An Application of the Umbral Calculus

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Submitted by R. P. Boas
Received January 4, 1988

The partial difference equation

\[ r(i, j) = r(i, j-1) + r(i-1, j) + r(i-1, j+1), \]

where \( r(i, j) \) are defined for integer numbers \( i \) and \( j, i \geq 0 \), by the conditions \( r(0, j) = 1 \) for all \( j \) and \( r(i, -1) = 0 \) for \( i > 0 \) is solved. For \( i > 0 \) and \( j > 0 \) a combinatorial meaning of numbers \( r(i, j) \) is given. The solution is obtained by the modern classical umbral calculus.

1. INTRODUCTION

PROBLEM. Let \( S = \{(i, j): i, j = 0, 1, 2, \ldots \} \). Define in the set \( S \) the relation \( \rho \) by

\( (i, j) \rho (p, q) \) if and only if \( (p = i, q = j - 1) \) or \( (p = i - 1, q = j) \) or \( (p = i - 1, q = j + 1) \).

The point \( (i, j) \in S \) is said to be connected with the origin \( (0, 0) \in S \) if and only if there exist points \( (i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n) \) in \( S \), where \( (i_1, j_1) \rho (0, 0), (i_2, j_2) \rho (i_1, j_1), \ldots, (i, j) \rho (i_n, j_n) \). Our aim is to compute the number \( r(i, j) \) of different connections of the point \( (i, j) \in S \) with the origin \( (0, 0) \). If we put it in the language of the graph theory, our problem is to determine the number of linearly connected graphs with vertices in the set \( S \) and with edges oriented parallel to the vectors \((1, 0), (0, 1), \) and \((1, -1)\).

Figure 1 shows one of the possible connections of the point \((3, 2)\) with the origin.

It is clear that \( r(0, j) = 1 \) for \( j \geq 1 \). Define \( r(0, 0) = 1 \). By an easy combinatorial argument we get the partial difference equation

\[ r(i, j) = r(i, j-1) + r(i-1, j) + r(i-1, j+1), \quad i \geq 1, j \geq 0; \]

\[ r(0, j) = 1, j > 0; \quad r(i, -1) = 0, i > 0. \]

A simple computation gives us the numbers \( r(i, j) \) in Table I.
In the sequel we shall derive the formula for our numbers \( r(i, j) \), the generating functions for the rows \( r(\cdot, j) \), and the columns \( r(i, \cdot) \).

*The umbral calculus.* We repeat the basic facts following Niven and Roman (see \([1, 2]\)). Let \( F \) denote the algebra of formal power series in the variable \( t \) over the field \( \mathbb{C} \). An element in \( F \) has the form

\[
f(t) = \sum_{k=0}^{\infty} a_k t^k, \quad a_k \in \mathbb{C}.
\]  

The addition and multiplication are defined formally by

\[
\sum_{k=0}^{\infty} a_k t^k + \sum_{k=0}^{\infty} b_k t^k = \sum_{k=0}^{\infty} (a_k + b_k) t^k
\]

\[
\left( \sum_{k=0}^{\infty} a_k t^k \right) \left( \sum_{k=0}^{\infty} b_k t^k \right) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} a_j b_{k-j} \right) t^k.
\]

