Abstract

Natural $q$ analogues of classical statistics on the symmetric groups $S_n$ are introduced; parameters like: the $q$-length, the $q$-inversion number, the $q$-descent number and the $q$-major index. Here $q$ is a positive integer. MacMahon’s theorem (Combinatory Analysis I–II (1916)) about the equi-distribution of the inversion number and the reverse major index is generalized to all positive integers $q$. It is also shown that the $q$-inversion number and the $q$-reverse major index are equi-distributed over subsets of permutations avoiding certain patterns. Natural $q$ analogues of the Bell and the Stirling numbers are related to these $q$ statistics—through the counting of the above pattern-avoiding permutations.

1. Introduction

MacMahon’s celebrated theorem about the equi-distribution of the length (or the inversion-number) and the major index statistics on the symmetric group $S_n$ [10]—has received far-reaching refinements and generalizations through the last three decades. For a brief review on these refinements—see [12]. In [12] we extended the various classical $S_n$ statistics, in a natural way, to the alternating group $A_{n+1}$. This was done via the canonical presentations of the elements of these groups, and by a certain covering map $f: A_{n+1} \to S_n$.

Further refinements of MacMahon’s theorem were obtained in [12] by the introduction of the ‘delent’ statistics on these groups. Then these equi-distribution theorems for $S_n$ were ‘lifted’ back, via $f: A_{n+1} \to S_n$, thus yielding equi-distribution theorems for $A_{n+1}$. 

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This paper continues [12] and might be considered as its \textit{q-analogue}. Note that here \( q \) is a positive integer; the generalization to an arbitrary \( q \) is still open. We introduce the \textit{q-analogues} of the classical statistics on the symmetric groups: the \textit{q-length}, the \textit{q-inversion number}, the \textit{q-descent number}, the \textit{q-major index} and the \textit{q-reverse-major index} of a permutation. The \textit{q-delent} statistics are also introduced. We then extend classical properties to these \( q \)-analogues. For example, it is proved that the \( q \)-length equals the \( q \)-inversion number of a permutation; furthermore, it is proved that the \( q \)-inversion number and the \( q \)-reverse major index are equi-distributed on \( S_{n+q-1} \). See below.

It is realized that the above map \( f: A_{n+1} \to S_n \) is the restriction to \( A_{n+1} \) of a covering map \( f_2: S_{n+1} \to S_n \). More generally, we have similar covering maps \( f_q: S_{n+q-1} \to S_n \) for all positive integers \( q \). These maps are defined via the canonical presentations of the elements in \( S_{n+q-1} \). It is proved that the map \( f_q \) sends the \( q \)-statistics on \( S_{n+q-1} \) to the corresponding classical statistics on \( S_n \), see Proposition 8.6 below. For example, if \( \pi \in S_{n+q-1} \), it is proved there that the \( q \)-inversion number of \( \pi \) equals the inversion number of \( f_q(\pi) \).

Dashed patterns in permutations were introduced by Babson and Steingrimsson [2]. For example, a permutation \( \sigma \) contains the pattern \((1-32)\) if \( \sigma = [\ldots, a, \ldots, c, b, \ldots] \) for some \( a < b < c \); if no such \( a, b, c \) exist then \( \sigma \) is said to avoid \((1-32)\). Connections between the number of permutations avoiding \((1-32)\)—and various combinatorial objects, like the Bell and the Stirling numbers, as well as the number of left-to-right-minima in permutations were proved by Claesson [3]. Via the various \( q \)-statistics we obtain \( q \)-analogues for these connections and results.

For a permutation \( \pi \in S_{n+q-1} \) it is proved that the \textit{q-descent} and the \textit{q-delent} numbers of \( \pi \) are equal exactly when \( \pi \) avoids a certain collection of dashed patterns, and that the number of these permutations is \((q-1)! \sum k^q S(n, k)\), where \( S(n, k) \) are the Stirling numbers of the second kind, see Corollary 2.8. Also, the number of permutations in \( S_{n+q-1} \) for which the \textit{q-delent} number equals \( k-1 \) is \((q-1)!q^k c(n, k)\), where \( c(n, k) \) are the Stirling numbers of the first kind; see Proposition 2.9.

Equi-distribution of \( q \)-statistics is studied in Section 11. A \textit{q-analogue} of MacMahon’s classical equi-distribution theorem is given, see Theorem 2.5 below. Multivariate refinements of MacMahon’s theorem, due to Foata–Schützenberger and others [7, 12, 14], also have corresponding \( q \)-analogues. These analogues are described in Section 11.1, see Theorem 11.5 and its consequences.

An intensive study of equi-distribution over subsets of permutations avoiding patterns has been carried out recently, cf. [1, 5, 6, 13]. In Section 11.2 it is shown that certain \( q \)-statistics are equi-distributed on the above subsets of dashed-patterns-avoiding permutations. See Theorems 2.6 and 11.7 below.

2. The main results

Throughout the paper \( q \) is a positive integer. Recall the unique canonical presentation of a permutation in \( S_n \) as a product of shortest coset representatives along the principal flag, see Section 3.1 below. The \textit{q-length} of a permutation \( \pi \in S_n \), \( \ell_q(\pi) \), is the number of Coxeter generators in the canonical presentation of \( \pi \), where the generators \( s_1, \ldots, s_{q-1} \) are not counted.
\[ \text{inv}_q(\pi) := \sum_{i=q+1}^{n} m_q(i), \]

where
\[ m_q(i) := \min\{i - q, \#\{j < i \mid \pi(j) > \pi(i)\}\}. \]

Also \( \text{inv}_q(\pi) := 0 \) if \( n \leq q \). Thus \( \ell_1(\pi) = \ell(\pi) \) and \( \text{inv}_1(\pi) = \text{inv}(\pi) \).

As in the (classical) case \( q = 1 \), we have

**Proposition 2.1** (See Proposition 4.2). For every \( \sigma \in S_n \)
\[ \ell_q(\sigma) = \ell(\sigma). \]

**Proposition 2.2** (See Proposition 6.1). For every \( \pi \in A_n \), \( \ell_2(\pi) \) is the length with respect to the set of generators \( \{a_1, \ldots, a_{n-2}\} \subset A_n \), where \( a_i := s_1 s_{i+1} \).

Define the \( q \)-delent number, \( \text{del}_q(\pi) \), to be the number of times \( s_q \) appears in the canonical presentation of \( \pi \).

For \( 0 \leq k \leq n - 1 \) define the \( k \)th almost left-to-right-minima in a permutation \( \pi \in S_n \) (denoted \( a^k \text{l.l.r.min} \)) as the set of indices
\[ \text{Del}_{k+1}(\pi) := \{i \mid k + 2 \leq i \leq n, \#\{j < i \mid \pi(j) < \pi(i)\} \leq k\}. \]

Thus \( \text{Del}_q(\pi) \) is the set of \( a^{q-1} \text{l.l.r.min} \) in \( \pi \). See Example 5.10 below.

**Proposition 2.3** (See Proposition 5.2). The number of occurrences of \( s_{k+1} \) in the canonical presentation of \( \pi \in S_n \), \( \text{del}_{k+1}(w) \), equals the number of \( a^k \text{l.l.r.min} \) in \( \pi \).

The second delent statistics \( \text{del}_2 \) on even permutations in \( A_{n+1} \) and the first delent statistics \( \text{del}_1 \) on \( S_n \) have analogous interpretations. See, for example, Proposition 6.1.

The \( q \)-descent set of \( \pi \in S_{n+q-1} \) is defined as
\[ \text{Des}_q(\pi) := \{i \mid i \text{ is a } q \text{-descent in } \pi\}, \]

and the \( q \)-descent number is defined as
\[ \text{des}_q(\pi) := \#\text{Des}_q(\pi). \]

For \( \pi \in S_{n+q-1} \) define the \( q \)-major index
\[ \text{maj}_q(\pi) := \sum_{i \in \text{Des}_q(\pi)} i \]

and the \( q \)-reverse major index
\[ \text{rmaj}_{q,m}(\pi) := \sum_{i \in \text{Des}_q(\pi)} (m - i), \]

where \( m = n + q - 1 \).

Thus \( \text{Des}_1 \) is the standard descent set of a permutation in \( S_n \). The definition of the \( q \)-descent set is justified by the following phenomena:
(1) \( \text{Des}_2 \) is the descent set on the alternating group \( A_n \) with respect to the distinguished set of generators \( \{a_1, \ldots, a_{n-2}\} \), where \( a_i := s_{i}s_{i+1} \), see Proposition 6.1.

(2) The \( q \)-descent set, \( \text{Des}_q \), is strongly related with pattern avoiding permutations, see Proposition 9.3.

(3) \( \text{Des}_q \) is involved in the definition of the \( q \)-(reverse) major index, and thus in the \( q \)-analogue of MacMahon’s equi-distribution theorem (Theorem 11.2).

Given \( q \), denote by

\[ \text{Pat}(q) = \{(\sigma_1 - \sigma_2 - \cdots - \sigma_q - (q + 2), (q + 1)) \mid \sigma \in S_q\} \]

the set with these \( q! \) dashed patterns. For example, \( \text{Pat}(1) = \{(1 - 32)\} \), \( \text{Pat}(2) = \{(1 - 2 - 43), (2 - 1 - 43)\} \).

Denote by \( \text{Avoid}_q(n + q - 1) \) the set of permutations in \( S_{n+q-1} \) avoiding all the \( q! \) patterns in \( \text{Pat}(q) \).

**Proposition 2.4** (See Proposition 9.3). A permutation \( \pi \in S_{n+q-1} \) avoids \( \text{Pat}(q) \) exactly when \( \text{Del}_q(\pi) - 1 = \text{Des}_q(\pi) \):

\[ \text{Avoid}_q(n + q - 1) = \{\pi \in S_{n+q-1} \mid \text{Del}_q(\pi) - 1 = \text{Des}_q(\pi)\} \]

The following is a \( q \)-analogue of MacMahon’s equi-distribution theorem.

**Theorem 2.5** (See Theorem 11.2).

\[
\sum_{\pi \in S_{n+q-1}} t^{\text{maj}_{q,n+q-1}}(\pi) = \sum_{\pi \in S_{n+q-1}} t^{\text{inv}_q}(\pi)
\]

\[ = q!(1 + tq)(1 + t + t^2 q) \cdots (1 + t + \cdots + t^{n-2} + t^{n-1} q). \]

Far reaching multivariate refinements of MacMahon’s theorem, which imply equi-distribution on subsets of permutations, were given by Foata and Schützenberger and others, cf. [7, 8, 12, 14]. In Section 11.1 we describe some \( q \)-analogues of these refinements, see Theorem 11.4 and Corollary 11.6 below.

The above \( q \)-statistics are equi-distributed on permutations avoiding \( \text{Pat}(q) \).

**Theorem 2.6** (See Corollary 11.8).

\[ \sum_{\pi^{-1} \in \text{Avoid}_q(n + q - 1)} t_1^{\text{maj}_{q,n+q-1}}(\pi) t_2^{\text{des}_q}(\pi) = \sum_{\pi^{-1} \in \text{Avoid}_q(n + q - 1)} t_1^{\text{inv}_q}(\pi) t_2^{\text{des}_q}(\pi). \]

For example, for \( q = 1 \)

\[ \sum_{\pi^{-1} \in \text{Avoid}(1-32)} t_1^{\text{maj}_1}(\pi) t_2^{\text{des}(\pi)} = \sum_{\pi^{-1} \in \text{Avoid}(1-32)} t_1^{\text{inv}(\pi)} t_2^{\text{des}(\pi)}. \]

For \( q = 2 \)

\[ \sum_{\pi^{-1} \in \text{Avoid}(1-2-43, 2-1-43)} t_1^{\text{maj}_{2,n+1}}(\pi) t_2^{\text{des}_2}(\pi) \]

\[ = \sum_{\pi^{-1} \in \text{Avoid}(1-2-43, 2-1-43)} t_1^{\text{inv}_2}(\pi) t_2^{\text{des}_2}(\pi). \]
Bell and Stirling numbers (of both kinds) appear naturally in the enumeration of permutations with respect to their $q$-statistics.

