THE MODIFIED MELLIN TRANSFORM AND CONVOLUTION

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Abstract

The modified Mellin transform and its inverse on the spaces $L^{\alpha}_{\infty}$, $\alpha > -1$, as well as the modified Mellin convolution and its properties over these spaces are investigated. The spaces $L^{\alpha}_{\infty}$, $\alpha > -1$, are inspected through the spaces of Newton’s series using an isomorphism between them. Remarks are given on the domain of convergence of some Dirichlet’s series. Finally, the modified Mellin convolution is applied in solving an integro-differential equation.

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1. Introduction

In the theory of integral transforms of generalized functions the monograph of Zemanian [11] takes a remarkable place. In this monograph are presented methods for constructing spaces of generalized functions which correspond to the appropriate differential operators and integral transforms. The investigation of transforms is the most effective procedure for solving many problems concerning partial differential equations and functional equations.
Usually, it is the case that inversion formulas cannot be performed analytically and then suitable numerical techniques have to be found. There are a number of related methods in use, such as [1], [8].

The Laplace and the Mellin transform are studied and applied in [6] on the space of tempered distributions through the Laguerre expansions of its elements. Some useful results are obtained. The properties of the Mellin transform which concern the differentiation, and multiplication by \( z \) are examined. Also given are two inversion formulas for the \( M \)-transform which are important in operational calculus. The first inversion formula is a new technique of inverting the Mellin transform using series of Laguerre polynomials. This approach is different from that given in [11] which can be considered as a distributional approach. It needs less operational calculus than the generalised version of Zemanian.

In this paper the generalization of results of [6] for the spaces \( L^0_{\alpha} \), \( \alpha > -1 \), are given. This paper is organised as follows:

In the first part we shall give definitions of the spaces involved. Then, we shall define the modified Mellin transform \( M^* \) in \( L^0_{\alpha} \) - spaces for later use. In Section 4 we shall include the definitions of the modified Mellin convolution in the spaces \( L^0_{\alpha} \) and \( L^\infty_{\alpha} \), \( \alpha > -1 \). A new numerical method for inverting modified Mellin transform is then detailed in Section 5 where the generalised function version of the inverse of the modified Mellin transform is also given. In Section 6 is given the characterization of spaces \( L^0_{\alpha} \), \( \alpha > -1 \), by using their isomorphisms with the spaces of Newton's series \( N_\alpha \). Remarks on the convergence of Dirichlet's series are given in Section 7. At the end, we shall give an algorithm for solving an integro-differential equation which uses the theoretical predictions given in Sections 3, 4 and 5.

2. Basic spaces

We shall consider the expansions of the spaces of generalised functions \( L^0_{\alpha} \), \( \alpha > -1 \), with respect to the Laguerre orthonormal systems \( e_{\alpha, n, \alpha} \), \( \alpha > -1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). When \( \alpha = 0 \) we have the space \( L^0_{\alpha} \), the space of tempered distributions with supports in \( \mathbb{R}_+ = (0, \infty) \) ([6]), known as the space \( S^\prime \). Basic references for the space \( L^0_{\alpha} \) are [2], [6], [7], [11]. For the properties of the spaces \( L^0_{\alpha} \) we refer to [2], [7], [8], [9], [11].

We shall repeat some basic properties of the spaces \( L^0_{\alpha} \), and its most
Let $\alpha > -1$. The generalised Laguerre orthonormal system in $L^2(\mathbb{R}_+)$ is given by

$$L_{n,\alpha}(t) = \tau_n e^{\alpha t/2} L_n^\alpha(t) e^{-t/2}, \, t \in \mathbb{R}_+,$$

where $\tau_n = (\Gamma(n+1)/\Gamma(n+\alpha+1))^{1/2}$, and $L_n^\alpha(t)$ are generalised Laguerre polynomials, defined by

$$L_n^\alpha(t) = \sum_{m=0}^{\infty} \left( \begin{array}{c} n + \alpha \cr n - m \end{array} \right) \frac{(-t)^m}{m!}, \, t > 0, \, n \in \mathbb{N}_0.$$

The functions $L_{n,\alpha}(t)$ are eigenfunctions for the operator $L_\alpha = (\partial^2 - 2\alpha \partial + 1) e^{2\alpha t} e^{-t}, \, R_\alpha = R_\alpha = R_\alpha(\tau_{\alpha+1}), \, k \in \mathbb{N}_0, \, R_{\alpha}^n$ is the identity operator, and the corresponding eigenvalues are $\lambda_n = -n, \, n \in \mathbb{N}_0$.