**TABLE 1**

The Numbers \( r(i, j) \) for \( i \geq 0, j \geq 0 \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
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<td>1</td>
<td>16</td>
<td>160</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
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<td>1</td>
<td>14</td>
<td>126</td>
<td>938</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
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<td>1</td>
<td>12</td>
<td>96</td>
<td>652</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
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<td>1</td>
<td>10</td>
<td>70</td>
<td>430</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
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<td>1</td>
<td>8</td>
<td>48</td>
<td>264</td>
<td>1408</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
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<td>1</td>
<td>6</td>
<td>30</td>
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<td>714</td>
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<td>...</td>
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<tr>
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<td>1</td>
<td>4</td>
<td>16</td>
<td>68</td>
<td>304</td>
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<td>...</td>
</tr>
<tr>
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<td>1</td>
<td>2</td>
<td>6</td>
<td>22</td>
<td>90</td>
<td>394</td>
<td>1806</td>
</tr>
</tbody>
</table>
Two formal power series are equal if and only if \( a_k = b_k \) for all \( k \). Let \( F_0 \) denote the set of all formal power series (2) where \( a_0 \neq 0 \) and \( F_1 \) the set of all formal power series (2) where \( a_0 = 0 \) and \( a_1 \neq 0 \). If \( f(t) \in F_0 \) then \( f(t) \) is invertible, and the formal inverse will be denoted by \( f(t)^{-1} \). The coefficients of the inverse can be computed solving a simple triangular system of equations. If \( f \) belong to the set \( F_1 \), then a compositional inverse \( f(t) \) exists, such that \( f(f(t)) = t \).

The formal derivative of the series (2) is defined as

\[
D_t f(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}.
\]

Let \( P \) denote the algebra of polynomials in the single variable \( x \) over the field \( \mathbb{C} \). Let \( P^* \) be the vector space of all linear functionals on \( P \). The action of the functional \( L \in P^* \) on the polynomial \( p(x) \in P \) will be denoted by

\[
\langle L | p(x) \rangle.
\]

Each formal power series

\[
f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k
\]

defines a linear functional on \( P \) if we set

\[
\langle f(t) | x^n \rangle = a_n \quad \text{for } n \geq 0.
\]

For any linear functional \( L \in P^* \) we have a formal power series

\[
f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L | x^k \rangle}{k!} t^k
\]

which has the form (3) and satisfies the relation

\[
\langle f_L(t) | x^n \rangle = \langle L | x^n \rangle \quad \text{for } n \geq 0.
\]

The map \( L \rightarrow f_L(t) \) is a vector space isomorphism from \( P^* \) to \( F \).

In the sequel we shall need the formulas

\[
\langle t^k | p(x) \rangle = p^{(k)}(0), \quad k \geq 0, p(x) \in P \quad (4)
\]

\[
\langle f(t) g(t) | x^n \rangle = \sum_{k=0}^{n} \binom{n}{k} \langle f(t) | x^k \rangle \langle g(t) | x^{n-k} \rangle \quad (5)
\]

\[
\langle f(t) | xp(x) \rangle = \langle D_t f(t) | p(x) \rangle. \quad (6)
\]
Any power series defines a linear operator on $P$. If $f(t)$ has the form (3), then we define
\[ f(t)x^n = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k} \quad \text{for } n \geq 0. \] (7)

Especially, for $f(t) = t^k$ we get
\[ t^k x^n = k! \binom{n}{k} x^{n-k}, \]
the $k$th derivative of the power $x^n$. Using the relation (5) we obtain
\[ \langle f(t) g(t) | p(x) \rangle = \langle g(t) | f(t) p(x) \rangle. \] (8)

Sheffer sequences. For each series $f(t) \in F_1$ and each series $g(t) \in F_0$ there exists a unique sequence of polynomials $s_n(x)$ such that
\[ \langle g(t) f(t)^k | s_n(x) \rangle = n! \delta_{n,k}, \]
where $\delta_{n,k}$ denotes the Kronecker delta function and the polynomial $s_n(x)$ has degree $n$. We say that the sequence $s_n(x)$ is Sheffer for the pair $(g(t), f(t))$. If $s_n(x)$ is Sheffer for the pair $(1, f(t))$ then $s_n(x)$ is associated to $f(t)$. The Sheffer sequence $s_n(x)$ of the pair $(g(t), f(t))$ admits the generating function
\[ g(\mathcal{F}(t))^{-1} e^{yf(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \] (9)
where $y \in \mathbb{C}$.

From (8) it follows that the sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ if and only if the sequence $g(t) s_n(x)$ is associated to $f(t)$.

A sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ for some $g(t) \in F_0$ if and only if the relation
\[ f(t) s_n(x) = ns_{n-1}(x) \] (10)
holds for all $n \geq 0$.

The sequence $s_n(x)$ is associated to $f(t)$ if and only if $\langle t^0 | s_n(x) \rangle = \delta_{n,0}$ and $f(t) s_n(x) = ns_{n-1}(x)$. 
APPLICATION OF UMbral Calculus

For the series \( f(t) = a_1 t + a_2 t^2 + \ldots, a_1 \neq 0 \), denote
\[
\frac{f(t)}{t} = a_1 + a_2 t + \ldots.
\]

It is clear that \( f(t) \in F_1 \) and \( f(t)/t \in F_0 \). The inverse of the series \( f(t)/t \) will be denoted by \( t/f(t) \).

We compute the associated sequence of the series \( f(t) \in F_1 \) by the transfer formula
\[
s_n(x) = x \left( \frac{t}{f(t)} \right)^n x^{n-1}
\]
for \( n \geq 1 \). Note that \( s_0(x) = 1 \).

These are the results of the excellent monograph [2]. We return now to our problem.

2. MAIN RESULTS

Since the simple power series \( 1 + t \) and \( 2 + t \) are formally invertible the formal power series
\[
f(t) = (1+t)^{-1} (2+t)^{-1}
\]
belongs to the set \( F_1 \). For each series \( g(t) \in F_0 \) we have the unique sequence of polynomials \( s_n(x) \) which are Sheffer for \( (g(t), f(t)) \). Denote \( p_n(x) \) as the associated sequence for \( f(t) \). It is clear that
\[
s_n(x) = g(t)^{-1} p_n(x)
\]
for all \( n \geq 0 \).

**Lemma 1.** Let \( s_n(x) \) be Sheffer for \( (g(t), f(t)) \), where \( f(t) \) is given by (12) and \( g(t) \) is an arbitrary invertible formal power series. Then the double sequence
\[
q(i,j) = \frac{1}{i!} \langle (1 + t)^i | s_j(x) \rangle, \quad i \geq 0, j \in \mathbb{Z},
\]
satisfies the partial difference equation
\[
q(i,j) = q(i,j-1) + q(i-1,j) + q(i-1,j+1)
\]
for \( i \geq 1 \) and \( j \in \mathbb{Z} \).
For every \( j \in \mathbb{Z} \) and \( i \geq 1 \) we have

\[
q(i, j) - q(i, j - 1) - q(i - 1, j) - q(i - 1, j + 1)
= \frac{1}{i!} \langle (1 + t)^i | s_i(x) \rangle - \frac{1}{i!} \langle (1 + t)^{i-1} | s_i(x) \rangle
- \frac{1}{(i-1)!} \langle (1 + t)^i | s_{i-1}(x) \rangle - \frac{1}{(i-1)!} \langle (1 + t)^{i+1} | s_{i-1}(x) \rangle
= \frac{1}{i!} \langle (1 + t)^{i-1} t | s_i(x) \rangle - \frac{1}{(i-1)!} \langle (1 + t)^i (2 + t) | s_{i-1}(x) \rangle.
\]

Using the relation (10) we obtain

\[
q(i, j) - q(i, j - 1) - q(i - 1, j) - q(i - 1, j + 1)
= \frac{1}{i!} \langle (1 + t)^{j-1} (t - (1 + t)(2 + t)f(t)) | s_i(x) \rangle = 0
\]

because of (12).

**Lemma 2.** If the invertible series \( g(t) \) in Lemma 1 has the form

\[
g(t) = 1 + a_1 t + a_2 t^2 + \ldots
\]

then the sequence \( q(i, j) \) has the property

\[
q(0, j) = 1
\]

for all \( j \in \mathbb{Z} \).