Let $c(n, k)$ be the $k$th Stirling number of the first kind and $S(n, k)$ be the $k$th Stirling number of the second kind. Let the $n$th $q$-Bell number be $b_q(n) := \sum_{k} q^k S(n, k)$. Let

$$B_q(x) := \sum_{n=0}^{\infty} b_q(n) \frac{x^n}{n!}$$

denote the exponential generating function of $\{b_q(n)\}$. Then

$$B_q(x) = \exp(qe^x - q).$$

The classical formula $b_1(n) = \frac{1}{e} \sum_{r=0}^{\infty} \frac{r^n}{r!}$ [4] (see also [15, (1.6.10)]) generalizes as follows:

$$b_q(n) = \frac{1}{e^q} \sum_{r=0}^{\infty} \frac{q^r r^n}{r!},$$

see Remark 10.4.

**Proposition 2.7** (See Proposition 10.8).

$$\#\{\sigma \in S_{n+q-1} | \text{Del}_q(\sigma) - 1 = \text{Des}_q(\sigma) \text{ and } \text{del}_q(\sigma) = k - 1\} = (q - 1)!q^k S(n, k).$$

**Corollary 2.8** (See Propositions 9.3 and 10.5).

$$(q - 1)!b_q(n) = \#\{\pi \in S_{n+q-1} | \text{Del}_q(\pi) - 1 = \text{Des}_q(\pi)\} = \text{Avoid}_q(n + q - 1).$$

**Proposition 2.9** (See Proposition 10.10).

$$\#\{\pi \in S_{n+q-1} | \text{del}_q(\pi) = k - 1\} = c_q(n, k),$$

where $c_q(n, k) = q^k (q - 1)!c(n, k)$.

### 3. Preliminaries

#### 3.1. The $S_n$ canonical presentation

A basic tool, both in [12] and in this paper, is the canonical presentation of a permutation, which we now describe.

Recall that the transpositions $s_i = (i, i + 1)$, $1 \leq i < n - 1$, are the Coxeter generators of the symmetric group $S_n$. For each $1 \leq j \leq n - 1$ define

$$R^S_j = \{1, s_j, s_j s_{j-1}, \ldots, s_j s_{j-1} \cdots s_1\}$$

and note that $R^S_1, \ldots, R^S_{n-1} \subseteq S_n$.

The following is a classical theorem; see for example [9, pp. 61–62]. See also [12, Theorem 3.1].

**Theorem 3.1.** Let $w \in S_n$, then there exist unique elements $w_j \in R^S_j$, $1 \leq j \leq n - 1$, such that $w = w_1 \cdots w_{n-1}$. Thus, the presentation $w = w_1 \cdots w_{n-1}$ is unique; it is called the canonical presentation of $w$. 

Note that \( R_j^S \) is the complete list of representatives of minimal length of right cosets of \( S_j \) in \( S_{j+1} \). Thus, the canonical presentation of \( w \in S_n \) is the unique presentation of \( w \) as a product of shortest coset representatives along the principal flag

\[ \{e\} = S_1 < S_2 < \cdots < S_n. \]

We remark that a similar canonical presentation for the alternating groups \( A_n \) is given in [12], see Section 3.2 below.

The descent set \( \text{Des}(\pi) \) of a permutation \( \pi \in S_n \) is a classical notion. In [12] the ‘delent’ statistic was introduced: \( \text{Del}(\pi) \) is the set of indices \( i \) which are left-to-right-minima of \( \pi \), and \( \text{del}(\pi) = \#\text{Del}(\pi) \). By Proposition 7.2 of [12], \( \text{del}(\pi) \) equals the number of times that \( s_1 = (1, 2) \) appears in the canonical presentation of \( \pi \).

Theorem 9.1 is the main theorem of [12] and we now state its part about \( S_n \) (it also has a similar part about \( A_n \)).

**Theorem 3.2.** For every subset \( D_1 \subseteq [n-1] \) and \( D_2 \subseteq [n-1] \)

\[
\sum_{\{\pi \in S_n | \text{Des}_S(\pi^{-1}) \subseteq D_1, \text{Del}_S(\pi^{-1}) \subseteq D_2\}} q^{\text{maj}_S(\pi)} = \sum_{\{\pi \in S_n | \text{Des}_S(\pi^{-1}) \subseteq D_1, \text{Del}_S(\pi^{-1}) \subseteq D_2\}} q^{\ell_S(\pi)}.
\]

In the following case, a simple explicit generating function is given.

**Theorem 3.3 ([12, Theorem 6.1]).**

\[
\sum_{\sigma \in S_n} q^{\ell_S(\sigma)} t^{\text{del}_S(\sigma)} = \sum_{\sigma \in S_n} q^{\text{maj}_S(\sigma)} t^{\text{del}_S(\sigma)} = (1 + qt)(1 + q + q^2t) \cdots (1 + q + \cdots + q^{n-1}t).
\]

### 3.2. The alternating group

The alternating group serves as a motivating example. Here are some results from [12], which are applied in Section 6 and Appendix and in the formulation and proof of Proposition 8.5. The reader who is not interested in this motivating example may skip this subsection.

Let

\[ a_i := s_1 s_{i+1} \quad (1 \leq i \leq n-1) \quad \text{and} \quad A := \{a_i | 1 \leq i \leq n-1\}. \]

The set \( A \) generates the alternating group on \( n+1 \) letters \( A_{n+1} \). This generating set and its following properties appear in [11].

**Proposition 3.4 ([11, Proposition 2.5]).** The defining relations of \( A \) are

\[
(a_i a_j)^2 = 1 \quad (|i - j| > 1); \quad (a_i a_{i+1})^3 = 1 \quad (1 \leq i < n-1); \quad a_i^3 = 1 \quad \text{and} \quad a_i^2 = 1 \quad (1 < i \leq n-1).
\]
For each $1 \leq j \leq n - 1$ define
\begin{equation}
R^A_j = \{1, a_j, a_j a_{j-1}, \ldots, a_j \cdots a_2, a_j \cdots a_2 a_1, a_j \cdots a_2 a_1^{-1}\}
\end{equation}
and note that $R^A_1, \ldots, R^A_{n-1} \subseteq A_{n+1}$.

**Theorem 3.5.** Let $v \in A_{n+1}$, then there exist unique elements $v_j \in R^A_j$, $1 \leq j \leq n - 1$, such that $v = v_1 \cdots v_{n-1}$, and this presentation is unique.

This presentation is called the $A$ canonical presentation of $v$.

For $\sigma \in A_{n+1}$ let $\ell_A(\sigma)$ be the length of the $A$ canonical presentation of $\sigma$. Let
\[
\text{Des}_A(\sigma) := \{ i \mid \ell_A(\sigma) \leq \ell_A(\sigma a_i) \}
\]
and $\text{des}_A(\sigma) := \#\text{Des}_A(\sigma)$, define $\text{maj}_A(\sigma) := \sum_{i \in \text{Des}_A(\sigma)} i$, and $\text{rmaj}_A_{n+1}(\sigma) := \sum_{i \in \text{Des}_A(\sigma)} (n - i)$. Let $\ell_1(\sigma)$ be the number of appearances of $a_i^{\pm 1}$ in its $A$ canonical presentation. It is proved in [12] that this number equals the number of almost-left-to-right-minima in $\sigma$.

**Theorems 3.1 and 3.5** allow us to introduce in [12] the following covering map:

**Definition 3.6.** Define $f: A_{n+1} \to S_n$ as follows.
\[
f(a_1) = f(a_1^{-1}) = s_1 \quad \text{and} \quad f(a_i) = s_i, \quad 2 \leq i \leq n - 1.
\]
Now extend $f: R^A_j \to R^S_j$ via
\[
f(a_j a_{j-1} \cdots a_\ell) = s_j s_{j-1} \cdots s_\ell, \quad f(a_j \cdots a_1) = f(a_j \cdots a_1^{-1}) = s_j \cdots s_1.
\]
Finally, let $v \in A_{n+1}$, $v = v_1 \cdots v_{n-1}$ its unique $A$ canonical presentation, then
\[
f(v) = f(v_1) \cdots f(v_{n-1})
\]
which is clearly the $S$ canonical presentation of $f(v)$.

**Proposition 3.7** ([12, Propositions 5.3–5.4]). For every $\pi \in A_{n+1}$,
\[
\ell_A(\pi) = \ell_S(f(\pi)), \quad \text{Des}_A(\pi) = \text{Des}_S(f(\pi)), \quad \text{Del}_A(\pi) = \text{Del}_S(f(\pi))
\]
Thus $\text{des}_A(\pi) = \text{des}_S(f(\pi))$, $\text{maj}_A(\pi) = \text{maj}_S(f(\pi))$, $\text{rmaj}_A_{n+1}(\pi) = \text{rmaj}_S_{n+1}(f(\pi))$ and $\text{del}_A(\pi) = \text{del}_S(f(\pi))$.

4. **Basic concepts I**

Let $\pi \in S_n$. Recall that its length $\ell(\pi)$ equals the number of the Coxeter generators $s_1, \ldots, s_{n-1}$ in its canonical presentation. It is well known that $\ell(\pi)$ also equals $\text{inv}(\pi)$, the number of inversions of $\pi$. Also, it is easily seen that $\text{inv}(\pi)$ can be written as
\[
\text{inv}(\pi) = \sum_{i=2}^{n} m(i),
\]
Thus, the following definition is a natural $q$-analogue of these two classical statistics.

**Definition 4.1.** Let $\pi \in S_n$.

1. $(\ell_q(\ell_q(\pi))$ as follows:
   
   $\ell_q(\pi) :=$ the number of Coxeter generators in the canonical presentation of $\pi$, where $s_1, \ldots, s_{q-1}$ are not counted (thus, for example, $\ell_2(s_1) = 0$ and $\ell_2(s_1s_2s_1s_3s_2s_1) = 3$).

2. ($\text{inv}_q$) beginning of Section 2.

Thus $\ell_1(\pi) = \ell(\pi)$ and $\text{inv}_1(\pi) = \text{inv}(\pi)$.

As in the (classical) case $q = 1$, we have

**Proposition 4.2.** For every $\sigma \in S_n$

$$\ell_q(\sigma) = \text{inv}_q(\sigma).$$

**Proof.** We may assume that $q < n$. Let $\sigma = w_1 \cdots w_{n-1}$ with $w_j \in R_j$ be the canonical presentation of $\sigma$, and denote $\pi = w_1 \cdots w_{n-2}$, then $\pi \in S_{n-1}$, hence $\pi = [b_1, \ldots, b_{n-1}, n]$. If $w_{n-1} = 1$ then $\sigma \in S_{n-1}$ and we are done by induction. Hence assume $w_{n-1} \neq 1$, so that $w_{n-1} = s_{n-1} \cdots s_k$ for some $1 \leq k \leq n - 1$, and therefore $\sigma = [b_1, \ldots, b_{k-1}, n, b_k, \ldots, b_{n-1}]$.

**Case 1.** $1 \leq k \leq q$, in which case

$$\ell_q(w_{n-1}) = n - q \quad \text{and} \quad \sigma = [b_1, \ldots, b_{k-1}, n, b_k, \ldots, b_q, \ldots, b_{n-1}].$$

Then for $q \leq i \leq n - 1$,

$$\{j < i + 1 \mid \sigma(j) > \sigma(i + 1)\} = \{j < i \mid b_j > b_i\} + 1$$

(the ‘+1’ comes from $n > b_i$). Therefore $m_q(i + 1, \sigma) = m_q(i, \pi) + 1$, since

$$m_q(i + 1, \sigma) = \min\{i + 1 - q; \{j < i + 1 \mid \sigma(j) > \sigma(i + 1)\}\} \begin{array}{c} = \min\{i + 1 - q; \{j < i \mid b_j > b_i\} + 1\} \end{array} \begin{array}{c} = \min\{i - q; \{j < i \mid b_j > b_i\} + 1\} \end{array} = m_q(i, \pi) + 1.$$

Thus

$$\text{inv}_q(\sigma) = \sum_{i=q+1}^{n} m_q(i, \sigma) = \sum_{i=q}^{n-1} m_q(i + 1, \sigma) = \sum_{i=q}^{n-1} m_q(i, \pi) + (n - q)$$

(by induction)

$$= \ell_q(\pi) + n - q = \ell_q(\pi) + \ell_q(w_{n-1}) = \ell_q(\sigma).$$

**Case 2.** $q + 1 \leq k$, hence $\ell_q(w_{n-1}) = \ell_1(w_{n-1}) = n - k$, $\sigma = [b_1, \ldots, b_q, \ldots, b_{k-1}, n, b_k, \ldots, b_{n-1}]$. Here
1. $m_q(i, \sigma) = m_q(i, \pi)$ if $q + 1 \leq i \leq k - 1$,
2. $m_q(k, \sigma) = 0 (i = k)$, and, as in Case 1,
3. $m_q(i + 1, \sigma) = m_q(i, \pi) + 1$ if $k \leq i \leq n - 1$.