The spaces $L_{\alpha}$ are the spaces of all the smooth functions $\phi \in C^\infty(\mathbb{R}_+)$, such that for every $k \in \mathbb{N}_0$, the seminorms $\| \phi \|_k = \| R_\alpha^n \phi \|_L^2 = (\int_{\mathbb{R}_+} |R_\alpha^n \phi(t)|^2 dt)^{1/2}$, are finite, and for every $n \in \mathbb{N}_0$ and every $k \in \mathbb{N}_0$, $R_\alpha^n \phi_k, \phi_k \in \mathbb{N}_0$ is given in [7].

The following relation between the spaces $L_{\alpha}$ is given in [7]:

$$L_{\alpha} = x^{\alpha/2} L_{\alpha} = \left\{ \psi \in C^\infty(\mathbb{R}_+); \psi = x^{\alpha/2} \phi \text{ for some } \phi \in L_{\alpha} \right\}.$$

We shall repeat some equivalent definitions for $L_{\alpha}$:

(i) $L_{\alpha} = \left\{ \psi \in C^\infty(\mathbb{R}_+); \sup_{x \in \mathbb{R}_+} |x^{j} \psi^{(k)}(x)|, r \in \mathbb{N}_0 \right\} < \infty$ [10],

$$j, k \leq r$$

(ii) $L_{\alpha} = \left\{ \psi \in C^\infty(\mathbb{R}_+); \sup_{x \in \mathbb{R}_+} |x^{j} \psi^{(k)}(x)|, r \in \mathbb{N}_0 \right\} < \infty$ [7],

$$k, j \leq r$$
Note that \( L^2(\mathbb{R}_+) = L_{G_0} \).

Let \( k \in \mathbb{N}_0, \alpha > -1 \), \( L_{k,\alpha} \) is the space defined as follows:

\[
L_{k,\alpha} = \left\{ \phi \in C^\infty(\mathbb{R}_+) : \|\phi\|_{k,\alpha} < \infty \right\},
\]

where

\[
\|\phi\|_{k,\alpha} = \left( \|b_{0,\alpha}\|^2 + \sum_{n=1}^{\infty} \|b_{n,\alpha}\|^2 n^{-2\alpha} \right)^{1/2},
\]

\[
L_k = \left\{ f = \sum_{n=0}^{\infty} b_{n,\alpha} e^{\lambda_n z} \text{ formal series}, \|f\|_k < \infty \right\},
\]

where

\[
\|f\|_k^2 = \left( \|b_{0,\alpha}\|^2 + \sum_{n=1}^{\infty} \|b_{n,\alpha}\|^2 n^{-2\alpha} \right)^{1/2}.
\]

Obviously, spaces \( L_{k,\alpha} \) and \( L_k \) can be defined for \( k \in \mathbb{R} \). Note, for \( k > 0 \), \( L_k^{\alpha} = L_{-k,\alpha} \).

\[
L_{G_0} = \text{proj lim}_{k \to \infty} L_{k,\alpha}, \quad L_{G_0} = \text{ind lim}_{k \to \infty} L_k^{\alpha}.
\]

The connections between the spaces \( L_{k,\alpha} \) and \( L_k^{\alpha}, k \in \mathbb{R} \), and consequently between \( L_{G_0} \) and \( L_{G_0} \) are given in the next proposition.