**Proof.** By (13) we have

\[
q(0, j) = \langle (1 + t)^j | s_0(x) \rangle = \langle (1 + t)^{j'} g(t)^{-1} p_0(t) \rangle
= \langle (1 + t)^j g(t)^{-1} | 1 \rangle = \langle h(t) | 1 \rangle,
\]

where the formal power series \( h(t) \) has the form

\[
h(t) = 1 + b_1 t + b_2 t^2 + \ldots.
\]

By the definition of the power series as a linear functional on the vector space \( P \) we get \( q(0, j) = 1 \).

**Lemma 3.** The unique invertible series \( g(t) \), such that the double sequence \( q(i, j) \) in Lemma 1 has properties
(i) \( q(0, j) = 1 \) for all \( j \in \mathbb{Z} \),
(ii) \( q(i, -1) = 0 \) for all \( i \in \mathbb{N} \)
is the series \( g(t) = (1 + t)^{-1} \).

Proof. According to Lemma 2 we must prove only (ii). Let
\[ g(t)^{-1} = 1 + c_1 t + c_2 t^2 + \cdots . \]
We have
\[ q(i, -1) = \frac{1}{i!} \langle (1 + t)^{-1} g(t)^{-1} | p_i(x) \rangle \]
\[ = \frac{1}{i!} \langle h(t) | p_i(x) \rangle , \]
where
\[ h(t) = 1 + b_1 t + b_2 t^2 + \cdots \]
with
\[ b_n = \sum_{k=0}^{n} (-1)^k c_{n-k}, \quad b_0 = c_0 = 1. \]
For \( i = 1, 2, 3, \ldots \), we deduce, using the relation (4), that
\[ q(i, -1) = \frac{1}{i!} \sum_{k=0}^{i} b_k p_i^{(k)}(0). \]

Note that \( p_i(x) \) is a polynomial of degree \( i \), thus \( p_i^{(i)}(0) \neq 0 \). The relation
\[ \langle f^0 | p_i(x) \rangle = p_i(0) = \delta_{i,0} \]
implies, according to (ii), the system of equations for \( b_k \):
\[ b_1 p_1(0) = 0 \]
\[ b_1 p_2'(0) + b_2 p_2''(0) = 0 \]
\[ b_1 p_3'(0) + b_2 p_3''(0) + b_3 p_3'''(0) = 0 \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
Step by step we conclude that \( b_1 = b_2 = b_3 = \cdots = 0 \). From the other system
\[ b_1 = c_1 - 1 \]
\[ b_2 = c_2 - c_1 + 1 \]
\[ b_3 = c_3 - c_2 + c_1 - 1 \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
we get $c_1 = 1$, $c_2 = c_3 = c_4 = \cdots = 0$. Thus $g(t)^{-1} = 1 + t$ respectively $g(t) = (1 + t)^{-1}$.

Lemmas 1, 2, and 3 imply the following result:

**Theorem 1.** The unique solution of the partial difference equation

$$r(i, j) = r(i, j - 1) + r(i - 1, j) + r(i - 1, j + 1)$$

with conditions

$$r(0, j) = 1 \text{ for all } j \quad \text{and} \quad r(i, -1) = 0 \text{ for } i \geq 1$$

is given by the formula

$$r(i, j) = \sum_{k=0}^{\min(i, j)} \binom{i+j}{i} \binom{i}{k} \binom{j}{k} 2^{i+j-k}.$$

**Proof.** It is clear that $r(i, j) = q(i, j)$ in Lemma 1 for $g(t) = (1 + t)^{-1}$. For $i \geq 0$ and every $j \in \mathbb{Z}$ we have

$$r(i, j) = \sum_{k=0}^{\min(i, j)} \binom{i+j}{i} \binom{i}{k} \binom{j}{k} 2^{i+j-k}.$$

By the transfer formula (11) we find an explicit form for the associated sequence $p_n(x)$ of the series (12):

$$p_n(x) = x(1 + t)^n (2 + t)^n x^{n-1}, \quad n \geq 1.$$

Using formula (6) we obtain

$$r(i, j) = \sum_{k=0}^{\min(i, j)} \binom{i+j}{i} \binom{i}{k} \binom{j}{k} 2^{i+j-k}.$$

for all $i \geq 1$ and $j$. This concludes the proof.

