Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 10, NUMBER 2 (2007) ISSN 1311-0454

ON q-LAPLACE TRANSFORMS OF THE q-BESSEL FUNCTIONS

S.D. Purohit [1] **and S.L. Kalla** [2]

Abstract

The present paper deals with the evaluation of the q-Laplace transforms of a product of basic analogues of the Bessel functions. As applications, several useful special cases have been deduced.

Mathematics Subject Classification: 33D15, 44A10, 44A20

Key Words and Phrases: q-Laplace transforms, basic hypergeometric functions, basic analogue of the Bessel function

1. Introduction

Recently, Yadav and Purohit [12]-[14] evaluated the q-Laplace images of a number of q-polynomials and generalized basic hypergeometric functions of one and more variables, including the basic analogue of Fox's H-function (due to Saxena, Modi and Kalla [9]), and Purohit, Yadav and Vyas [7] obtained q-Laplace transform of a basic analogue of the I-function (due to Saxena and Kumar [8]).

Hahn [6] defined the q-analogues of the well-known classical Laplace transform

$$\phi(s) = \int_0^\infty e^{-st} f(t) dt, \tag{1.1}$$

by means of the following q-integrals:

$$_qL_s \{f(t)\} = \frac{1}{(1-q)} \int_0^{s^{-1}} E_q(qst) f(t) d(t;q), \tag{1.2}$$

and

$$_qL_s\{f(t)\} = \frac{1}{(1-q)}\int_0^\infty e_q(-st)f(t)d(t;q), \tag{1.3}$$

where the q-exponential series (analogues of the classical exponential function) are defined by

$$e_q(t) = \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}, \tag{1.4}$$

and

$$E_q(t) = \sum_{n=0}^{\infty}\frac{(-1)^n\, q^{n(n-1)/2}t^n}{(q;q)_n}. \tag{1.5}$$

The basic integration (cf. Gasper and Rahman [4]) is defined by

$$\int_0^x f(t)d(t;q) = x(1-q)\sum_{k=0}^{\infty}q^k f(xq^k). \tag{1.6}$$

By virtue of (1.6), the operator (1.2) can be expressed as

$$\phi(s) \equiv \ _qL_s\{f(t)\} = \frac{(q;q)_\infty}{s}\sum_{j=0}^{\infty}\frac{q^j f(s^{-1}q^j)}{(q;q)_j}. \tag{1.7}$$

The correspondence defined by operators (1.2) and (1.7) shall be denoted symbolically by

$$f(t) \supset_q \phi(s),$$

where the function $f(t)$ is called the original function, and $\phi(s)$ is named as the q-Laplace transform, or q-image of the original function $f(t)$.

For real or complex a and $|q| < 1$, the q-shifted factorial is defined as:

$$(a;q)_n = \begin{cases} 1 & ;\text{if} \quad n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & ;\text{if} \quad n \in N, \end{cases} \tag{1.8}$$

also

$$(x-y)_\nu = x^\nu \prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{\nu+n}}\right], \tag{1.9}$$

and

$$\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty(1-q)^{a-1}} = \frac{(q;q)_{a-1}}{(1-q)^{a-1}}, \tag{1.10}$$

where $a \neq 0, -1, -2, \cdots$.

The generalized basic hypergeometric series ${}_r\Phi_s(.)$ (cf. Slater [10]), is:

$$ {}_r\Phi_s\left[\begin{array}{cc} a_1,\cdots,a_r; & \\ & q,x \\ b_1,\cdots,b_s; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1,\cdots,a_r;q)_n\ x^n}{(q,b_1,\cdots,b_s;q)_n}\ , \tag{1.11} $$

where for the convergence of the series (1.11), we require $|q|<1$ and $|x|<1$ if $r=s+1$.

The q-analogue of the Bessel function (cf. Gasper and Rahman [4]) is defined as

$$ J_\nu(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu {}_2\Phi_1\left[\begin{array}{cc} 0,0; & \\ & q,-x^2/4 \\ q^{\nu+1}; & \end{array}\right]. \tag{1.12} $$

Abdi [1] has investigated the fundamental properties of the q-Laplace transforms and established several theorems related to q-images of basic functions.

The main motivation for this paper is to evaluate the q-Laplace transform of product of basic analogues of the Bessel functions. Interesting special cases of the main result are also discussed.