**Proposition 1.** Let \( k \in \mathbb{R}, \alpha > -1 \). Then

\[
L_{k,\alpha} = z^{\alpha/2} L_k = \left\{ \psi \in C^\infty(\mathbb{R}_+) : \psi = z^{\alpha/2} \phi \text{ for some } \phi \in L_k \right\}.
\]

**Proof.** Let \( \psi = \sum_{n=0}^{\infty} a_n e^{\lambda_n z} \in L_{k,\alpha} \). From [3], p. 192(39)

\[
L_k^\alpha(z) = \sum_{m=0}^{\infty} (1/m!) (\Gamma(m+\alpha)/\Gamma(\alpha)) L_{m-\alpha-}(x), \quad z \in \mathbb{R}_+, \text{ we have}
\]

\[
z^{-\alpha/2} \psi = \sum_{n=0}^{\infty} a_n e^{\lambda_n z} = \sum_{n=0}^{\infty} \alpha_n e^{\lambda_n z}. \]

\[
\cdot \sum_{m=0}^{\infty} 1/m! (\Gamma(\alpha + m)/\Gamma(\alpha)) L_{m-\alpha-}(z) =
\]

\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} \alpha_n (1/m!) (\Gamma(\alpha + m)/\Gamma(\alpha)) e^{\lambda_n m} \right) L_m(z).
\]
Let \( b_n = \sum_{n=0}^{\infty} r_n (1/n!) (\Gamma(n+\alpha)/\Gamma(\alpha)) a_{n+m,m}, n \in N_0 \). We have to prove that \( \sum_{n=0}^{\infty} \lVert b_n \rVert_m m^{2k} < \infty \). By Cauchy's inequality

\[
\sum_{n=0}^{\infty} m^{2k} \sum_{n=0}^{\infty} \left( \frac{r_n (1/n!) (\Gamma(n+\alpha)/\Gamma(\alpha)) a_{n+m,m}}{m^{2k}} \right)^2 \leq C \sum_{m=0}^{\infty} m^{2k} \sum_{n=0}^{\infty} r_n (1/n!)^2.
\]

So we prove if \( \varphi \in L_{k,\alpha} \), then \( z^{-\alpha/2} \varphi \in L_{k,\alpha} \). By using the formula \( L_n(x) = \sum_{n=0}^{\infty} \frac{1}{n!} (-\alpha)_n L_{n+\alpha}^\alpha (x) \), \( (3) \), p.129(4) similarly as above we can prove if \( \varphi \in L_{k,\alpha} \) then \( z^{-\alpha/2} \varphi \in L_{k,\alpha} \).

Define the space \( \mathcal{H}_a \) as follows:

\( \mathcal{H}_a = \{ \phi; \phi = e^{t/2} \phi(t) \text{ for some } \psi \in L_{G,0} \} \)

with the topology generalised by the norms \( \lVert \cdot \rVert_{k,\alpha}, k \in N_0, \alpha > -1 \), defined by

\[
\lVert \phi \rVert_{k,\alpha} = \sup_{t \in \mathbb{R}_+} t^k |(e^{-t/2} e^{-\alpha t/2} \phi(t))^{(n)}|, \quad l \leq k.
\]

This sequence of norms defines the convergence structure in \( \mathcal{H}_a \).

Let the sequence \( \varphi_n \) belong to \( \mathcal{H}_a \). Then \( \varphi_n \rightarrow 0 \) in \( \mathcal{H}_a \) iff there exists a sequence \( \psi_n \) in \( L_{G,0} \) such that \( \psi_n \rightarrow 0 \) in \( L_{G,0} \) and \( \varphi_n = e^{t/2} \psi_n, n \in N_0 \).

Thus, \( (\mathcal{H}_a, \lVert \cdot \rVert_{k,\alpha}, k \in N_0) \) is an F-space.

Put \( H_{k,\alpha} = \{ e^{t/2} \psi; \psi \in L_{k,\alpha} \}, k \in \mathbb{R}, \) and transport the convergence structure from \( L_{k,\alpha} \) to \( H_{k,\alpha} \).