**Theorem 2.** The explicit form for the numbers $r(i, j)$ for $i \geq 1$ and $j \in \mathbb{Z}$ is

$$r(i, j) = \sum_{k=0}^{\min(i, j)} \binom{i+j}{i} \binom{i}{k} \binom{j}{k} 2^{i+j-k}.$$
Proof. Formula (4) implies $\langle t^k | x^n \rangle = n! \delta_{k,n}$. Using formula (5) we get from (16)

$$r(i, j) = \frac{j + 1}{i!} \sum_{k=0}^{i-1} \left( \frac{i - 1}{k} \right) \langle (1 + t)^{i+j} | x^k \rangle \langle (2 + t)^i | x^{i-1-k} \rangle.$$  

Since

$$\langle (1 + t)^{i+j} | x^k \rangle = \binom{i+j}{k} k! \quad \text{and} \quad \langle (2 + t)^i | x^{i-1-k} \rangle = \binom{i}{k+1} 2^{k+1} (i-1-k)!$$

the desired result follows from a simple computation.

In our case formula (10) gives the recurrence formula for the associated polynomials

$$p_n(x) = p_{n-1}(x) + 3p_{n-1}(x) + 2p_{n-1}(x)$$

for $n \geq 1$ and the initial conditions $p_0(0) = \delta_{n,0}$. We find

$$p_0(x) = 1, p_1(x) = 2x, p_2(x) = 4x^2 + 12x, p_3(x) = 8x^3 + 72x^2 + 132x.$$

3. Generating Functions

The generating function for the sequence of polynomials $p_n(x)$ follows immediately from the expansion (9)

$$e^{yf(t)} = \sum_{n=0}^{\infty} \frac{p_n(y)}{n!} t^n. \quad (17)$$

If we differentiate this relation with respect to $y$, we obtain after setting $y = 0$

$$f'(t) = \sum_{n=1}^{\infty} \frac{p'_n(0)}{n!} t^n. \quad (18)$$

For $n > 0$ we have from (14)

$$r(n, 0) = \frac{1}{n!} \langle (1 + t) | p_n(x) \rangle = \frac{1}{n!} (p_n(0) + p'_n(0)) = \frac{1}{n!} p'_n(0).$$
and so

\[ f(t) = \sum_{n=1}^{\infty} r(n, 0) t^n. \quad (19) \]

Since \( r(0, 0) = 1 \) we have the generating function for the row \( r(\cdot, 0) \):

\[ 1 + f(t) = \sum_{n=0}^{\infty} r(n, 0) t^n. \quad (20) \]

Recall that for every formal power series (see [1])

\[ h(t) = 1 + a_1 t + a_2 t^2 + \cdots \]

there is a unique formal power series \( h(t)^{1/2} \) of the form

\[ h(t)^{1/2} = 1 + b_1 t + b_2 t^2 + \cdots \]

such that \( (h(t)^{1/2})^2 = h(t) \). From (12) we obtain the candidate for the series \( f(t) \):

\[ f(t) = \frac{1 - 3t - (1 - 6t + t^2)^{1/2}}{2t}. \quad (21) \]

We must show that the numerator in (21) has the correct form. Let

\[ (1 - 6t + t^2)^{1/2} = 1 + b_1 t + b_2 t^2 + \cdots. \]

We get the system of equations for the coefficients \( b_n \):

\[ \begin{align*}
2b_1 &= -6 \\
2b_2 + b_1^2 &= 1 \\
2b_3 + 2b_1 b_2 &= 0 \\
2b_4 + 2b_1 b_3 + b_2^2 &= 0 \\
\end{align*} \quad (22) \]