It follows that

$$\text{inv}_q(\sigma) = \sum_{i=q+1}^{n} m_q(i, \sigma) = \sum_{i=q+1}^{k-1} m_q(i, \pi) + \sum_{i=k}^{n-1} m_q(i, \pi) + n - k$$

$$= \sum_{i=q}^{n-1} m_q(i, \pi) + (n - k) \text{ (by induction)}$$

$$= \ell_q(\pi) + n - k = \ell_q(\pi) + \ell_q(w_{n-1}) = \ell_q(\sigma). \quad \square$$

The following lemma was proved in [12].

**Lemma 4.3** ([12, Lemma 3.7]). Let $w = s_i \cdots s_{i_p}$ be the canonical presentation of $w \in S_n$. Then the canonical presentation of $w^{-1}$ is obtained from the presentation $w^{-1} = s_{i_p} \cdots s_i$ by commuting moves only—without any braid moves. Similarly for $v, v^{-1} \in A_{n+1}$.

**Proposition 4.4.** For every $\sigma \in S_n$, 

$$\ell_q(\sigma^{-1}) = \ell_q(\sigma), \quad \text{hence also } \text{inv}_q(\sigma^{-1}) = \text{inv}_q(\sigma).$$

**Proof.** Lemma 4.3 easily implies that $\ell_q(\sigma^{-1}) = \ell_q(\sigma)$, while this, together with Proposition 4.2 implies the equality $\text{inv}_q(\sigma^{-1}) = \text{inv}_q(\sigma)$. \quad \square

5. Basic concepts II

A natural $q$-analogue of the del statistics from [12] is introduced in this section. This allows us to introduce below a (less intuitive) $q$-analogue of the descent statistics.

5.1. The del statistics

Recall the definitions of Del and del (of types $S$ and $A$) from [12]: given a permutation $w$ in $S_n$, $\text{Del}_S(w)$ is the set of indices which are left-to-right-minima (l.t.r.min) in $w$, and $\text{Del}_A(w)$ is the set of indices which are almost left-to-right-minima (a.l.t.r.min) in $w$. Let $s_i = (i, i + 1), i = 1, \ldots, n - 1$, denote the Coxeter generators of $S_n$. The following classical fact is of fundamental importance in this paper.

Let $R_j = \{1, s_j, s_js_j^{-1}, \ldots, s_js_{j-1} \cdots s_1\}$ and let $w \in S_n$, then there exist unique elements $w_j \in R_j, 1 \leq j \leq n - 1$, such that $w = w_1 \cdots w_{n-1}$; this is the (unique) canonical presentation of $w$, see Theorem 3.1 in [12].

Similarly $a_i = s_1s_{i+1}, i = 1, \ldots, n - 1$, are the corresponding generators for the alternating group $A_{n+1}$, and there is a corresponding unique canonical presentation for the elements of $A_{n+1}$, see Section 3 in [12]. The following was observed in [12]:
1. The number of times \( s_1 \) appears in the canonical presentation of \( w \) (i.e. \( \text{del}_S(w) \)) equals the number of l.t.r.min in \( w \) (hence \( \#\text{Del}_S(w) = \text{del}_S(w) \)), see [12, Proposition 7.2].

2. The number of times \( s_2 \) appears in \( w \) equals the number of a.l.t.r.min in \( w \). Moreover, if \( w \in A_{n+1} \), that number equals the number of times \( a_{i}^{\pm 1} \) appears in the \( A \)-canonical presentation of \( w \), which by definition is \( \text{del}_A(w) \), and \( \text{del}_A(w) = \#\text{Del}_A(w) \), see [12, Proposition 7.6].

In this paper, ‘sub \( S \)’ is replaced by ‘sub \( 1 \)’: \( \text{Del}_S = \text{Del}_1 \) and \( \text{del}_S = \text{del}_1 \), etc. Similarly (in \( A_n \) ‘sub \( A \)’ is replaced by ‘sub \( 2 \)’. We shall also encounter ‘sub \( q \)’ for every positive integer \( q \).

**Definition 5.1.** Let \( \pi \in S_n \) and let \( 1 \leq q \leq n - 1 \).

1. Define \( \text{del}_q(\pi) \) to be the number of times \( s_q \) appears in the canonical presentation of \( \pi \).
2. For \( 0 \leq k \leq n-1 \) define the \( k \)th *almost-left-to-right-minima* in a permutation \( \pi \in S_n \) (denoted \( a^k.l.t.r.min \)) as the set of indices

\[
\text{Del}_{k+1}(\pi) := \{ i \mid k + 2 < i \leq n, \#\{ j < i \mid \pi(j) < \pi(i) \} \leq k \}.
\]

Thus \( \text{Del}_q(\pi) \) is the set of \( a^{q-1}.l.t.r.min \) in \( \pi \).

See Example 5.10 below.

*Note* that if \( i \leq k + 1 \) then, trivially, \( \#\{ j < i \mid \pi(j) < \pi(i) \} \leq k \), however these indices are not counted as \( a^k.l.t.r.min \). Also note that \( a^0.l.t.r.min \) is simply l.t.r.min.

**Proposition 5.2.** Let \( w \in S_n \). Then for every nonnegative integer \( k \), the number of occurrences of \( s_{k+1} \) in the canonical presentation of \( w \), \( \text{del}_{k+1}(w) \), equals the number of \( a^k.l.t.r.min \) in \( w \). Writing \( k + 1 = q \) we have

\[
\#\text{Del}_q(w) = \text{del}_q(w).
\]

**Proof.** (Generalizes the Proof of Proposition 7.6 in [12]). We first need the following two lemmas.

**Lemma 5.3.** Let \( 1 \leq k + 1 \leq n \), let \( w \in S_n \) and let \( \pi \in S_{k+1} \). Also let \( i \leq n \). Then \( i \) is \( a^k.l.t.r.min \) of \( w \) if and only if \( i \) is \( a^k.l.t.r.min \) of \( \pi w \). In particular, the number of \( a^k.l.t.r.min \) of \( w \) equals the number of \( a^k.l.t.r.min \) of \( \pi w \).

**Proof.** Denote \( w = [b_1, \ldots, b_n] \) (namely \( w(r) = b_r \)), and compare \( w \) with \( \pi w: \pi \) permutes only the \( b_r \)'s in \( \{1, \ldots, k + 1\} \). If \( b_i \in \{1, \ldots, k + 1\} \), the total number of \( b_j \)'s smaller than \( b_i \) is \( \leq k \); in particular such \( i \) is \( a^k.l.t.r.min \) in both \( w \) and \( \pi w \), provided \( i \geq k + 2 \). If on the other hand \( b_i \notin \{1, \ldots, k + 1\} \) then \( b_i \) is greater than all the elements in that subset; thus such \( i \) is \( a^k.l.t.r.min \) of \( w \) if and only if \( i \) is \( a^k.l.t.r.min \) of \( \pi w \). This implies the proof. \( \square \)

**Lemma 5.4.** Let \( 1 \leq k \leq n - 1 \) and denote \( s_{[k,n-1]} = s_k s_{k+1} \cdots s_{n-1} \). Let \( \sigma \in S_n \) and write \( \sigma = [b_1, \ldots, b_{n-1}, n] \). Then \( s_{[k,n-1]}\sigma = [c_1, \ldots, c_{n-1}, k] \), and the two tuples \( (b_1, \ldots, b_{n-1}) \) and \( (c_1, \ldots, c_{n-1}) \) are order-isomorphic, namely for all \( i, j \), \( b_i < b_j \) if and only if \( c_i < c_j \).
Proof. Comparing $\sigma$ with $s_{[k,n-1]}\sigma$, we see that

1. the (position with) $n$ in $\sigma$ is replaced in $s_{[k,n-1]}\sigma$ by $k$;
2. each $j$ in $\sigma$, $k \leq j \leq n - 1$, is replaced by $j + 1$ in $s_{[k,n-1]}\sigma$;
3. each $j$, $1 \leq j \leq k - 1$ is unchanged.

This implies the proof. □

The Proof of Proposition 5.2 is by induction on $n$. If $n \leq k + 1$, the number of $a^k{.l.t.r.min}$ of any permutation in $S_n$ is zero, and also $s_{k+1} \notin S_n$, hence 5.2 holds in that case.

Next assume 5.2 holds for $n - 1$ and prove for $n$. Let $w = w_1 \cdots w_{n-1}$ be the canonical presentation of $w \in S_n$ and denote $\sigma = w_1 \cdots w_{n-2}$, then $\sigma \in S_{n-1}$. If $w_{n-1} = 1$ then $w \in S_{n-1}$ and the proof follows by induction. So let $w_{n-1} \neq 1$, then we can write $w_{n-1} = s_{n-1}s_{n-2} \cdots s_d v$, where $d \geq k + 1$ and $v \in \{1, s_k, s_k s_{k-1}, \ldots, s_k s_{k-1} \cdots s_1\}$ hence $v \in S_k + 1$. If $d \geq k + 2$ then necessarily $v = 1$ and in that case the number of times $s_{k+1}$ appears in $w$ and in $\sigma$ is the same. If $d = k + 1$, that number in $w$ is one more than in $\sigma$. We show that the same holds for the number of $a^k{.l.t.r.min}$ for these two permutations $\sigma$ and $w$.

By Lemma 3.4 of [12], it suffices to prove that statement for the inverse permutations $w^{-1}$ and $\sigma^{-1}$. Now, $w^{-1} = \pi s_{[d,n-1]}\sigma^{-1}$, where $\pi = v^{-1} \in S_{k+1}$, hence by Lemma 5.3 it suffices to compare the number of $a^k{.l.t.r.min}$ in $\sigma^{-1}$ with that in $s_{[d,n-1]}\sigma^{-1}$. By Lemma 5.4 $\sigma^{-1} = [b_1, \ldots, b_{n-1}, n]$ and $s_{[d,n-1]}\sigma^{-1} = [c_1, \ldots, c_{n-1}, d]$ where the $b$’s and the $c$’s are order isomorphic.

The case $d \geq k + 2$. Here the two last positions—$n$ in $\sigma^{-1}$ and $d$ in $s_{[d,n-1]}\sigma^{-1}$—are not $a^k{.l.t.r.min}$, and the above order isomorphism implies the proof in that case.

The case $d = k + 1$. By a similar argument, now the last position in $s_{[d,n-1]}\sigma^{-1}$ (which is $k + 1$) is one additional $a^k{.l.t.r.min}$.

The proof now follows. □

Proposition 5.5. For every positive integer $q$ and every permutation $\pi \in S_{n+q-1}$

$$\text{del}_q(\pi) = \text{del}_q(\pi^{-1}).$$

Proof. This is a straightforward consequence of Lemma 3.7 of [12], which says the following: let $\pi \in S_n$ and let $\pi = s_{i_1} \cdots s_{i_r}$ be its canonical presentation. Then the canonical presentation of $\pi^{-1}$ is obtained from the equation $\pi^{-1} = s_{i_r} \cdots s_{i_1}$ by commuting moves only, without any braid moves. Thus, the number of times a particular $s_j$ appears in $\pi$ and in $\pi^{-1}$ is the same. This clearly implies the proof. □

Corollary 5.6. For every positive integer $q$ and every permutation $\pi \in S_{n+q-1}$ the number of $a^{q-1}l.t.r.min$ in $\pi$ equals the number of $a^{q-1}l.t.r.min$ in $\pi^{-1}$.

Proof. Combining Proposition 5.2 with Proposition 5.5. □

Remark 5.7. Setting $q = k + 1$ in Lemma 5.3, deduce that for any two permutations $\sigma$ and $\eta$ in $S_{n+q-1}$, if $\sigma$ and $\eta$ belong to the same right coset of $S_q$, i.e. $\eta \in S_q \sigma$, then

$$\text{Del}_q(\eta) = \text{Del}_q(\sigma) \quad \text{(and therefore \text{del}_q(\eta) = \text{del}_q(\sigma)).}$$
The same is also true for the left cosets: let \( \eta \in \sigma S_q \) then again

\[
\text{Del}_q(\eta) = \text{Del}_q(\sigma) \quad \text{(and therefore \( \text{del}_q(\eta) = \text{del}_q(\sigma) \)).}
\]

This easily follows from Definition 5.1, since if \( \sigma = [b_1, \ldots, b_q, \ldots, b_n] \), \( \tau \in S_q \) and \( \eta = \sigma \tau \), then \( \eta = [b_{\tau(1)}, \ldots, b_{\tau(q)}, b_{q+1}, \ldots, b_n] \).