We have:

\[
\mathcal{H}_a = \text{proj lim } H_{k,\alpha}, \quad \mathcal{H}_a' = \text{ind lim } H_{k,\alpha}',
\]

\[
L_{G,0} = \{ e^{-t/2} \varphi; \varphi \in \mathcal{H}_a \},
\]

A duality argument gives the following proposition:
Proposition 2. We have $\mathcal{H}_a = \{ e^{-t/2}f; f \in L_{k,\alpha} \}$, 

\[ L_{k,\alpha} = \{ e^{-t/2}f; f \in \mathcal{H}_{k,\alpha} \}, k \in \mathbb{N}_0. \]

Proof. If $f \in L_{k,\alpha}$ then $e^{-t/2}f \in \mathcal{H}_a$ and conversely, via the dual pairing 

\[
\mathcal{H}_a^* \{ e^{-t/2}f(t), \phi(t) \} = L_{k,\alpha} \{ f(t), e^{-t/2} \phi(t) \} L_{k,\alpha} = \\
= L_{k,\alpha} \{ e^{t/2}f(t), e^{-t/2+\alpha} \phi(t) \} L_{k,\alpha}, \phi \in \mathcal{H}_a.
\]

The second assertion follows similarly. \( \square \)

Note that for $Re(s), t \to t^{-1}, t \in \mathbb{R}_+$, belongs to $\mathcal{H}_{k,0}$, and $t \to e^{t/2+\alpha+1}$, $t \in \mathbb{R}_+$, belongs to $\mathcal{H}_{k,\alpha}$, $k \in \mathbb{R}, (\mathcal{H}_{k,\alpha} = e^{z/2})$.

3. Modified Mellin transform

We present the modified Mellin transform in $L_{k,\alpha}, \alpha > -1$.

Let 

\[ \varphi_{\alpha}(t) = (1/\Gamma(s+\alpha))^{1/2}e^{-t/2}t^{-1}, t \in \mathbb{R}_+, \]

and 

\[ D_{k,\alpha/2} = \{ s \in \mathbb{C}; Res > k + (1 - \alpha)/2 \}, \\
k \in \mathbb{R}. \]

Then the mapping

\[ s \to (M_{\alpha}f)(s) = (f(t), \varphi_{\alpha}(t)), s \in D_{k,\alpha/2}, \]

is the modified Mellin transform (hereafter referred to as MMT) of an $f \in L_{k,\alpha}$, $k \in \mathbb{R}$.

The expansion of $\varphi_{\alpha, s}$ with respect to the generalised Laguerre orthonormal system is given by $\varphi_{\alpha, s} = \sum_{n=0}^{\infty} a_{n,\alpha} \ell_{n,\alpha}$, where $a_{n,\alpha} = \langle \varphi_{s,\alpha}(t), \ell_{n,\alpha}(t) \rangle$ 

\[ = (-1)^n \binom{s-1}{n} r_{n, n} \in \mathbb{N}_0 \] (11, p.32).

Since $|a_{n,\alpha}| \leq C_n^{-1}(Res+\alpha/2)$, $n > n(s, \alpha), C = C(s, \alpha)$ [6], we have $\varphi_{\alpha, s} \in L^2(\mathbb{R}_+)$ if $Res > (1 - \alpha)/2$ and $\varphi_{\alpha, s} \in L_{k,\alpha}$ if $Res > (1 - \alpha)/2 + k$.

Proposition 3. The mapping $s \to \varphi_{\alpha, s}$ is a holomorphic mapping from $D_{k,\alpha/2}$ into $L_{k,\alpha}$, $k \in \mathbb{N}_0, \alpha > -1$. 
Proof. Since $\mathcal{H}_a \varphi_{\alpha, \alpha} = (s - \alpha) \varphi_{\alpha-1, \alpha} - \varphi_{\alpha, \alpha}$, by similar arguments as in [6] it is easy to show the assertion of Proposition 3.

Similarly as in [6] we have

Proposition 4. The mapping $s \rightarrow \varphi_{\alpha, \alpha}$ from $D_{k, \alpha} / \mathbb{Z}$ into $L_{k, \alpha}, k \in \mathbb{R}, \alpha > -1$, is holomorphic.