Successively we compute:

\[ b_1 = -3, \ b_2 = -4, \ b_3 = -12, \ b_4 = -44, \ldots \]

The numerator in (21) is the formal power series

\[ 4t^2 + 12t^4 + 44t^4 + \cdots \]

and the compositional inverse \( f(t) \) of the series \( f(t) \) should be

\[ f(t) = 2t + 6t^2 + 22t^3 + \cdots. \]
A straightforward computation shows that the right side in (21) is really $\dot{f}(t)$ in the sense of the formal power series theory. We omit the proof.

The solution of the system (22) is connected with the numbers $r(i, 0)$, namely,

$$r(0, 0) = 1, \quad r(1, 0) = -b_2/2, \quad r(2, 0) = -b_3/2, \ldots$$

We can find the row $r(\cdot, 0)$ independently of the other rows and columns.

Denote by $G_n(t)$ the generating functions of the $n$th row in Table I

$$G_n(t) = \sum_{i=0}^{\infty} r(i, n) t^i, \quad n \in \mathbb{Z}. \quad (23)$$

We have the result

$$G_0(t) = 1 + \dot{f}(t) = \frac{1 - t - (1 - 6t + t^2)^{1/2}}{2t}. \quad (24)$$

**Theorem 3.** The generating functions $G_n(t)$ of the $n$th row of the numbers $r(i, j)$ are given by

$$G_n(t) = (1 + \dot{f}(t))^{n+1} = \left(\frac{1 - t - (1 - 6t + t^2)^{1/2}}{2t}\right)^{n+1}. \quad (25)$$

**Proof.** If we differentiate the relation (17) $k$ times with respect to $v$, we get

$$\dot{f}(t)^k = \sum_{n=0}^{\infty} \frac{p_n^{(k)}(0)}{n!} t^n.$$ 

By the binomial formula we have

$$(1 + \dot{f}(t))^{n+1} = \sum_{k=0}^{n+1} \binom{m+1}{k} \dot{f}(t)^k = \sum_{k=0}^{n+1} \binom{m+1}{k} \sum_{n=0}^{\infty} \frac{p_n^{(k)}(0)}{n!} t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n+1} \binom{m+1}{k} \langle t^k | p_n(x) \rangle \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \langle (1 + t)^{n+1} | p_n(x) \rangle \frac{t^n}{n!} = \sum_{n=0}^{\infty} r(n, m) t^n.$$ 

**Corollary.** For every integer $p$ the relation

$$r(n, m) = \sum_{k=0}^{n} r(k, p-1) r(n-k, m-p) \quad (26)$$
holds. Especially
\[ r(n, m) = \sum_{k=0}^{n} r(k, m - 1) r(n - k, 0). \]
In other words, the convolution product of the \( p \)th and \( q \)th rows gives the \((p + q + 1)\)th row in the table of the numbers \( r(i, j) \).

Similarly, we define the generating functions for the columns. Let
\[ H_i(t) = \sum_{j=0}^{\infty} \frac{r(i, j)}{j!} t^j. \]
Note that only the numbers \( r(i, j), i \geq 0, j \geq 0 \), enter in this formal power series.

**THEOREM 4.** For every non-negative number \( i \) the generating function for columns of the numbers \( r(i, j) \) can be written in the form
\[ H_i(t) = \frac{1}{i!} e^{s_i(t)}. \] (27)

**Proof.** The definition of a formal power series as a linear functional on \( P \) implies that
\[ \langle H_i(t) | x^n \rangle = r(i, n) \]
for \( i \geq 0 \) and \( n \geq 0 \).