Let now \( \sigma \) and \( \eta \) belong to the same left coset or right coset of \( S_q \), then by the same reasoning, for any \( q \leq d \), \( \text{del}_d(\eta) = \text{del}_d(\sigma) \) since \( S_q \subseteq S_d \). Since

\[
\ell_q(\eta) = \sum_{d=1}^{n-1} \text{del}_d(\eta), \quad \text{and} \quad \ell_q(\sigma) = \sum_{d=1}^{n-1} \text{del}_d(\sigma),
\]

deduce that in that case \( \ell_q(\eta) = \ell_q(\sigma) \).

### 5.2. The q-descent set

Recall that \( i \) is a descent of \( \pi \) if \( \pi(i) > \pi(i+1) \), and let \( \text{Des}(\pi) \) denote the (‘classical’) descent-set of \( \pi \). The following definition seems to be the appropriate \( q \)-analogue for descents.

**Definition 5.8.** \( i \) is a \( q \)-descent in \( \pi \in S_{n+q-1} \) if \( i \geq q \) and at least one of the following two conditions holds:

1. \( i \in \text{Des}(\pi) \);
2. \( i + 1 \) is an \( a^{q-1} \)-l.t.r.min in \( \pi \).

Thus \( \text{Des}_q(\pi) = (\text{Des}(\pi) \cap \{q, q+1, \ldots, n-1\}) \cup (\text{Del}_q(\pi) - 1) \), hence for all \( q \), \( \text{Del}_q(\pi) - 1 \subseteq \text{Des}_q(\pi) \) where \( \text{Del}_q(\pi) - 1 = \{i - 1 \mid i \in \text{Del}_q(\pi)\} \).

Note that when \( q = 1 \), condition (2) says that \( i + 1 \) is l.t.r.min, which implies that \( i \) is a descent. Thus, a 1-descent is just a descent in the classical sense.

**Definition 5.9.**

1. **The \( q \)-descent set** of \( \pi \in S_{n+q-1} \) is defined as

\[
\text{Des}_q(\pi) := \{i \mid i \text{ is a } q \text{-descent in } \pi\}.
\]

2. **The \( q \)-descent number** of \( \pi \) is defined as \( \text{des}_q(\pi) := \#\text{Des}_q(\pi) \).

3. The \( q \)-major index and the \( q \)-reverse major index of \( \pi \in S_{n+q-1} \) are defined as

\[
\text{maj}_q(\pi) := \sum_{i \in \text{Des}_q(\pi)} i \quad \text{and} \quad \text{rmaj}_{q,m}(\pi) := \sum_{i \in \text{Des}_q(\pi)} (m - i),
\]

where \( m = n + q - 1 \).

**Example 5.10.** Let \( \sigma = [7, 8, 6, 5, 2, 9, 4, 1, 3] \).

When \( q = 2 \), \( \text{Del}_2(\sigma) = \{3, 4, 5, 7, 8\} \) and \( \text{Des}_2(\sigma) = \text{Del}_2(\sigma) - 1 = \{2, 3, 4, 6, 7\} \).

When \( q = 3 \), \( \text{Del}_3(\sigma) = \{4, 5, 7, 8, 9\} \), hence \( \text{Des}_3(\sigma) = \{3, 4, 6, 7\} \cup \{3, 4, 6, 7, 8\} = \{3, 4, 6, 7, 8\} \).

Also, \( \text{Des}_4(\sigma) = \{4, 6, 7, 8\} \), etc.
6. Motivating examples

When \( q = 1 \), the corresponding statistics are classical. By definition, for every \( \pi \in S_n \), \( \ell_1(\pi) = \ell_S(\pi) \), \( \text{Des}_1(\pi) = \text{Des}_S(\pi) \), and \( \text{Del}_1(\pi) = \text{Del}_S(\pi) \). It follows that for every \( \pi \in S_n \), \( \text{des}_1(\pi) = \text{des}_S(\pi) \), \( \text{maj}_1(\pi) = \text{maj}_S(\pi) \), \( \text{ram}_{j,n}(\pi) = \text{rmaj}_{S_n}(\pi) \), and \( \text{del}_1(\pi) = \text{del}_S(\pi) \). The delent statistics, \( \text{del}_S \), were introduced in [12].

The corresponding \( A \)-statistics were also studied in [12]; these \( A \)-statistics correspond to the case \( q = 2 \) and are restricted to the alternating groups. This is the following proposition.

**Proposition 6.1.** For every even permutation \( \pi \in S_{n+1} \)

1. \( \ell_2(\pi) = \ell_A(\pi) \),
2. \( \text{Des}_2(\pi) = \text{Des}_A(\pi) \), and
3. \( \text{Del}_2(\pi) = \text{Del}_A(\pi) \).

**Proof.** (1) follows from [12, Proposition 4.5]. (2) follows from Lemma A.1 in the Appendix. For (3) see [12, Proposition 7.5]. \( \square \)

An alternative and more conceptual proof is given below (see Remark 8.9).

**Corollary 6.2.** For every even permutation \( \pi \in S_{n+1} \), \( \text{des}_2(\pi) = \text{des}_A(\pi) \), \( \text{maj}_2(\pi) = \text{maj}_A(\pi) \), \( \text{ram}_{j,n}(\pi) = \text{rmaj}_{A_n}(\pi) \), and \( \text{del}_2(\pi) = \text{del}_A(\pi) \).

7. The double cosets of \( S_q \subseteq S_n \)

Let \( S_q \) be the subgroup of \( S_{n+q-1} \) generated by \( \{s_1, \ldots, s_{q-1}\} \). It is shown here that the previous \( q \)-statistics are invariant on the double cosets of \( S_q \) in \( S_{n+q-1} \).

**Proposition 7.1.** For any two permutations \( \pi \) and \( \sigma \) in \( S_{n+q-1} \), if \( \pi \) and \( \sigma \) belong to the same double coset of \( S_q \) (namely, \( \pi \in S_q \sigma S_q \)), then

1. \( \text{Del}_q(\pi) = \text{Del}_q(\sigma) \), hence \( \text{del}_q(\pi) = \text{del}_q(\sigma) \);
2. \( \text{Des}_q(\pi) = \text{Des}_q(\sigma) \), hence \( \text{des}_q(\pi) = \text{des}_q(\sigma) \);
3. \( \text{inv}_q(\pi) = \text{inv}_q(\sigma) = \ell_q(\pi) = \ell_q(\sigma) \).

**Proof.** It suffices to prove that if there exists \( \tau \in S_q \), such that \( \pi = \tau \sigma \) or \( \pi = \sigma \tau \), then equalities 1–3 hold.

(1) Part 1 was proved in Remark 5.7.

(2) Denote \( \sigma = [b_1, \ldots, b_{n+q-1}] \) and \( \pi = [b'_1, \ldots, b'_{n+q-1}] \). Since \( \text{Des}_q(\pi) = (\text{Des}(\pi) \cap \{q, q+1, \ldots, n\}) \cup (\text{Del}_q(\pi) - 1) \), and the same for \( \text{Des}_q(\sigma) \), it suffices to prove the following: let \( i \geq q \) and \( i \in \text{Des}(\sigma) \), then either \( i \in \text{Des}(\pi) \) or \( i + 1 \in \text{Del}_q(\pi) \).

We prove first the case of the right cosets: \( \pi = \tau \sigma \). It is given that \( b_i > b_{i+1} \).

**Case 1.** \( b_i, b_{i+1} \notin \{1, \ldots, q\} \). Then \( b_i = b'_i \) and \( b_{i+1} = b'_{i+1} \) and we are done.

**Case 2.** \( b_i \notin \{1, \ldots, q\} \) and \( b_{i+1} \in \{1, \ldots, q\} \). Then \( b_i = b'_i > q \) while \( b'_{i+1} \in \{1, \ldots, q\} \) and we are done.

The map $f_q$ is invariant on the double cosets of $S_n$: Let $\tau \in S_q$.

3. Denote $\pi = s_{11} \cdots s_{it}$ be its canonical presentation, then define $f_q: S_{n+q-1} \rightarrow S_n$ as follows:

$$f_q(\pi) = f_q(s_{i_1}) \cdots f_q(s_{i_t}),$$

where $f_q(s_1) = \cdots = f_q(s_{q-1}) = 1$, and $f_q(s_j) = s_{j-q+1}$ if $j \geq q$.

Remark 8.2. It is easy to verify that for any $q_1, q_2$, $f_{q_1} \circ f_{q_2} = f_{q_1+q_2-1}$. Thus, for every natural $q$, $f_q = f_2^{q-1}$.

Proposition 8.3. The map $f_q$ is invariant on the double cosets of $S_q$: Let $\sigma \in S_{n+q-1}$ and $\pi \in S_q \sigma S_q$, then $f_q(\sigma) = f_q(\pi)$.

Proof. It suffices to prove that if $\sigma \in S_{n+q-1}$ and $\tau \in S_q$ then $f_q(\sigma \tau) = f_q(\tau \sigma) = f_q(\sigma)$. By Remark 8.2, it suffices to prove when $q = 2$ and hence when $\tau = s_1$. As usual, let $\sigma = w_1 \cdots w_n \in S_{n+1}$ be the canonical presentation of $\sigma$. By analysing the two cases $w_1 = 1$ and $w_1 = s_1$, it easily follows that $f_2(s_1 \sigma) = f_2(\sigma)$.

We now show that $f_2(s_1 \sigma) = f_2(\sigma)$. The proof in that case follows from the definition of $f_2$ and by induction on $n$, by analysing the following cases:

$w_n = 1$;

$w_n = s_n s_{n-1} \cdots s_k$ with $k \geq 3$;

$w_n = s_n s_{n-1} \cdots s_2$, and

$w_n = s_n s_{n-1} \cdots s_2 s_1$.

We verify, for example, the case $k \geq 3$. Denote $\pi = w_1 \cdots w_{n-1}$, so $\sigma = \pi w_n$. Now $f_2(\sigma s_1) = f_2(\pi s_1 \cdot w_n) = f_2(\pi s_1) f_2(w_n) = (by \ induction) = f_2(\pi) f_2(w_n) = f_2(\sigma)$.

The proof in the last two cases follows similarly, and from the fact that $f_2(s_n s_{n-1} \cdots s_2) = f_2(s_n s_{n-1} \cdots s_2 s_1) = s_{n-1} \cdots s_2 s_1$. □
Note that $f_q$ is not a group homomorphism. For example, let $q = 2$, $g = s_2$ and $h = s_1s_2$. Then $f_2(g) = f_2(h) = s_1$ so $f_2(g)f_2(h) = 1$, but $gh = s_1s_2s_1$, hence $f_2(gh) = s_1$. Nevertheless we do have the following

**Proposition 8.4.** For any permutation $\pi$, $f_q(\pi^{-1}) = (f_q(\pi))^{-1}$.

**Proof.** Again by Remark 8.2, it suffices to prove for $q = 2$. The proof is based on Lemma 4.3. Denote $s_0 := 1$, then note that if $s_is_j = s_js_i$ then also $s_{i-1}s_{j-1} = s_{j-1}s_{i-1}$ (the converse is false, as $s_1s_2 \neq s_2s_1$).

Let $\pi = s_{i_1} \cdots s_{i_r}$ be the canonical presentation of $\pi$. By commuting moves, $\pi^{-1} = s_{i_r} \cdots s_{i_1} = \cdots = s_{p_1} \cdots s_{p_r}$ where the right hand side is the canonical presentation of $\pi^{-1}$. By definition, $f_2(\pi^{-1}) = s_{p_1-1} \cdots s_{p_r-1}$. Now by the same commuting moves $s_{i_r-1} \cdots s_{i_1-1} = \cdots = s_{p_1-1} \cdots s_{p_r-1}$ and the left hand side equals $(f_q(\pi))^{-1}$, which completes the proof. □

**Proposition 8.5.** Recall from [12] and Section 3.2 the map $f : A_{n+1} \rightarrow S_n$. Then $f$ is the restriction $f = f_2|_{A_{n+1}}$ of $f_2$ to $A_{n+1}$.