2. q-Laplace image of product of q-Bessel functions

THEOREM. *Consider a $t^{\nu-1}$-weighted product of n different q-Bessel functions $J_{2\mu_j}(2\sqrt{a_jt};q), j=1,\ldots,n$. Then, their q-Laplace transform is given by*

$$ {}_qL_s\{t^{\nu-1}J_{2\mu_1}(2\sqrt{a_1t};q)\cdots J_{2\mu_n}(2\sqrt{a_nt};q)\} = \frac{\Gamma_q(\nu+M)(1-q)^{\nu-M-1}a_1^{\mu_1}\cdots a_n^{\mu_n}}{s^{\nu+M}\Gamma_q(2\mu_1+1)\cdots\Gamma_q(2\mu_n+1)} $$

$$ \times\ \Psi_2^{(n)}\left(q^{\nu+M};\ q^{2\mu_1+1},\cdots,q^{2\mu_n+1};\ q;\ \frac{-a_1}{s},\cdots,\frac{-a_n}{s}\right), \tag{2.1} $$

where $M=\mu_1+\cdots+\mu_n$, $Re(\nu+M)>0$ and $Re(s)>0$.

P r o o f. To prove (2.1), we put

$$ f(t) = t^{\nu-1}J_{2\mu_1}(2\sqrt{a_1t};q)\cdots J_{2\mu_n}(2\sqrt{a_nt};q) $$

into definition (1.7) and use (1.12), to obtain

$$ {}_qL_s\left\{t^{\nu-1}J_{2\mu_1}(2\sqrt{a_1t};q)\cdots J_{2\mu_n}(2\sqrt{a_nt};q)\right\} = \frac{(q;q)_\infty}{s}\sum_{j=0}^{\infty}\frac{q^j}{(q;q)_j}\left(\frac{q^j}{s}\right)^{\nu-1} $$

$$\times \frac{(q^{2\mu_1+1};q)_\infty}{(q;q)_\infty}\left(\frac{a_1q^j}{s}\right)^{\mu_1}\cdots\frac{(q^{2\mu_n+1};q)_\infty}{(q;q)_\infty}\left(\frac{a_nq^j}{s}\right)^{\mu_n}$$

$$\times \sum_{m_1,\cdots,m_n=0}^{\infty}\frac{\left(-a_1q^j/s\right)^{m_1}}{(q^{2\mu_1+1};q)_{m_1}(q;q)_{m_1}}\cdots\frac{\left(-a_nq^j/s\right)^{m_n}}{(q^{2\mu_n+1};q)_{m_n}(q;q)_{m_n}}. \tag{2.2}$$

Interchanging the order of summations in the right-hand side of the above equation (2.2), this yields

$$\frac{(q^{2\mu_1+1};q)_\infty}{(q;q)_\infty}\cdots\frac{(q^{2\mu_n+1};q)_\infty}{(q;q)_\infty}\frac{(q;q)_\infty a_1^{\mu_1}\cdots a_n^{\mu_n}}{s^{\nu+\mu_1+\cdots+\mu_n}}$$

$$\times \sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(-a_1/s)^{m_1}}{(q^{2\mu_1+1};q)_{m_1}(q;q)_{m_1}}\cdots\frac{(-a_n/s)^{m_n}}{(q^{2\mu_n+1};q)_{m_n}(q;q)_{m_n}}$$

$$\times \sum_{j=0}^{\infty}\frac{q^{j(\nu+\mu_1+\cdots+\mu_n+m_1+\cdots+m_n)}}{(q;q)_j}.$$

Using equation (1.10), and then summing the inner ${}_0\Phi_0(.)$-series with the help of the formula

$${}_0\Phi_0(-;-;q;t)=\frac{1}{(t;q)_\infty}, \tag{2.3}$$

the above expression yields:

$$\frac{(q;q)_\infty(1-q)^{-2(\mu_1+\cdots+\mu_n)}a_1^{\mu_1}\cdots a_n^{\mu_n}}{s^{\nu+\mu_1+\cdots+\mu_n}\Gamma_q(2\mu_1+1)\cdots\Gamma_q(2\mu_n+1)}$$

$$\times \sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(-a_1/s)^{m_1}}{(q^{2\mu_1+1};q)_{m_1}(q;q)_{m_1}}\cdots$$

$$\times \frac{(-a_n/s)^{m_n}}{(q^{2\mu_n+1};q)_{m_n}(q;q)_{m_n}}\frac{1}{(q^{\nu+\mu_1+\cdots+\mu_n+m_1+\cdots+m_n};q)_\infty}$$

$$=\frac{(q;q)_\infty(1-q)^{-2(\mu_1+\cdots+\mu_n)}a_1^{\mu_1}\cdots a_n^{\mu_n}}{s^{\nu+\mu_1+\cdots+\mu_n}\Gamma_q(2\mu_1+1)\cdots\Gamma_q(2\mu_n+1)(q^{\nu+\mu_1+\cdots+\mu_n};q)_\infty}$$

$$\times \sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(q^{\nu+\mu_1+\cdots+\mu_n};q)_{m_1+\cdots+m_n}}{(q^{2\mu_1+1};q)_{m_1}\cdots(q^{2\mu_n+1};q)_{m_n}}\frac{(-a_1/s)^{m_1}}{(q;q)_{m_1}}\cdots\frac{(-a_n/s)^{m_n}}{(q;q)_{m_n}}.$$