Define
\[ f_i(t) = \frac{1}{i!} e^{s_i(t)}. \]
We have
\[ \langle f_i(t) | x^n \rangle = \frac{1}{i!} \langle e^{s_i(t)} | x^n \rangle = \frac{1}{i!} \langle s_i(t) | e^{x^n} \rangle = \frac{1}{i!} \langle (t + 1)^n | s_i(x) \rangle = r(i, n). \]

It follows that \( \langle H_i(t) | x^n \rangle = \langle f_i(t) | x^n \rangle \) which implies \( H_i(t) = f_i(t) \). Note that the series
\[ e^{yt} = 1 + \frac{yt}{1!} + \frac{y^2t^2}{2!} + \ldots \]
implies \( \langle e^{xt} | p(x) \rangle = p(y) \) and \( e^{xt} p(x) = p(x + y) \) for every \( y \in \mathbb{C} \) and every polynomial \( p(x) \in P \). It is also easy to see that \( \langle p(t) | q(x) \rangle = \langle q(t) | p(x) \rangle \) for any two polynomials \( p(x) \) and \( q(x) \). The proof is complete.

We now go a step further. It is possible to construct a generating function for \( H_s(t) \). For a fixed \( s \in \mathbb{C} \) we define

\[
\mathcal{G}(s, t) = \sum_{i=0}^{\infty} H_i(s) t^i.
\]

The function \( \mathcal{G}(s, t) \) can be written in the closed form. Recall that

\[
(1 + f(t)) e^{sf(t)} = \sum_{i=0}^{\infty} \frac{s^i}{i!} t^i
\]

because of the expansion (9). We obtain

\[
\mathcal{G}(s, t) = \sum_{i=0}^{\infty} e^s \frac{s^i}{i!} t^i = e^s(1 + f(t)) e^{sf(t)} = e^s G_0(t) e^{sf(t)}.
\]

Note that the form \( e^{s(1 + f(t))} \) is not correct because the series \( 1 + f(t) \) has the zeroth coefficient different from 0.

Differentiating with respect to \( s \) we get

\[
D_s \mathcal{G}(s, t) = e^s G_0(t)^2 e^{sf(t)},
\]

\[
D_s^2 \mathcal{G}(s, t) = e^s G_0(t)^3 e^{sf(t)}.
\]

Since \( f(f(t)) = t \) we have an equation for the function \( G_0(t) \):

\[
t G_0(t)(1 + G_0(t)) = G_0(t) - 1.
\]

**Theorem 5.** The function \( \mathcal{G}(s, t) \) is a formal solution of the equation

\[
(tD_s^2 + (t - 1) D_s + 1) \mathcal{G}(s, t) = 0
\]

with the boundary condition

\[
D_s^2 \mathcal{G}(0, t) = \mathcal{G}(0, t)^2 \neq 0.
\]

**Proof.** A simple verification.
4. The Group Structure

Table I contains the numbers \( r(i, j) \) for \( j \geq 0 \) only. But we also can write these for \( j < 0 \). One method is by using generating functions \( G_j(i) \). The other, simplest, way to compute \( r(i, j) \) is with the recurrence relation

\[
r(i, j - 1) = r(i, j) - r(i - 1, j) - r(i - 1, j + 1).
\]

Table II the central part of the extended table for numbers \( r(i, j) \). Denote \( r(\cdot, j) = a_j \). Define the convolution product \( x * y \) of sequences \( x = (x_0, x_1, x_2, \ldots) \) and \( y = (y_0, y_1, y_2, \ldots) \):

\[
(x * y)_n = \sum_{k=0}^{n} x_k y_{n-k}.
\]

It is easy to see that \((x * y) * z = x * (y * z)\) for all sequences \( x, y, \) and \( z \). For our sequences \( a_k \) we find the following properties:

\[
a_i * a_j = a_{i+j}, \quad a_i * a_0 = a_i, \quad a_1 * a_{-1} = a_0.
\]

