this article we work towards the following generalization: How many permutations of length \(n\) avoid patterns \(p_i\), for \(i \geq 0\), and contain \(r_j p_j\)-patterns, for \(j \geq 1, r_j \geq 1\)? We will first consider the permutations of length \(n\) which avoid 132-patterns, but contain exactly one 123-pattern. We then define a natural bijection between these permutations and the permutations of length \(n\) which avoid 123-patterns, but contain exactly one 132-pattern. Finally, we will calculate the number of permutations which contain one 123-pattern and one 132-pattern. These results address questions first raised in [NZ].

\footnote{webpage: www.math.temple.edu/~aaron/}

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Known Results

For completeness, two results which are already known are given below.

**Lemma 1:** The number of permutations of length $n$ with one 12-pattern is $n - 1$.

**Proof:** Induct on $n$. The base case is trivial. A permutation, $\phi$, of length $n$ with one 12-pattern must have $n = \phi(1)$ or $n = \phi(2)$. If $n = \phi(1)$, by induction we get $n - 2$ permutation. If $n = \phi(2)$, then we must have $n - 1 = \phi(1)$ (or we would have more than one 12-pattern). The rest of the entries of $\phi$ must be decreasing. Hence we get 1 more permutation from this second case, for a total of $n - 1$.

**Lemma 2:** The number of permutations which avoid both the pattern 123 and 132 is $2^{n-1}$.

**Proof:** Let $f_n$ denote the number of permutations we are interested in. Then $f_n = \sum_{i=1}^{n} f_{n-i} + 1$ with $f_0 = 0$. To see this, let $\rho$ be a permutation of length $n - 1$. Insert the element $n$ into the $i^{th}$ position of $\rho$. Call this new permutation of length $n \pi$. To assure that $\pi$ avoids the 132-pattern, we must have all entries preceding $n$ in $\pi$ be larger than the entries following $n$. To assure that $\pi$ avoids the 123-pattern, the entries preceding $n$ must be in decreasing order. This argument gives the sum in the recursion. The recursion holds by noting that if $n = 1$, there is one permutation which avoids both patterns. To complete the proof note that $f_n = 2^{n-1}$.

**One 123-pattern, but no 132-pattern**

**Theorem 1:** The number of permutations of length $n$ which have exactly one 123-pattern, and avoid the 132-pattern is $(n - 2)2^{n-3}$.

**Proof:** Let $g_n$ denote the number of permutation we desire to count. Call a permutation good if it has exactly one 123-pattern and avoids the 132-pattern. Let $\gamma$ be a permutation of length $n - 1$. Insert the element $n$ into the $i^{th}$ position of $\gamma$. Call this newly constructed permutation of length $n \pi$. To assure that $\pi$ avoids the 132-pattern, we must have all elements preceding $n$ in $\pi$ be larger than the elements following $n$ in $\pi$. For $\pi$ to be a good permutation, we must consider two disjoint cases.

**Case I:** The pattern 123 appears in the elements following $n$ in $\pi$. This forces the elements preceding $n$ to be in decreasing order. Summing over $i$, this case accounts for $\sum_{i=1}^{n} g_{n-i}$ permutations.

**Case II:** The pattern 123 appears in the elements preceding and including $n$ in $\pi$. This forces the 3 in the pattern to be $n$. Hence the elements preceding $n$ must contain exactly one 12-pattern. (Further there must be at least 2 elements. Hence $i$ must be at least 3). From Lemma 1, this number is $i - 2$. We are also forced to avoid both patterns in the elements following $n$. Lemma 2 implies that there are $2^{n-i-1}$ such permutations. Summing over $i$, this case accounts for $\sum_{i=3}^{n-1} (i - 2)2^{n-i-1} + n - 2$ permutations.
We have established that the recurrence relation

\[ g_n = \sum_{i=1}^{n} g_{n-i} + \sum_{i=3}^{n-1} (i-2)2^{n-i-1} + n - 2, \]

which holds for \( n \geq 3 \) \((g_0 = 0, g_1 = 0, g_2 = 0)\), enumerates the permutations of length \( n \) which avoid the pattern 132 and contain one 123-pattern.

The obvious way to proceed would be to find the generating function of \( g_n \). However, in this article we would like to employ a different, and in many circumstances more powerful, tool. We will use the Maple procedure \texttt{findrec} in Doron Zeilberger’s Maple package \texttt{EKHAD}\(^2\). (The Maple shareware package \texttt{gfun} could have also been used.) Instructions for its use are available online. To use \texttt{findrec} we compute the first few terms of \( g_n \). These are (for \( n \geq 4 \)) 4, 12, 32, 80, 192, 448, 1024. We type \texttt{findrec([4,12,32,80,192,448,1024],0,2,n,N)} and are given the recurrence

\[ h_n = 4(h_n - 1 - h_{n-2}) \]

for \( n \geq 4 \). Define \( h_0 = 0, h_1 = 0, h_2 = 0, \) and \( h_3 = 1 \), and it is routine to verify that \( g_n = h_n \) for \( n \geq 0 \). Another routine calculation shows us that \( h_n = (n - 2)2^{n-3} \) for \( n \geq 3 \), thereby proving the statement of the theorem.