Further simplifications lead to

$$\frac{\Gamma_q(\nu+M)(1-q)^{\nu-M-1}a_1^{\mu_1}\cdots a_n^{\mu_n}}{s^{\nu+M}\Gamma_q(2\mu_1+1)\cdots\Gamma_q(2\mu_n+1)}$$

$$\times\ \Psi_2^{(n)}\left(q^{\nu+M};\ q^{2\mu_1+1},\cdots,q^{2\mu_n+1};\ q;\ \frac{-a_1}{s},\cdots,\frac{-a_n}{s}\right), \quad (2.4)$$

where $M=\mu_1+\cdots+\mu_n$ and $\Psi_2^{(n)}(.)$ denotes the confluent q-hypergeometric function defined as

$$\Psi_2^{(n)}\left(a;c_1,\cdots,c_n;q;x_1,\cdots,x_n\right)$$
$$=\sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(a;q)_{m_1+\cdots+m_n}}{(c_1;q)_{m_1}\cdots(c_n;q)_{m_n}}\frac{x_1^{m_1}}{(q;q)_{m_1}}\cdots\frac{x_n^{m_n}}{(q;q)_{m_n}}. \quad (2.5)$$

This completes the proof of (2.1). ■

3. Special cases

In this section we evaluate the q-Laplace transforms involving the q-Bessel functions as applications of our main result (2.1). First, we put $n=2$ in (2.1) and this yields a q-Laplace image of a product of two q-Bessel functions, namely:

$${}_qL_s\left\{t^{\nu-1}J_{2\mu_1}(2\sqrt{a_1t};q)J_{2\mu_2}(2\sqrt{a_2t};q)\right\}=\frac{\Gamma_q(\nu+M)(1-q)^{\nu-M-1}a_1^{\mu_1}a_2^{\mu_2}}{s^{\nu+M}\Gamma_q(2\mu_1+1)\Gamma_q(2\mu_2+1)}$$
$$\times\ \Psi_2\left(q^{\nu+M};\ q^{2\mu_1+1},q^{2\mu_2+1};\ q;\ \frac{-a_1}{s},\frac{-a_2}{s}\right), \quad (3.1)$$

where $M=\mu_1+\mu_2$, $Re(\nu+M)>0$ and $Re(s)>0$.

If we put $n=1,\ \mu_1=\nu,\ \nu=\mu$ and $a_1=a$ in (2.1), we obtain a Laplace q-image of basic analogue of the Bessel function as below:

$${}_qL_s\left\{t^{\mu-1}J_{2\nu}(2\sqrt{at};q)\right\}=\frac{\Gamma_q(\mu+\nu)(1-q)^{\mu-\nu-1}a^{\nu}}{s^{\mu+\nu}\Gamma_q(2\nu+1)}\,{}_1\Phi_1\left[\begin{matrix}q^{\mu+\nu}; & \\ & q,-a/s\\ q^{2\nu+1}; & \end{matrix}\right], \quad (3.2)$$

for $Re(\mu+\nu)>0$ and $Re(s)>0$.

Replacing μ and ν by $\frac{\nu}{2}+1$ and $\frac{\nu}{2}$ respectively in (3.2), yields

$${}_qL_s\left\{t^{\nu/2}J_{\nu}(2\sqrt{at};q)\right\}=a^{\nu/2}s^{-\nu-1}\ e_q(-a/s),\quad Re(s)>0. \quad (3.3)$$

Next, we put $\nu=1$ in (3.3) and get

$${}_qL_s\left\{t^{1/2}J_1(2\sqrt{at};q)\right\}=a^{1/2}s^{-2}\ e_q(-a/s),\quad Re(s)>0. \quad (3.4)$$

Similarly, for $\nu = 0$ the equation (3.3) reduces to

$$ {}_qL_s\left\{J_0(2\sqrt{at};q)\right\} = s^{-1}\ e_q(-a/s), \quad Re(s) > 0. \tag{3.5} $$

Again, on replacing μ and ν by $1-\frac{\nu}{2}$ and $\frac{\nu}{2}$ respectively, (3.2) yields

$$ {}_qL_s\left\{t^{-\nu/2}J_\nu(2\sqrt{at};q)\right\} = \frac{(1-q)^{-\nu}a^{\nu/2}}{s\ \Gamma_q(\nu+1)}\, {}_1\Phi_1\left[\begin{matrix} q; \\ q^{\nu+1}; \end{matrix}\ q, -a/s\right]. $$