We have

**Theorem 6.** The set \( \{ \ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots \} \) with the convolution product is an infinite cyclic group. The unit in this group is the sequence \( a_0 = (1, 0, 0, \ldots) \). The convolutional inverse of the sequence \( a_i \) is the sequence \( a_{-i} \). For any sequence \( y \) the equation \( a_i * x = y \) has the unique solution \( x = a_{-i} * y \). The group generator is the sequence \( a_1 \).

| \( r(\cdot, 2) \) |  1  |  6  |  30  |  146 |  714 | \ldots | \( a_1 \) |
| \( r(\cdot, 1) \) |  1  |  4  |  16  |   68 |  304 | \ldots | \( a_2 \) |
| \( r(\cdot, 0) \) |  1  |  2  |   6  |   22 |   90 | \ldots | \( a_1 \) |
| \( r(\cdot, -1) \) |  1  |  0  |   0  |   0  |   0  | \ldots | \( a_0 \) |
| \( r(\cdot, -2) \) |  1  | -2  | -2  | -2  | -22  | \ldots | \( a_{-1} \) |
| \( r(\cdot, -3) \) |  1  | -4  |  0  | -4  | -16  | \ldots | \( a_{-2} \) |
| \( r(\cdot, -4) \) |  1  | -6  |  6  | -2  | -6   | \ldots | \( a_{-3} \) |
The generating function of numbers in \( a_i \) is given by (24) and (20). From the relation

\[ 1 - t - (1 - 6t + t^2)^{1/2} = 2 \sum_{n=1}^{\infty} r(n-1, 0) t^n \] (28)

we obtain after formal derivation

\[ -1 - (t - 3)(1 - 6t + t^2)^{-1/2} = 2 \sum_{n=0}^{\infty} (n+1) r(n, 0) t^n. \] (29)

Multiply (29) by \( 1 - 6t + t^2 \). We get, again using (28), the relation

\[ 1 + t + (t - 3) \sum_{n=1}^{\infty} r(n-1, 0) t^n = (1 - 6t + t^2) \sum_{n=0}^{\infty} (n+1) r(n, 0) t^n. \]

The equality principle of formal power series gives a new result:

**Theorem 7.** The numbers \( r(n, 0) \) admit a three-term recurrent formula

\[ (n + 1) r(n, 0) - 3(2n - 1) r(n - 1, 0) + (n - 2) r(n - 2, 0) = 0, \quad n \geq 2 \] (30)

with the initial conditions \( r(0, 0) = 1 \) and \( r(1, 0) = 2 \).

We can now get the numbers \( r(n, 0) \) very quickly using (30):
\[ r(6, 0) = 1806, \quad r(7, 0) = 8558, \quad r(8, 0) = 41586, \quad r(9, 0) = 206098, \quad r(10, 0) = 1037718, \quad r(11, 0) = 5293446, \quad r(12, 0) = 27297738. \]

We also can express the numbers \( r(n, 0) \) by Legendre polynomials \( P_k(x) \). The formal power series

\[ (1 - 2xt + t^2)^{-1/2} = \sum_{k=0}^{\infty} P_k(x) t^k \]

gives us the numbers \( r(n, 0) \) in a closed form. It is easy to see that

\[ 2 \sum_{n=0}^{\infty} r(n, 0) t^{n+1} = 1 - t - \sum_{n=0}^{\infty} P_n(3) t^{n+2} \]

\[ + 6 \sum_{n=0}^{\infty} P_n(3) t^{n+1} - \sum_{n=1}^{\infty} P_n(3) t^n. \]

It follows that

\[ 2r(n, 0) = -P_{n-1}(3) + 6P_n(3) - P_{n+1}(3) \quad \text{for} \quad n \geq 1. \]
Theorem 8. The numbers $r(n, 0)$ can be written in the form

$$r(n, 0) = -\frac{1}{2}(P_{n-1}(3) - 6P_n(3) + P_{n+1}(3))$$

for every $n \geq 1$, where $P_k(x)$ denote the Legendre polynomials.

References