**One 132-pattern, but no 123-pattern**

**Theorem 2:** The number of permutations of length \( n \) which have exactly one 132-pattern, and avoid the 123-pattern is \((n - 2)2^{n-3}\).

**Proof:** We prove this by exhibiting a (natural) bijection from the permutations counted in Theorem 1 to the permutations counted in this theorem. Define \( S := \{ \pi : \pi \) avoids 132-pattern and contains one 123-pattern\} and \( T := \{ \pi : \pi \) avoids 123-pattern and contains one 132-pattern\}. We will show that \(|S| = |T|\), by using the following bijection:

Let \( \phi : S \rightarrow T \). Let \( s \in S \), and let \( abc \) be the 123-pattern in \( s \). Then \( \phi \) acts on the elements of \( s \) as follows: \( \phi(x) = x \) if \( x \not\in \{b,c\} \), \( \phi(b) = c \), and \( \phi(c) = b \). In other words, all elements keep their positions except \( b \) and \( c \) switch places. An easy examination of several cases shows that this is a bijection, thereby proving the theorem.

**One 132-pattern and one 123-pattern**

**Theorem 3:** The number of permutations of length \( n \) which have exactly one 132-pattern and one 123-pattern is \((n - 3)(n - 4)2^{n-5}\).

**Proof:** We use the same insertion technique as in the proof of Theorem 1. Let \( g_n \) denote the number of permutation we desire to count. Call a permutation \textit{good} if it has exactly one 123-pattern and exactly one 132-pattern. Let \( \gamma \) be a permutation of length \( n - 1 \). Insert the element \( n \) into the \( i^{th} \) position of \( \gamma \). Call this newly constructed permutation of length \( n, \pi \).

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\(^2\)Available for download at \url{www.math.temple.edu/~zeilberg/}
We note that the 132-pattern cannot consist of elements only preceding \( n \). If this were the case, we would have two 123-patterns ending with \( n \). For \( \pi \) to be a **good** permutation, we must consider the following disjoint cases.

**Case I:** The 132-pattern consists of elements following \( n \). In this case all elements preceding \( n \) must be larger than the elements following \( n \).

**Subcase A:** The 123-pattern consists of elements following \( n \). Summing over \( i \) we get \( \sum_{i=1}^{n} g_{n-i} \) good permutations in this subcase.

**Subcase B:** The elements preceding \( n \) have exactly one 12-pattern. This gives a 123-pattern where the 3 in the pattern is \( n \). We must also avoid the 123-pattern in the elements following \( n \). Summing over \( i \) and using Lemma 1 and Theorem 1, we get \( \sum_{i=3}^{n-3} (i-2)(n-i-3)2^{n-i-2} \) good permutations in this subcase.

**Case II:** The 132-pattern has the first element preceding \( n \), the last element following \( n \), and \( n \) as the middle element. The elements preceding \( n \) must be \( n-1, n-2, \ldots, n-1+2, n-i \), where \( n-i \) immediately precedes \( n \) in \( \pi \). See [B] for a more detailed argument as to why this must be true.

**Subcase A:** The elements preceding \( n \) have exactly one 12-pattern. This gives a 123-pattern where the last element of the pattern is \( n \). We must also avoid both the 123 and the 132 pattern in the elements following \( n \). Summing over \( i \) and using Lemma 1 and Lemma 2 we have \( \sum_{i=4}^{n-1} (i-3)2^{n-i-1} \) good permutations in this subcase.

**Subcase B:** The 123-pattern consists of elements following \( n \). We must have the elements preceding \( n \) in \( \pi \) be decreasing to avoid another 123-pattern. Further, the elements following \( n \) must not contain a 132-pattern. Using Theorem 1 and summing over \( i \), we get a total of \( \sum_{i=2}^{n-3} (n-i-2)2^{n-i-3} \) good permutations in this subcase.

In total, we find that the following recurrence enumerates the permutations of length \( n \) which contain exactly one 123-pattern and one 132-pattern.

\[
g_n = \sum_{i=1}^{n} g_{n-i} + \sum_{i=1}^{n-4} (2i(n-i-4) + n-3)2^{n-i-4}
\]

for \( n \geq 5 \) and \( g_1 = g_2 = g_3 = g_4 = 0 \).

Using `findrec` again by typing `findrec([2,12,48,160,480,1344,3584],1,1,n,N)` (where the list is the first few terms of our recurrence for \( n \geq 5 \)) we get the recurrence \( f_{n+1} = \frac{2(n+2)}{n} f_n \), with \( f_1 = 2 \). After reindexing, another routine calculation shows that \( f_n = g_n \). Solving \( f_n \) for an explicit answer, we find that \( g_n = (n-3)(n-4)2^{n-5} \).

We conjecture that the number of 132-avoiding permutations with \( r \) 123-patterns is always a sum of powers of 2. For more evidence, and further extensions see [ERZ].
References