**Proof.** Let $\pi \in A_{n+1}$, and let $\pi = a_{i_1}^{\epsilon_1} \cdots a_{i_r}^{\epsilon_r}$ be its $A$-canonical presentation, where all $\epsilon_j = \pm 1$. By definition, $f(\pi) = s_{i_1} \cdots s_{i_r}$. Replace each $a_j$ in the above presentation by $a_j = s_1s_{j+1}$ then, by commuting moves ‘push’ each $s_1$ as much as possible to the left. After some cancellations, an $s_1$ cannot move any more to the left if it is already the left-most factor, or if it is preceded by an $s_2$ on its left. It follows that

$$\pi = bs_{i_1+1} \cdots s_{2s_1} \cdots s_{2s_1} \cdots s_{i_r+1} \cdots$$

where $b \in \{1, s_1\}$, and this is an $S$-canonical presentation. Then $f_2(\pi) = s_{i_1} \cdots s_{i_r}$ and the proof follows. □

Restricting the maps $f_q$ to $A_{n+q-1}$ we get more ‘$f$-pairs’ (see [12, Section 5]) with corresponding statistics, equi-distributions and generating-functions-identities for the alternating groups.

The main result here is

**Proposition 8.6.** For every $\pi \in S_{n+q-1}$

(1) $\text{Del}_q(\pi) - q + 1 = \text{Del}_1(f_q(\pi))$, and in particular, $\text{del}_q(\pi) = \text{del}_1(f_q(\pi))$.

(2) $\text{Des}_q(\pi) - q + 1 = \text{Des}_1(f_q(\pi))$, and in particular, $\text{des}_q(\pi) = \text{des}_1(f_q(\pi))$.

(3) $\text{inv}_q(\pi) = \text{inv}_1(f_q(\pi)) = \ell_q(\pi) = \ell_1(f_q(\pi))$.

Here $\text{Del}_q(\pi) - r = \{i - r \mid i \in \text{Del}_q(\pi)\}$ and similarly for $\text{Des}_q(\pi) - r$.

The proof is given below.

**Remark 8.7.** Recall that $R_j = \{1, s_j, s_js_{j-1}, \ldots, s_js_{j-1} \cdots s_1\}$.

(1) Let $w = w_1 \cdots w_{n+q-2}$ where all $w_j \in R_j$ be the canonical presentation of $w \in S_{n+q-1}$. Then $f_q(w) = f_q(w_1) \cdots f_q(w_{n+q-2})$ is the canonical presentation of $f_q(w)$. Note that $f_q(w_1) = \cdots = f_q(w_{q-1}) = 1$.

(2) In addition, let also $w' = w'_1 \cdots w'_{n+q-2}$, where also $w'_j \in R_j$. It is obvious that $f_q(w) = f_q(w')$ if and only if $f_q(w_j) = f_q(w'_j)$ for all $j$. 
(3) The definition of $a^k\text{l.t.r.min}$ in $\sigma = [b_1, \ldots, b_n]$—and therefore also the definition of the set $\text{Del}_q(\sigma)$—applies whenever the integers $b_1, \ldots, b_n$ are distinct.

(4) Let $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$ be two sets of distinct integers, let $M$ be an integer satisfying $b_j, c_j < M$ for all $j$, let $1 \leq k \leq n$ and denote

$$\sigma = [b_1, \ldots, b_n], \quad \sigma^* = [b_1, \ldots, b_{k-1}, M, b_k, \ldots, b_n]$$

and

$$\eta = [c_1, \ldots, c_n], \quad \eta^* = [c_1, \ldots, c_{k-1}, M, c_k, \ldots, c_n].$$

Then it is rather easy to verify that $\text{Del}_q(\sigma) = \text{Del}_q(\eta)$ if and only if $\text{Del}_q(\sigma^*) = \text{Del}_q(\eta^*)$.

**Lemma 8.8.** Let $w, w' \in S_{n+q-1}$ satisfy $f_q(w) = f_q(w')$, then

1. $\text{Del}_q(w) = \text{Del}_q(w')$.
2. $\text{Des}_q(w) = \text{Des}_q(w')$.

**Proof.** Since $f_j(w) = w$, we assume that $q \geq 2$.

By the definition of $f_q$ and by Remark 8.7 it suffices to prove the following claim:

Let $w_j, w'_j \in R_j$ satisfy $f_q(w_j) = f_q(w'_j)$, $q \leq j \leq n + q - 2$, and let $w = w_q \cdots w_{n+q-2}$ and $w' = w'_q \cdots w'_{n+q-2}$. Then $\text{Del}_q(w) = \text{Del}_q(w')$.

The proof is by induction on $n \geq 1$. If $n = 1$, $w = w'$.

The induction step:

Denote $m = n + q - 1$, so $w = w_q \cdots w_{m-1}$ and $w' = w'_q \cdots w'_{m-1}$, then denote $\sigma = w_q \cdots w_{m-2}$ and $\sigma' = w'_q \cdots w'_{m-2}$. Since both permutations are in $S_{m-1} \subseteq S_m$, we have

$$\sigma = [b_1, \ldots, b_{m-1}, m] \quad \text{and} \quad \sigma' = [c_1, \ldots, c_{m-1}, m].$$

By induction, $\text{Del}_q(\sigma) = \text{Del}_q(\sigma')$. If $w_{m-1} = 1$ then also $w'_{m-1} = 1$ and we are done.

Thus, assume both are $\neq 1$. Recall that $f_q(w_{m-1}) = f_q(w'_{m-1})$ and let $w_{m-1} = s_{m-1} \cdots s_k$ and $w'_{m-1} = s_{m-1} \cdots s_{k'}$. If $k > q$, it follows that $w_{m-1} = w'_{m-1}$ and we are done. So let $k, k' \leq q$. By comparing both cases with the case $k = q$ we may assume that $k = q$ and $k' \leq q$, hence $w'_{m-1} = w_{m-1}s_{q-1} \cdots s_{k'}$.

Compare first $\sigma w_{m-1}$ with $\sigma' w_{m-1}$:

$$\sigma w_{m-1} = [b_1, \ldots, b_{q-1}, m, b_q, \ldots, b_{m-1}],$$
$$\sigma' w_{m-1} = [c_1, \ldots, c_{q-1}, m, c_q, \ldots, c_{m-1}].$$

and by induction and Remark 8.7(4), $\text{Del}_q(\sigma w_{m-1}) = \text{Del}_q(\sigma' w_{m-1})$. Compare now $\sigma' w_{m-1}$ with $\sigma' w'_{m-1} = (\sigma' w_{m-1})s_{q-1} \cdots s_{k'}$:

$$\sigma' w_{m-1} = [c_1, \ldots, c_{q-1}, m, c_q, \ldots, c_{m-1}]$$
$$\sigma' w'_{m-1} = [c_1, \ldots, c'_{k'-1}, m, c'_k, \ldots, c_{q-1}, \ldots, c_{m-1}].$$

A simple argument now shows that $q < i$ is a $a^{q-1}\text{l.t.r.min}$ $\text{Del}_q(\sigma' w_{m-1}) = \text{Del}_q(\sigma' w'_{m-1})$ and the proof of part 1 is complete.

(2) The proof is similar to that of part 1. Denote $m = n + q - 1$, then write $w = w_1 \cdots w_{m-1} = \sigma w_{m-1}$ where $\sigma = w_1 \cdots w_{m-2}$, and similarly $w' = w'_1 \cdots w'_{m-1} = \sigma' w'_{m-1}$. 


We assume that \( f_q(w_j) = f_q(w'_j) \) for all \( j \). Thus \( f_q(\sigma) = f_q(\sigma') \) and by induction, \( \text{Des}_q(\sigma) = \text{Des}_q(\sigma') \). By an argument similar to that in the proof of part 1, it follows that \( \text{Des}_q(\sigma w_m) = \text{Des}_q(\sigma' w_m) \) and it remains to show that \( \text{Des}_q(\sigma' w_m) = \text{Des}_q(\sigma w_m) \). Again as in the proof of part 1, we may assume that \( w_m = s_m \cdots s_q \) and \( w'_m = s_m \cdots s_t \) where \( t < q \). We prove the case \( t = q - 1 \), the other cases being proved similarly.

Write \( \sigma' = [a_1, \ldots, a_{m-1}, m] \). Now \( \sigma' w_m = \sigma' w_m s_{q-1} \), hence

\[
\begin{align*}
\sigma' w_m &= [a_1, \ldots, a_{q-2}, a_q, a_{q+1}, a_{q+2}, \ldots, a_{m-1}] \\
\sigma' w_m &= [a_1, \ldots, a_{q-2}, m, a_q, \ldots, a_{m-1}].
\end{align*}
\]

Clearly, \( q \in \text{Des}(\sigma' w_m) \) (therefore \( q \in \text{Des}(\sigma w_m) \)), but it is possible that \( q \notin \text{Des}(\sigma' w_m) \). However, at most all the \( q - 1 \) integers \( a_1, \ldots, a_{q-1} \) are smaller than \( a_q \) (but \( m > a_q \)), hence \( q + 1 \in \text{Del}_q(\sigma' w_m) \), which implies that \( q \in \text{Des}(\sigma' w_m) \).

For all other indices \( i \neq q \) it is easy to check that \( i \in \text{Des}_q(\sigma' w_m) \) if and only if \( i \in \text{Des}_q(\sigma w_m) \), and the proof is complete. \( \square \)

The Proof of Proposition 8.6. (1) Let \( \pi \in S_{n+q-1} \) and let \( \pi' \) denote the permutation obtained from \( \pi \) by erasing—in the canonical presentation of \( \pi \)—all the appearances of the Coxeter generators \( s_1, \ldots, s_{q-1} \). Clearly, \( f_q(\pi) = f_q(\pi') \), hence suffices to prove that

(a) \( \text{Del}_q(\pi) = \text{Del}_q(\pi') \), and

(b) \( \text{Del}_q(\pi') - q + 1 = \text{Del}(f_q(\pi')) \), i.e. \( \text{Del}_q(\pi') = \text{Del}(f_q(\pi')) + q - 1 \).

Let \( \pi = w_1 w_2 \cdots w_{q-1} w_q \cdots w_{m-1} (m = n + q - 1) \) be the canonical presentation of \( \pi: w_j \in R_j \). Denote \( \tau = w_1 \cdots w_{q-1} \) and \( \sigma = w_q \cdots w_{m-1} \), then both are given in their canonical presentations. Clearly, \( f(\pi) = 1 \) and \( \pi' = \sigma' = w'_q \cdots w'_{m-1} \), where for each \( w'_j \) is obtained from \( w_j \) by erasing all the appearances of \( s_1, \ldots, s_{q-1} \), and therefore \( f_q(w_j) = f_q(w'_j) \). By Lemma 8.8, \( \text{Del}_q(\pi) = \text{Del}_q(\sigma') = \text{Del}_q(\pi') \). Since \( \pi = \tau \sigma \) and \( \tau \in S_q \), by Remark 5.7 \( \text{Del}_q(\pi) = \text{Del}_q(\sigma) \)—and (a) is proved.

Part (b) follows from the following fact:

Let \( \pi' = s_{i_1} \cdots s_{i_k} \) be the canonical presentation of the above \( \pi' \) (therefore all \( i_j \geq q \)), then \( f_q(\pi') = s_{i_1+1} \cdots s_{i_k} \). If \( f_q(\pi') = [a_1, \ldots, a_n] \), it follows that \( \pi' = [1, \ldots, q - 1, a_1 + q - 1, \ldots, a_n + q - 1] \). If \( 2 \leq i \), it then follows that \( i \) is a l.t.r.min of \( f_q(\pi') \) if and only if \( i + q - 1 \) is \( a^{q-1} \). l.t.r.min of \( f_q(\pi') \), which proves (b). \( \square \)

(2) Recall that

\[
\text{Des}_q(\pi) = (\text{Des}(\pi) \cap \{q, q + 1, \ldots, n\}) \cup (\text{Del}_q(\pi) - 1).
\]

Special Case: Assume \( \pi \) does not involve any of \( s_1, \ldots, s_{q-1} \). As above, if \( f_q(\pi) = [a_1, \ldots, a_n] \) then \( \pi = [1, \ldots, q - 1, a_1 + q - 1, \ldots, a_n + q - 1] \), hence

\[
\text{Des}(\pi) \cap \{q, q + 1, \ldots, n + q - 1\} = \text{Des}(f_q(\pi)) + q - 1.
\]

By part 1

\[
\text{Des}_q(\pi) = ([\text{Des}(f_q(\pi))] \cup \text{Del}(f_q(\pi)) - 1) + q - 1.
\]
Since for any $\sigma \in S_n \text{Des}(\sigma) \supseteq \text{Del}(\sigma) - 1$, it follows that the right hand side equals \( \text{Des}(f_q(\pi)) + q - 1 \), and this completes the proof of this case.