Now we shall introduce the Null-Mellin transform in $\mathcal{H}_a$ by following the method of Zemanian. ([11], Ch.4.)

Since for $\text{Res} > k$,

$$R \ni t \rightarrow t^{\alpha / 2} e^{-t} \in \mathcal{H}_a, t \in \mathbb{R}, (k \in \mathbb{R})$$

the definition which is to follow is correct. The mapping

$$s \rightarrow (\tilde{M}_a(f))(s) = (f(t), t^{\alpha / 2} e^{-t}), \text{Res} > k,$$

for an $f \in \mathcal{H}_a$ is Null-Mellin transform. When $\alpha = 0$ we obtain the Zemanian Mellin transform $(\tilde{M}_0(f))(s)$ (with our modification).

The relation between these two transforms is given in

Proposition 5. For an $f \in \mathcal{L}_a$, Res $> k$,

$$(\tilde{M}_a(e^{-t^2 / 2} f))(s) = \Gamma(s + \alpha)(\tilde{M}_a(f))(s)$$

Proof. If $f \in \mathcal{L}_a$, then $e^{-t^2 / 2} f(t) \in \mathcal{H}_a$, for some $k \in \mathbb{R}$, and for Res $> k$

$$(\tilde{M}_a(e^{-t^2 / 2} f))(s) = (e^{-t^2 / 2} f(t), t^{\alpha / 2} e^{-t / 2}) = (f(t), e^{-t^2 / 2} / t^{\alpha / 2} e^{-t / 2}) = \Gamma(t + \alpha)(\tilde{M}_a(f))(s).$$

The properties of MMT are given in

Proposition 6. Let $s \in D_{k, \alpha} / \mathbb{Z}, k \in \mathbb{R},$ and $f = \sum_{n=0}^{\infty} h_n e^{\alpha} \in L_{k, \alpha}$. Then

$$(i \{\tilde{M}_a(e^{\alpha})\})(s) = (-1)^n \left( \begin{array}{c} s - 1 \\ n \end{array} \right) \tau_n, n \in \mathbb{N}_0, (\text{we take } \left( \begin{array}{c} s - 1 \\ 0 \end{array} \right) = 1);$$
This series converges absolutely and uniformly in $D_{k,\alpha/2}$.

The following holds:

$$(\mathcal{M}_\alpha(Ef))(s+1) = -(s+\alpha/2)/(s+\alpha)(\mathcal{M}_\alpha f)(s) + \frac{1}{2}(\mathcal{M}_\alpha f)(s+1);$$

$$(\mathcal{M}_\alpha(tf))(s-1) = (s+\alpha-1)(\mathcal{M}_\alpha f)(s);$$

$$(\mathcal{M}_\alpha(e^{-t/\alpha}(e^{-t/\alpha})^\prime))(s) = -(s+\alpha/2)(\mathcal{M}_\alpha f)(s);$$

$$(\mathcal{M}_\alpha(\alpha e^{-t^2}(\alpha e^{-t^2})^\prime))(s) = -(s-1+\alpha/2)/(s+\alpha-1)(\mathcal{M}_\alpha f)(s-1);$$

$$(\mathcal{M}_\alpha(R_\alpha f))(s) = (s-1)(\mathcal{M}_\alpha f)(s-1) - (\mathcal{M}_\alpha f)(s).$$

**Proof.** (i), (ii) follows from ([1], p.32). Since

$$\left|\frac{s-1}{\alpha}\right| < e^{-\alpha}, \tau_n < n^{-\alpha/2}, n \in \mathbb{N},$$

this series converges in $D_{k,\alpha/2}$. (iii) follows from the definition of the MMT.