By further simplifications this reduces to

$$ {}_qL_s\left\{t^{-\nu/2}J_\nu(2\sqrt{at};q)\right\} = \frac{(-1)^\nu\ s^{\nu-1}}{a^{\nu/2}\ \Gamma_q(\nu)} e_q(-a/s)\Gamma_q(\nu,-a/s), \tag{3.6} $$

where $\Gamma_q(\alpha,x)$ denotes the q-extension of the incomplete gamma function $\gamma(\alpha,x)$ (cf. Gupta [5], eqn.(9), p.253):

$$ \Gamma_q(\alpha,x) = \frac{x^\alpha(x;q)_\infty(1-q)^{1-\alpha}}{(1-q^\alpha)}\, {}_1\Phi_1\left[\begin{matrix} q; \\ q^{\alpha+1}; \end{matrix}\ q, x\right]. \tag{3.7} $$

Further, for $\nu = 0$ and $a = 0$, we obtain a q-extension of the well-known result for the Laplace transform, namely:

$$ {}_qL_s\left\{t^{\mu-1}\right\} = \frac{\Gamma_q(\mu)(1-q)^{\mu-1}}{s^\mu}\ , \quad Re(s) > 0. \tag{3.8} $$

Finally, it is interesting to observe that in view of the limit formulae

$$ \lim_{q\to1^-}\Gamma_q(a) = \Gamma(a) \quad \text{and} \quad \lim_{q\to1^-}\frac{(q^a;q)_n}{(1-q)^n} = (a)_n\ , \tag{3.9} $$

where

$$ (a)_n = a(a+1)\cdots(a+n-1), \tag{3.10} $$

the main result (2.1) and the results (3.2) to (3.6), give q-extensions of the known results mentioned in Erdélyi, Magnus, Oberhettinger and Tricomi [3](table number (4.14), pp. 182-187), namely:

$$ L\left\{t^{\nu-1}J_{2\mu_1}(2\sqrt{a_1t})\cdots J_{2\mu_n}(2\sqrt{a_nt})\right\} = \frac{\Gamma(\nu+M)a_1^{\mu_1}\cdots a_n^{\mu_n}}{s^{\nu+M}\Gamma(2\mu_1+1)\cdots\Gamma(2\mu_n+1)} $$

$$ \times\ \Psi_2^{(n)}\left(\nu+M;\ 2\mu_1+1,\cdots,2\mu_n+1;\ \frac{-a_1}{s},\cdots,\frac{-a_n}{s}\right), \tag{3.11} $$

where $M = \mu_1 + \cdots + \mu_n, Re(\nu + M) > 0$ and $Re(s) > 0$;

$$L\left\{t^{\mu-1}J_{2\nu}(2\sqrt{at})\right\} = \frac{\Gamma(\mu+\nu)a^\nu}{s^{\mu+\nu}\Gamma(2\nu+1)}\,{}_1F_1\left[\begin{array}{cc}\mu+\nu; & \\ & -a/s \\ 2\nu+1; & \end{array}\right], \quad (3.12)$$

where $Re(\mu+\nu) > 0$ and $Re(s) > 0$;

$$L\left\{t^{\nu/2}J_\nu(2\sqrt{at})\right\} = a^{\nu/2}s^{-\nu-1}\ e^{-a/s}, \quad Re(s) > 0; \quad (3.13)$$

$$L\left\{t^{1/2}J_1(2\sqrt{at})\right\} = a^{1/2}s^{-2}\ e^{-a/s}, \quad Re(s) > 0; \quad (3.14)$$

$$L\left\{J_0(2\sqrt{at})\right\} = s^{-1}\ e^{-a/s}, \quad Re(s) > 0; \quad (3.15)$$

$$L\left\{t^{-\nu/2}J_\nu(2\sqrt{at})\right\} = \frac{e^{i\pi\nu}\ s^{\nu-1}}{a^{\nu/2}\ \Gamma(\nu)}e^{-a/s}\gamma(\nu, \frac{e^{i\pi}a}{s}). \quad (3.16)$$

The results proved in this paper give some contributions to the theory of the q-series, especially q-Bessel functions, and may find applications to solutions of certain q-difference and q-integral equations associated with various q-Bessel functions. In this regard, one can refer to the work of Abdi [2].

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[1] *Department of Basic-Sciences (Mathematics)*
College of Technology and Engineering
M.P. University of Agriculture and Technology
Udaipur - 313001, INDIA
e-mail: sunil_a_purohit@yahoo.com

Received: September 24, 2007

[2] *Department of Mathematics and Computer Science*
Faculty of Science, Kuwait University
P.O. Box-5969, Safat 13060, KUWAIT
e-mail: shyamkalla@yahoo.com