The general case. Let $\pi \in S_{n+q-1}$ be arbitrary. Let $\pi'$ be the permutation obtained from $\pi$ by deleting all the appearances of $s_1, \ldots, s_{q-1}$ from its canonical presentation. Then $f_q(\pi) = f_q(\pi')$ and the proof easily follows from the above special case and from Lemma 8.8.(2).

(3) By Proposition 4.2, $\text{inv}_q(\pi) = \ell_q(\pi)$. By the definitions of $\ell_q$ and $f_q$, $\ell_q(\pi) = \ell(f_q(\pi))$, and finally, $\ell(\sigma) = \text{inv}(\sigma)$ for any permutation $\sigma$. \( \square \)

Remark 8.9. Proposition 6.1 now follows from Proposition 8.6, combined with Propositions 3.7 and 8.5.

Lemma 8.10. For every $\pi \in S_n$

\[
\# f_q^{-1}(\pi) = q! \cdot q^{\text{del}_1(\pi)} = (q-1)! \cdot q^{\text{del}_1(\pi)+1}.
\]

Moreover, let $g_q : A_{n+q-1} \to S_n$ be the restriction $g_q = f_q |_{A_{n+q-1}}$ of $f_q$ to $A_{n+q-1}$. Then

\[
\# g_q^{-1}(\pi) = \frac{1}{2} \# f_q^{-1}(\pi).
\]

Proof. Denote $m = n + q - 1$, so $f_q : S_m \to S_n$. Consider the canonical presentation of $\pi \in S_n$ and write it as $\pi = \pi^{(n-1)} \cdot v_{n-1}$, where $\pi^{(n-1)} \in S_{n-1}$ and $v_{n-1} \in R_{n-1} = \{1, s_{n-1}, s_{n-1}s_{n-2}, \ldots, s_{n-1} \cdot s_{n-2} \cdots s_1\}$. Thus

\[
\# f_q^{-1}(\pi) = \# f_q^{-1}(\pi^{(n-1)}) \cdot \# f_q^{-1}(v_{n-1}) = q! \cdot q^{\text{del}_1(\pi^{(n-1)})} \# f_q^{-1}(v_{n-1})
\]

(by induction). If $\text{del}_1(v_{n-1}) = 0$ then $\# f_q^{-1}(v_{n-1}) = 1$. If $\text{del}_1(v_{n-1}) = 1$ then $\# f_q^{-1}(v_{n-1}) = q$, since in that case $v_{n-1} = s_{n-1} \cdots s_1$ and

\[
f_q^{-1}(v_{n-1}) = \{w_{m-1}, w_{m-1}s_{q-1}, \ldots, w_{m-1}s_{q-1} \cdots s_1\},
\]

where $w_{m-1} = s_{m-1} \cdots s_{q-2} \cdots s_1$. The proof now follows.

The argument for $g_q$ is similar. The factor 1/2 comes from the fact that $\# f_q^{-1}(1) = #S_q$ while $\# g_q^{-1}(1) = #A_q$. \( \square \)

Following [12], we introduce

Definition 8.11. Let $m_1$ and $m_q$ be two statistics on the symmetric groups. We say that $(m_1, m_q)$ is an $f_q$-pair if for all $n$ and $\pi \in S_{n+q-1}$, $m_q(\pi) = m_1(f_q(\pi))$.

As a corollary of Proposition 8.6 and Remark 11.1, we have

Corollary 8.12. The following are $f_q$-pairs:

$(\text{inv}_1, \text{inv}_q), (\ell_1, \ell_q), (\text{del}_1, \text{del}_q), (\text{des}_1, \text{des}_q)$, and $(\text{rmaj}_1, \text{rmaj}_q, \text{rmaj}_{n+q-1})$.

The same argument as in the proof of Proposition 5.6 in [12], together with Lemma 8.10, now proves
Proposition 8.13. Let \((m_1, m_q)\) be an \(f_q\)-pair of statistics on the symmetric groups. Then

\[
\sum_{\pi \in S_{n+q-1}} t_1^{m_q(\pi)} t_2^{\text{del}_q(\pi)} = q! \sum_{\sigma \in S_n} t_1^{m_1(\sigma)} t_2^{\text{del}_1(\sigma)}.
\]

Restricting \(f_q\) to \(A_{n+q-1}\) we obtain similarly, that

\[
\sum_{\pi \in A_{n+q-1}} t_1^{m_q(\pi)} t_2^{\text{del}_q(\pi)} = \frac{1}{2} q! \sum_{\sigma \in S_n} t_1^{m_1(\sigma)} t_2^{\text{del}_1(\sigma)}.
\]

Remark 8.14. As in [12], Proposition 8.13 allows us to lift equi-distribution theorems from \(S_n\) to \(S_{n+q-1}\), as well as to \(A_{n+q-1}\). This is demonstrated in Theorem 11.3. We leave the formulation and the proof of the corresponding \(A_{n+q-1}\) statement for the reader.

9. Dashed patterns

Dashed patterns in permutations were introduced in [2]. For example, the permutation \(\sigma\) contains the pattern \((1-32)\) if \(\sigma = [\ldots, a, \ldots, c, \ldots]\) for some \(a < b < c\); if no such \(a, b, c\) exist then \(\sigma\) is said to avoid \((1-32)\). In [3] the author shows connections between the number of permutations avoiding \((1-32)\) and various combinatorial objects, like the Bell and the Stirling numbers, as well as the number of left-to-right-minima in permutations. In this and in the next sections we obtain the \(q\)-analogues for these connections and results.

In Section 5.2 it was observed that, always, \(\text{Del}_q(\pi) - 1 \subseteq \text{Des}_q(\pi)\). It is proved in Proposition 9.3 that equality holds exactly for permutations avoiding a certain set of dashed-patterns.

Definition 9.1.

1. Given \(q\), denote by

\[
\text{Pat}(q) = \{ (\sigma_1 - \sigma_2 - \cdots - \sigma_q - (q+2)(q+1)) \mid \sigma \in S_q \}
\]

the set with these \(q!\) dashed patterns.

For example, \(\text{Pat}(2) = \{(1 - 2 - 43), (2 - 1 - 43)\}\).

2. Denote by \(\text{Avoid}_q(m), m = n + q - 1\), the set of permutations in \(S_m\) avoiding all the \(q!\) patterns in \(\text{Pat}(q)\), and let \(h_q(m)\) denote the number of the permutations in \(S_m\) avoiding \(\text{Pat}(q)\). Thus \(h_q(m) = \#\text{Avoid}_q(m)\) is the number of the permutations in \(S_{n+q-1}\) avoiding \(\text{Pat}(q)\). Note that \(h_q(m) = n!\) if \(m \leq q + 1\). As usual, define \(h_q(0) = 1\).

Connections between \(h_q(n)\) and the \(q\)-Bell and \(q\)-Stirling numbers are given in Section 10.

Remark 9.2. A permutation \(\pi \in S_{n+q-1}\) does satisfy one of the patterns in \(\text{Pat}(q)\) if and only if there exists a subsequence

\[
1 \leq i_1 < i_2 < \cdots < i_{q+1} < n + q - 1,
\]

such that \(\pi(i_{q+1}) > \pi(i_{q+1} + 1)\) and for every \(1 \leq j \leq q, \pi(i_j) < \pi(i_{q+1} + 1)\). In such a case, \(i_{q+1} + 1\) (namely, \(\pi(i_{q+1} + 1)\)) is not an \(a^{q-1}.l.t.r.min\) in \(\pi\).
Proposition 9.3. A permutation $\pi \in S_{n+q-1}$ avoids $\text{Pat}(q)$ exactly when $\text{Del}_q(\pi) - 1 = \text{Des}_q(\pi)$:

$$\text{Avoid}_q(n+q-1) = \{\pi \in S_{n+q-1} | \text{Del}_q(\pi) - 1 = \text{Des}_q(\pi)\}.$$ 

In particular,

$$h_q(n+q-1) = \#\{\pi \in S_{n+q-1} | \text{Del}_q(\pi) - 1 = \text{Des}_q(\pi)\}.$$

Proof. (1) Recall from Section 5.2 that, always, $\text{Del}_q(\pi) - 1 \subseteq \text{Des}_q(\pi)$. Let $\pi = [b_1, \ldots, b_{n+q-1}] \in S_{n+q-1}$ satisfy $\text{Del}_q(\pi) - 1 = \text{Des}_q(\pi)$, which implies that $\text{Des}(\pi) \cap \{q, \ldots, n+q-1\} \subseteq \text{Del}_q(\pi) - 1$, and show that $\pi$ avoids $\text{Pat}(q)$. If not, by Remark 9.2 we obtain a descent in $\pi$ at $i_{q+1}$, while $i_{q+1} + 1$ is not $aq-1$ l.t.r.min in $\pi$; thus $i_{q+1}$ is in $\text{Des}(\pi) \cap \{q, \ldots, n+q-1\}$ but not in $\text{Del}_q(\pi) - 1$, a contradiction.

(2) Denote $\pi = [b_1, \ldots, b_{n+q-1}]$. Assume now that $\pi \in \text{Avoid}_q(n)$, let $k \in \text{Des}(\pi) \cap \{q, \ldots, n+q-1\}$ (so $b_k > b_{k+1}$) and show that $k+1 \in \text{Del}_q(\pi)$, that is, $k+1$ (namely $b_{k+1}$) is $aq-1$ l.t.r.min in $\pi$. If not, there exist $q$ (or more) $b_j$'s in $\pi$, smaller than and left of $b_{k+1}$—hence also left of $b_k$. Together with $b_k > b_{k+1}$ this shows that $\pi \notin \text{Avoid}_q(n + q - 1)$, a contradiction. \(\square\)

Corollary 9.4. The covering map $f_q$ maps $\text{Avoid}_q(S_{n+q-1})$ to $\text{Avoid}_1(S_n)$:

$$f_q : \text{Avoid}_q(S_{n+q-1}) \to \text{Avoid}_1(S_n).$$

Similarly,

$$f_2 : \text{Avoid}_q(S_{n+q-1}) \to \text{Avoid}_{q-1}(S_{n+q-2}).$$

Proof. This follows straightforwardly from Propositions 8.6 and 9.3. \(\square\)

10. \textit{q}-Bell and \textit{q}-Stirling numbers

10.1. The $q$-Bell numbers

Recall that $S(n, k)$ are the Stirling numbers of the second kind, i.e. the numbers of $k$-partitions of the set $[n] = \{1, \ldots, n\}$. Recall also that the Bell number $b(n)$ is the total number of the partitions of $[n]$: $b(n) = \sum_k S(n, k)$.

Definition 10.1. Define the $q$-Bell numbers $b_q(n)$ by

$$b_q(n) = \sum_k q^k S(n, k).$$

Remark 10.2. Let $q \geq 1$ be an integer and consider partitions of $[n]$ into $k$ subsets, where each subset is coloured by one of $q$ colours. The number of such $q$-coloured $k$-partitions is obviously $q^k S(n, k)$. It follows that the total number of such $q$-coloured partitions of $[n]$ is the $n$th $q$-Bell number $b_q(n)$.

Proposition 10.8 below shows that

$$\#\{\sigma \in S_{n+q-1} | \text{Del}_q(\sigma) - 1 = \text{Des}_q(\sigma) \text{ and } \text{del}_q(\sigma) = k - 1\}
= (q - 1)! q^k S(n, k),$$

...
and therefore
\[(q - 1)b_q(n) = \#\{\pi \in S_{n+q-1} | \text{Del}_q(\pi) - 1 = \text{Des}_q(\pi)\}.
\]
The $q$-Bell numbers are studied first.