**Remark.** Let us note that $(\mathcal{M}_\alpha f)$ is also defined for $s \in \mathbb{C}$ and $s \leq k + (1-\alpha)/2$ by

$$(\mathcal{M}_\alpha f)(s) = \sum_{n=0}^{\infty} b_{n}\alpha (s-1)\tau_n \left(\begin{array}{c} s-1 \\ n \end{array}\right) (-1)^n \mathbf{we\ take} \left(\begin{array}{c} 0 \\ 0 \end{array}\right), \left(\begin{array}{c} -1 \\ 0 \end{array}\right) = 1$$

4. Modified Mellin convolution (MMC)

For generalized functions which are Mellin's transformable ([11]) there exists the Mellin convolution ([11]).

In this section we shall give for MMT the analogues for the theorems of Zemanian ([11], p.151-156).

First, we consider the modified convolution over the elements in the spaces $L^p_\mathbb{R}$ and $L^q_\mathbb{R}$ separately.
The modified Mellin transform and convolution

Proposition 7. Let $\Theta \in \mathcal{H}_0$. Then for $g \in \mathcal{H}_0$,
$$\phi(x) = (g(y), \Theta(x \cdot y)), \quad x \in \mathbb{R}_+,$$
belongs to $\mathcal{H}_0$.

The modified Mellin convolution is defined as follows. Let $f, g \in \mathcal{L}^2_0$. Then, by
$$\left( f \ast g, \Theta \right) = \left( f(x)e^{-x/2}, \Theta(x) \ast g(y) \right), \quad x \in \mathbb{R}_+,\ g \in \mathcal{H}_0,$$
the modified Mellin convolution (MMCC) of $f$ and $g$ is defined.

Proposition 8. For $f, g \in \mathcal{L}^2_0$, $f \ast g \in \mathcal{H}_0$.

Proof. It follows from Proposition 7, because $e^{-x/2}f, e^{-x/2}g \in \mathcal{H}_0$.

Proposition 9. Let $f \in \mathcal{L}^2_0$, $g \in D(R_+)$. Then
$$f \ast g = \left( f(x)e^{-x/2}, e^{-x/2}g(1/x) \right), \quad x \in \mathbb{R}_+,$$
in $D(R_+)$. (This means that for every $\Theta \in D(R_+)$,
$$\left( f \ast g, \Theta \right) = \left( \left( f(x)e^{-x/2}, e^{-x/2}g(1/x) \right), \Theta(x) \right).$$

Proof. By substitution of variables (2) can be expressed as
$$\left( f \ast g, \Theta \right) = \left( f(x)e^{-x/2}, e^{-x/2}g(1/x) \right).$$
The supports of $e^{-x/2}g(y)$ and $(1/x)g(y/x)$ have the intersection in a compact subset of the open quadrant $R_+ \times R_+$.

Then (2) has the form
$$
\begin{align*}
\left( f(x)e^{-x/2}, \Theta(y)e^{-y/2}, (1/x)g(y/x) \right) &= \\
&= \left( f(x)e^{-x/2} \Theta(y), (1/x)g(y/x)e^{-y/2} \right),
\end{align*}
$$
where $\Theta$ denotes the direct product.

From the commutativity of the direct product we have
$$\left( f \ast g, \Theta \right) = \left( \Theta(y), (f(x)e^{-x/2}, e^{-x/2}(1/x)g(y/x)) \right),$$
which implies the assertion.
Proposition 10. Let \( f, g \in L G_0^\alpha \). Then, \((f \in \text{domain which depends on } f \text{ and } g)\)

\[
\begin{align*}
\hat{M}_0(f \vee g)(s) &= (\Gamma(s))^2(\hat{M}_0f)(s)(\hat{M}_0g)(s), \\
\hat{M}_0(e^{-s/2}(f \vee g))(s) &= \Gamma(s)(\hat{M}_0f)(s)(\hat{M}_0g)(s)
\end{align*}
\]

Proof. \( \hat{M}_0(f \vee g)(s) = (f \vee g, x^{s-1}) = (f(x)e^{-s/2}, \)

\( (g(y)e^{-s/2}(xy)^{-1}) = (f(x)e^{-s/2}, x^{s-1})(g(y)e^{-s/2}, y^{s-1}) = \\
= (\Gamma(s))^2(\hat{M}_0f)(s)(\hat{M}_0g)(s). \square
\]