When $q = 1$, by considering the subset in a $k$-partition of $[n]$ which contains $n$, one easily deduces the well-known recurrence relation
\[b_1(n) = \sum_k \binom{n - 1}{k} b_1(n - k - 1).
\]
In the general $q$ colours case, apply the same argument, now taking into account that each subset—and in particular the one containing $n$—can be coloured by $q$ colours. This proves:

**Lemma 10.3.** For each integer $q \geq 1$ we have the following recurrence relation
\[b_q(n) = q \sum_k \binom{n - 1}{k} b_q(n - k - 1).
\]

**Remark 10.4.**
1. Let $B_q(x) = \sum_{n=0}^{\infty} b_q(n) \frac{x^n}{n!}$ denote the exponential generating function of $\{b_q(n)\}$. As in page 42 in [15], Lemma 10.3 implies that $B'(x) = qe^x B_q(x)$. Together with $B(0) = 1$ (since, by definition, $b_q(0) = 1$), this implies that
\[B_q(x) = \exp(qe^x - q).
\]
2. The classical formula
\[b_1(n) = \frac{1}{e} \sum_{r=0}^{\infty} \frac{r^n}{r!}
\]
generalizes as follows:
\[b_q(n) = \frac{1}{e^q} \sum_{r=0}^{\infty} \frac{q^r r^n}{r!}.
\]
The proof follows, essentially unchanged, the argument on page 21 in [15].

10.2. Connections with pattern-avoiding permutations

Recall that $\text{Pat}(q) = \{(\sigma_1 - \sigma_2 - \cdots - \sigma_q - (q + 2)(q + 1)) | \sigma \in S_q\}$ and that $h_q(n)$ denotes the number of the permutations in $S_n$ avoiding all these $q!$ patterns in $\text{Pat}(q)$.

**Proposition 10.5.** The $q$-Bell numbers $b_q(n)$ and the numbers $h_q(n + q - 1)$ of permutations in $S_{n+q-1}$ that avoid $\text{Pat}(q)$, satisfy
\[h_q(n + q - 1) = (q - 1)! \cdot b_q(n).
\]
By Proposition 9.3 this implies that
\[(q - 1)b_q(n) = \#\{\pi \in S_{n+q-1} | \text{Del}_q(\pi) - 1 = \text{Des}_q(\pi)\}.
\]
The proof requires the following recurrence.
Lemma 10.6. If \( n \geq q \) then
\[
h_q(n) = q \sum_{k=0}^{n-q} \binom{n-q}{k} h_q(n-k-1).
\]

Proof. The proof is by a rather standard argument.

Let \( K \subseteq \{q + 1, q + 2, \ldots, n \} \) be a subset, with \( |K| = k \), hence \( 0 \leq k \leq n - q \). Let \( \kappa \) be the word obtained by writing the numbers of \( K \) in an increasing order. Note that there are \( \binom{n-q}{k} \) such \( K \)'s—hence \( \binom{n-q}{k} \) such \( \kappa \)'s. Let \( 1 \leq i \leq q \) and let \( \sigma^{(i)} \) be a permutation of the set \( \{1, \ldots, i-1, i+1, \ldots, n\} \setminus K \), which avoids \( \text{Pat}(q) \). By definition, since there are \( n-1-k \) elements in that set, there are \( h_q(n-k-1) \) such \( \sigma^{(i)} \)'s. Now construct (the word) \( \eta^{(i)} = \sigma^{(i)} i \kappa \), then \( \eta^{(i)} \in S_n \) and it avoids \( \text{Pat}(q) \) since there is no descent in the part \( i \kappa \) of \( \eta^{(i)} \) (see Remark 9.2). For each \( 1 \leq i \leq q \), the number of \( \eta^{(i)} \)'s thus constructed is \( \sum_{k=0}^{n-q} \binom{n-q}{k} h_q(n-k-1) \), hence
\[
h_q(n) \leq q \sum_{k=0}^{n-q} \binom{n-q}{k} h_q(n-k-1).
\]

Conversely, assume \( \eta \in S_n \) avoids \( \text{Pat}(q) \). Among 1, \ldots, \( q \), let \( i \) appear the rightmost in \( \eta \) and write the word \( \eta = \sigma i \kappa \), then none of 1, \ldots, \( q \) appears in \( \kappa \). The numbers in \( \kappa \) are increasing since otherwise, if there is a descent in \( \kappa \), Remark 9.2 would imply that \( \eta \) does satisfy one of the dashed patterns in \( \text{Pat}(q) \), a contradiction. Since \( \eta \) avoids \( \text{Pat}(q) \), obviously the part \( \sigma \) of \( \eta \) also avoids \( \text{Pat}(q) \). It follows that \( \eta \) is the above permutation \( \eta = \eta^{(i)} \). This proves the reverse inequality and completes the proof. \( \square \)

The proof of Proposition 10.5 now follows by induction on \( n \geq 0 \). The case \( n = 0 \) is clear. Assume \( n \geq 1 \), then by Lemma 10.6
\[
h_q(n+q-1) = q \sum_{k=0}^{n-1} \binom{n-1}{k} h_q(n-1-k+q-1)
\]
(by induction)
\[
= q \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot (q-1)! \cdot b_q(n-k-1)
\]
\[
= (q-1)! \cdot \left[ q \sum_{k=0}^{n-1} \binom{n-1}{k} b_q(n-k-1) \right]
\]
(by Lemma 10.3)
\[
= (q-1)! \cdot b_q(n).
\]

This proves the first equation of the proposition. Together with Definition 9.1 and Proposition 9.3, this implies that \( h_q(n+q-1) = \#\{\pi \in S_{n+q-1} \mid \text{Del}_q(\pi) - 1 = \text{Des}_q(\pi)\} \), hence
\[
(q-1)!b_q(n) = \#\{\pi \in S_{n+q-1} \mid \text{Del}_q(\pi) - 1 = \text{Des}_q(\pi)\}. \quad \square
\]
In the case $q = 1$,

$$b_1(n) = b(n) = \#\text{Avoid}_1(n) = \#\{\sigma \in S_n \mid \text{Del}_1(\sigma) - 1 = \text{Des}_1(\sigma)\},$$

which appears in [3].

Let

$$H_q(x) = \sum_{n=0}^{\infty} h_q(n + q - 1) \frac{x^n}{n!}$$

be the exponential generating function of the $h_q(n + q - 1)$’s. By Remark 10.4(1) and Proposition 10.5 we have

**Corollary 10.7.**

$$H_q(x) = (q - 1)! \cdot \exp(qe^x - q).$$

10.3. Stirling numbers of the second kind

The following refinement of the second equation of Proposition 10.5 is proved in this subsection.

**Proposition 10.8.**

$$\#\{\sigma \in S_{n+q-1} \mid \text{Del}_q(\sigma) - 1 = \text{Des}_q(\sigma) \text{ and } \text{del}_q(\sigma) = k - 1\} = (q - 1)!q^k S(n, k).$$

Deduce that

$$\sum_{\{\pi \in S_n \mid \text{Del}_1(\pi) - 1 = \text{Des}_1(\pi)\}} q^{\text{del}_1(\pi)} = \frac{1}{q} \cdot b_q(n),$$

and more generally,

$$\sum_{\{\sigma \in S_{n+q-1} \mid \text{Del}_q(\sigma) - 1 = \text{Des}_q(\sigma)\}} q^{\text{del}_q(\sigma)} = \frac{(q - 1)!}{q} \cdot \sum_{k} q^{2k} S(n, k)$$

$$= \frac{(q - 1)!}{q} \cdot b_q^2(n).$$

**Proof.** We first prove the case $q = 1$ namely, that

$$\#\{\sigma \in S_n \mid \text{Del}_1(\sigma) - 1 = \text{Des}_1(\sigma) \text{ and } \text{del}_1(\sigma) = k - 1\} = S(n, k).$$

Recall that $S(n, k)$ is the number of partitions of $[n]$ into $k$ non-empty subsets. Given such a partition $D_1 \cup \cdots \cup D_k = [n]$, assume w.l.o.g. that the numbers in each $D_i$ are increasing: $D_i$ is \{d_{i,1} < d_{i,2} < \cdots\}, and also, the minimal elements $d_{1,1}, d_{2,1}, \ldots$ are decreasing: $d_{1,1} > d_{2,1} > \cdots > d_{k,1}$. Corresponding to that partition we construct the permutation $\sigma = [D_1, D_2, \ldots]$, namely $\sigma = [d_{1,1}, d_{1,2}, \ldots, d_{2,1}, d_{2,2}, \ldots, d_{k,1}, d_{k,2} \ldots]$. 

Now $\text{Del}_1(\sigma)$, the l.t.r.min of $\sigma$, are exactly at the $(k - 1)$ positions of $d_{2,1}, d_{3,1}, \ldots, d_{k,1}$, and obviously the descents occur at $\text{Del}_1(\sigma) - 1$. This establishes an injection of the set of the $k$ partitions of $[n]$ into the above set, which implies that

$$\text{card}\{\sigma \in S_n \mid \text{Del}_1(\sigma) - 1 = \text{Des}_1(\sigma) \text{ and } \text{del}_1(\sigma) = k - 1\} \geq S(n, k).$$

Since the sum on all $k$’s of both sides equals $b(n)$, this implies the case $q = 1$.

The general $q$ case follows from Proposition 8.6, and from Lemma 8.10:

Let $\pi \in S_n$. By Proposition 8.6,

$$\text{Del}_1(\pi) - 1 = \text{Des}_1(\pi) \text{ if and only if } \text{Del}_q(f_q^{-1}(\pi)) - 1 = \text{Des}_q(f_q^{-1}(\pi)),$$

and also, $\text{del}_1(\pi) = k - 1$ if and only if $\text{del}_q(f_q^{-1}(\pi)) = k - 1$. Denote $D_q(n, k) = \{\sigma \in S_{n+q-1} \mid \text{Del}_q(\sigma) - 1 = \text{Des}_q(\sigma) \text{ and } \text{del}_q(\sigma) = k - 1\}$, so that $D_1(n, k) = \{\pi \in S_n \mid \text{Del}_q(\pi) - 1 = \text{Des}_1(\pi) \text{ and } \text{del}_1(\pi) = k - 1\}$. It follows that

$$D_q(n, k) = \bigcup_{\pi \in D_1(n, k)} f_q^{-1}(\pi),$$

a disjoint union. By Lemma 8.10, $\#f_q^{-1}(\pi) = (q - 1)! \cdot q^k$ for all $\pi \in D_1(n, k)$, and the proof now follows easily from the case $q = 1$. □

10.4. Stirling numbers of the first kind

Let $c(n, k)$ be the signless Stirling numbers of the first kind.

**Proposition 10.9.** $c(n, k) = \#\{\pi \in S_n \mid \text{del}_S(\pi) = \text{del}_1(\pi) = k - 1\}$, namely, $c(n, k)$ equals the number of permutations in $S_n$ with $k - 1$ l.t.r.min.

For the proof, see Proposition 5.8 in [12].

The following is a $q$-analogue of Proposition 10.9.

**Proposition 10.10.**

$$\#\{\pi \in S_{n+q-1} \mid \text{del}_q(\pi) = k - 1\} = c_q(n, k),$$

where $c_q(n, k) = q^k(q - 1)!c(n, k)$.

**Proof.** The proof is essentially identical to the proof of Proposition 10.8, with the set $D_q(n, k)$ being replaced here by the set $H_q(n, k) = \{\pi \in S_{n+q-1} \mid \text{del}_q(\pi) = k - 1\}$. Then $H_1(n, k) = \{\pi \in S_n \mid \text{del}_1(\pi) = k - 1\}$, and by Proposition 5.8 in [12], $\#H_1(n, k) = c(n, k)$, the signless Stirling number of the first kind. The proof now follows. □

11. Equi-distribution

11.1. MacMahon type theorems for $q$-statistics

Recall the definition of $\text{rmaj}_{q,n+q-1}$ from Definition 5.9.
Remark 11.1. Note that for $\pi \in S_{n+q-1}$,
\[ \text{rmaj}_{q,n+q-1}(\pi) = \text{maj}_{1,n}(f_q(\pi)) = \text{maj}_{S_n}(f_q(\pi)). \]
This follows since by Proposition 8.6(2), $i \in \Des_q(\pi)$ if and only if $i - q + 1 \in \Des_1(f_q(\pi))$.

The following is a $q$-analogue of MacMahon’s equi-distribution theorem.

Theorem 11.2. For every positive integer $n$ and $q$
\[ \sum_{\pi \in S_{n+q-1}} t^{\text{maj}_{q,n+q-1}(\pi)} = \sum_{\pi \in S_{n+q-1}} t^{\text{inv}_q(\pi)} \]
\[ = q!(1 + tq)(1 + t + t^2q) \cdots (1 + t + \cdots + t^{n-2} + t^{n-1}q). \]

This theorem is obtained from the next one by substituting $t_2 = 1$.

Theorem 11.3. For every positive integer $n$ and $q$
\[ \sum_{\pi \in S_{n+q-1}} t_1^{\text{maj}_{q,n+q-1}(\pi)} t_2^{\text{del}_q(\pi)} = \sum_{\pi \in S_{n+q-1}} t_1^{\text{inv}_q(\pi)} t_2^{\text{del}_q(\pi)} \]
\[ = q!(1 + t_1t_2q)(1 + t_1 + t_1^2t_2q) \cdots \]
\[ \times (1 + t_1 + \cdots + t_1^{n-2} + t_1^{n-1}t_2q). \]

Proof. By Proposition 8.6 and Remark 11.1, $(\text{maj}_{S_n}, \text{maj}_{q,n+q-1})$ and $(\text{inv}, \text{inv}_q)$ are $f_q$-pairs. The proof now follows from Proposition 8.13 and Theorem 3.3.

The following is a $q$-analogue of Foata–Schützenberger’s equi-distribution theorem [7, Theorem 1].

Theorem 11.4. For every positive integer $n$ and $q$ and every subset $B \subseteq [q, n + q - 1]$
\[ \sum_{\{\pi \in S_{n+q-1} | \Des_q(\pi^{-1}) = B\}} t^{\text{inv}_q(\pi)} = \sum_{\{\pi \in S_{n+q-1} | \Des_q(\pi^{-1}) = B\}} t^{\text{maj}_{q,n+q-1}(\pi)}. \]

This theorem is obtained from the next one by substituting $B_2 = [q, n + q - 1]$.

Theorem 11.5. For every positive integer $n$ and $q$ and every subsets $B_1 \subseteq [q, n + q - 1]$ and $B_2 \subseteq [q, n + q - 1]$
\[ \sum_{\{\pi \in S_{n+q-1} | \Des_q(\pi^{-1}) = B_1, \Del_q(\pi^{-1}) = B_2\}} t^{\text{inv}_q(\pi)} \]
\[ = \sum_{\{\pi \in S_{n+q-1} | \Des_q(\pi^{-1}) = B_1, \Del_q(\pi^{-1}) = B_2\}} t^{\text{maj}_{q,n+q-1}(\pi)}. \]
**Proof.** By Proposition 8.6 and Remark 11.1, it suffices to prove that for every subset $B_1 \subseteq [n-1]$ and $B_2 \subseteq [n-1]$

$$\sum_{\{\pi \in S_{n+q-1} | \text{Des}_1(f_q(\pi^{-1})) = B_1, \text{Del}_1(f_q(\pi^{-1})) = B_2\}} t^{\text{inv}_1(f_q(\pi))}$$

$$= \sum_{\{\pi \in S_{n+q-1} | \text{Des}_1(f_q(\pi^{-1})) = B_1, \text{Del}_1(f_q(\pi^{-1})) = B_2\}} t^{\text{maj}_{1,n}(f_q(\pi))}.$$ 

By Proposition 8.4 $f_q(\pi^{-1}) = f_q(\pi)^{-1}$. Thus, denoting $\sigma = f_q(\pi)$, it suffices to prove that

$$\sum_{\{\sigma \in S_n | \text{Des}_1(\sigma^{-1}) = B_1, \text{Del}_1(\sigma^{-1}) = B_2\}} \# f_q^{-1}(\sigma) \cdot t^{\text{inv}_1(\sigma)}$$

$$= \sum_{\{\sigma \in S_n | \text{Des}_1(\sigma^{-1}) = B_1, \text{Del}_1(\sigma^{-1}) = B_2\}} \# f_q^{-1}(\sigma) \cdot t^{\text{maj}_{1,n}(\sigma)}.$$ 

By Propositions 5.2 and 5.5, for every $\sigma \in S_n$ with Del$_1(\sigma^{-1}) = B_2$, Del$_1(\sigma) = #B_2$. Thus, by Lemma 8.10, $\# f_q^{-1}(\sigma) = (q - 1)! \cdot q^{#B_2+1}$ for all permutations in the sums. Hence, the theorem is reduced to

$$(q - 1)! \cdot q^{#B_2+1} \sum_{\{\sigma \in S_n | \text{Des}_1(\sigma^{-1}) = B_1, \text{Del}_1(\sigma^{-1}) = B_2\}} t^{\text{inv}_1(\sigma)}$$

$$= (q - 1)! \cdot q^{#B_2+1} \sum_{\{\sigma \in S_n | \text{Des}_1(\sigma^{-1}) = B_1, \text{Del}_1(\sigma^{-1}) = B_2\}} t^{\text{maj}_{1,n}(\sigma)}.$$ 

**Theorem 3.2** completes the proof. \qed

**Corollary 11.6.** For every positive integer $n$ and $q$

$$\sum_{\pi \in S_{n+q-1}} t^{\text{inv}_q(\pi)} t^{\text{des}_q(\pi^{-1})} = \sum_{\pi \in S_{n+q-1}} t^{\text{maj}_{q,n+q-1}(\pi)} t^{\text{des}_q(\pi^{-1})},$$

and

$$\sum_{\pi \in S_{n+q-1}} t^{\text{inv}_q(\pi)} t^{\text{maj}_{q,n+q-1}(\pi^{-1})} = \sum_{\pi \in S_{n+q-1}} t^{\text{maj}_{q,n+q-1}(\pi)} t^{\text{maj}_{q,n+q-1}(\pi^{-1})}.$$ 

**11.2. Equi-distribution on Avoid$_q(n)$**

The main theorem on equi-distribution on permutations avoiding patterns is the following.

**Theorem 11.7.** For every positive integer $n$ and $q$ and every subset $B \subseteq [q, \ldots, n+q-2]$

$$\sum_{\{\pi^{-1} \in \text{Avoid}_q(n+q-1) | \text{Des}_q(\pi^{-1}) = B\}} t^{\text{maj}_{q,n+q-1}(\pi)}$$

$$= \sum_{\{\pi^{-1} \in \text{Avoid}_q(n+q-1) | \text{Des}_q(\pi^{-1}) = B\}} t^{\text{inv}_q(\pi)}.$$
Proof. Substituting $B_1 = B_2 - 1 = B$ in Theorem 11.5 we obtain, for every subset $B \subseteq [q, n + q - 1]$

$$
\sum_{\{\pi \in S_{n+q-1} | \text{Des}_q(\pi^{-1}) = \text{Del}_q(\pi^{-1}) - 1 = B\}} t^{\text{inv}_q(\pi)} = \\
\sum_{\{\pi \in S_{n+q-1} | \text{Des}_q(\pi^{-1}) = \text{Del}_q(\pi^{-1}) - 1 = B\}} t^{\text{maj}_{q,n+q-1}(\pi)}.
$$

By Proposition 9.3

$$
\{\pi \in S_{n+q-1} | \text{Des}_q(\pi^{-1}) = \text{Del}_q(\pi^{-1}) - 1 = B\} = \{\pi^{-1} \in \text{Avoid}_q(n + q - 1) | \text{Des}_q(\pi^{-1}) = B\}.
$$

This completes the proof. □

Theorem 11.7 implies

**Corollary 11.8.** For every positive integer $n$ and $q$

$$
\sum_{\pi^{-1} \in \text{Avoid}_q(n+q-1)} t^{\text{maj}_{q,n+q-1}(\pi)} \frac{\text{des}_q(\pi)}{t_1} = \\
\sum_{\pi^{-1} \in \text{Avoid}_q(n+q-1)} t^{\text{inv}_q(\pi)} \frac{\text{des}_q(\pi)}{t_2}.
$$

The following is an extension of MacMahon’s theorem to permutations avoiding patterns.

**Theorem 11.9.** For every positive integer $n$ and $q$

$$
\sum_{\pi^{-1} \in \text{Avoid}_q(n+q-1)} t^{\text{maj}_{q,n+q-1}(\pi)} = \\
\sum_{\pi^{-1} \in \text{Avoid}_q(n+q-1)} t^{\text{inv}_q(\pi)}
$$

Proof. Substitute $t_2 = 1$ in Corollary 11.8. □

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**Appendix. Des$_2 = \text{Des}_A$: the proof**

**Lemma A.1.** Let $w = [b_1, \ldots, b_{n+1}] \in A_{n+1}$. Let $1 \leq i \leq n - 1$, then $i \in \text{Des}_A(w)$ if and only if one of the following two conditions hold.

1. $b_{i+1} > b_{i+2}$, or
2. $b_{i+1} < b_{i+2}$ and $b_1, b_2, \ldots, b_i > b_{i+2}$.

In particular, $1 \in \text{Des}_A(w)$ if and only if $b_1 > b_3$ (and/) or $b_2 > b_3$. 
Proof. The basic tool is the formula
\[ \ell_A(w) = \ell_S(w) - \text{del}_S(w). \]
Assume first that \( 2 \leq i \leq n - 1 \), then \( v = wa_i = [b_2, b_1, \ldots, b_{i+2}, b_{i+1}, \ldots]. \) Now compare \( \ell_S(w) \) with \( \ell_S(v) \), and \( \text{del}_S(w) \) with \( \text{del}_S(v) \), then apply the above formula, and the proof follows. Here are the details.

The case \( 2 \leq i \leq n - 1 \) and \( b_{i+1} > b_{i+2} \).

If \( b_1 < b_2 \) then \( \ell_S(w) = \ell_S(v) \). Now, \( \text{del}(\sigma) \) is the number of l.t.r.min in \( \sigma \). Interchanging \( b_1 < b_2 \) in \( w \) adds one such l.t.r.min, while interchanging \( b_{i+1} > b_{i+2} \) reduces that (\( \text{del}_S \)) number by one, or leaves it unchanged. In particular, \( \text{del}_S(w) \leq \text{del}_S(v) \). It follows that \( \ell_A(w) = \ell_S(w) - \text{del}_S(w) \geq \ell_S(v) - \text{del}_S(v) = \ell_A(v) \), i.e. \( \ell_A(wa_i) \leq \ell_A(w) \), hence \( i \in \text{Des}_A(w) \).

Similarly for the other cases. If \( b_1 > b_2 \) (and \( b_{i+1} > b_{i+2} \)), verify that \( \ell_S(w) = \ell_S(v) + 2 \), while \( \text{del}_S(w) \leq \text{del}_S(v) + 2 \), and again this implies that \( i \in \text{Des}_A(w) \). This completes the proof of 2.a.

The case \( 2 \leq i \leq n - 1 \) and \( b_{i+1} < b_{i+2} \).

First, assume \( b_1 < b_2 \), then \( \ell_S(v) = \ell_S(w) + 2 \). If \( b_1, b_2, \ldots, b_i > b_{i+2} \), then \( \text{del}_S(w) = \text{del}_S(v) + 2 \), hence \( \ell_A(wa_i) = \ell_A(v) = \ell_A(w) \), and \( i \in \text{Des}_A(w) \). If \( b_j < b_{i+2} \) for some \( 1 \leq j \leq i \) then \( \text{del}_S(v) = \text{del}_S(w) + 1 \) and it follows that \( i \notin \text{Des}_A(w) \).

If \( b_1 > b_2 \) then \( \ell_S(v) = \ell_S(w) \). Assuming that \( b_1, b_2, \ldots, b_i > b_{i+2} \), deduce that \( \text{del}_S(v) = \text{del}_S(w) \), hence \( i \in \text{Des}_A(w) \). If \( b_j < b_{i+2} \) for some \( 1 \leq j \leq i \) then \( \text{del}_S(v) = \text{del}_S(w) - 1 \), so \( \ell_A(wa_i) = \ell_A(v) = \ell_A(w) - 1 \) and \( i \notin \text{Des}_A(w) \).

Finally assume that \( i = 1 \), then \( v = wa_1 = ws_1s_2 = [b_2, b_3, b_1, b_4, b_5, \ldots] \). Obviously, \( \ell_S(w) - \ell_S(v) \) depends only on the order relations among \( b_1, b_2, b_3 \), and similarly for \( \text{del}_S(w) - \text{del}_S(v) \). We can therefore assume that \( \{b_1, b_2, b_3\} = \{1, 2, 3\} \), then check the 3! = 6 possible cases of \( w = [b_1, b_2, b_3, \ldots] \). For example, assume \( w = [1, 3, 2, \ldots] \), then \( wa_1 = [3, 2, 1, \ldots] = v \), so \( \ell_S(v) = \ell_S(w) + 2 \) while \( \text{del}_S(v) = \text{del}_S(w) + 2 \), hence \( 1 \in \text{Des}_A(w) \).

Similarly for the remaining five cases. □

References

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