Proposition 11. Let \( f, g \in L G_0^\alpha \), then

\[
\hat{M}_0(f \vee g)(s) = \hat{M}_0(e^{-s/2}f)(s)\hat{M}_0(e^{-s/2}g)(s).
\]

Proof. It follows from (4) and Proposition 5. \( \square \)

The MMC has the following properties over the generalized functions in \( L G_0^\alpha \):

i. Commutativity: \( \hat{M}_0(f \vee g)(s) = \hat{M}_0(g \vee f)(s) \).

ii. Associativity: \( \hat{M}_0((f \vee g) \vee h)(s) = \hat{M}_0(f \vee (g \vee h))(s) \).

We shall introduce the MMC in the spaces \( L G_0^\alpha, \alpha > -1 \). The arguments are very similar to those for the space \( L G_0^\alpha \) given in the previous part. Here we shall only quote the necessary changes to be made in the arguments given there.

Using the relations between the spaces: \( L G_0^\alpha = \alpha^{-n/2}L G_0^\beta \{7\} \), and \( \mathcal{H}_0 = z^{s/2}\mathcal{H}_0 \) we have

Proposition 12. Let \( \Theta \in \mathcal{H}_0^\beta \). Then for \( g \in \mathcal{H}_0^\gamma \),

\[ \phi(x) = (g(y), \Theta(xz)x^{-\alpha}), \ z \in \mathbb{R}_+ \]

belongs to \( \mathcal{H}_0 \).

Let \( f, g \in L G_0^\alpha \). Then for \( \Theta \in \mathcal{H}_0^\alpha \),

\[ (f \vee g, \Theta) = (f(x)e^{-s/2}x^{-\alpha/2}, (g(y)e^{-s/2}y^{-\alpha/2}, \Theta(xy)(xy)^{-\alpha/2})) \]

is the MMC over the functions in \( L G_0^\beta \).
Proposition 13. For \( \phi \in \mathcal{H}_\alpha' \), \( f, g \in LG_\alpha' \).

**Proposition 14.** Let \( f \in LG_\alpha' \), \( g \in D(R_+) \). Then

\[
(f \ast g)(y) = \langle (f(x)e^{-x/2})g(y/z)/(1/z)e^{y/2z} \rangle_{R_+}, \quad \text{for all } \Theta \in D(R_+),
\]

\[
(f \ast g, \Theta) = \langle (f(x)e^{-x/2})g(y/z)/(1/z)e^{y/2z}, \Theta(y) \rangle.
\]

**Proposition 15.** For \( f, g \in LG_\alpha' \) (and \( s \) in a suitable domain)

\[
\hat{M}_\alpha(f \ast g)(s) = (1/(s + \alpha))\Gamma(\alpha)M_\alpha f(s)M_\alpha g(s).
\]

In particular when \( \alpha = 0 \) we obtain (4).

**Proposition 16.** Let \( f, g \in LG_\alpha' \). Then,

\[
\hat{M}_\alpha(f \ast g)(s) = \hat{M}_\alpha(fe^{-1/2})(s)\hat{M}_\alpha(ge^{-1/2})(s).
\]

For \( \alpha = 0 \) we have (6).

The MMC over the elements in \( LG_\alpha' \) has the properties of commutativity and associativity.

5. **Inversion of Transforms**

We shall give two inversion formulas for MMT. Firstly, we give the generalised function version of the inverse of MMT. Secondly, we give a numerical inversion of MMT using the series of generalised Laguerre polynomials appropriate for applications.

If \( f \in LG_\alpha' \), then there are \( m \in N_0 \) and a continuous function of slow growth \( F \) with \( \text{supp} F \subset R_+ \) such that

\[
f = e^{-\alpha x/2}F(m)
\]
in the case of dual pairing 
\[ L_{\Omega}^m(f, \varphi) \mid_{\Omega_{\theta}} = (z^{-\alpha/2} F^{(m)}, \varphi) = L_{\Omega}^m(\langle F^{(m)}, z^{-\alpha/2} \varphi \rangle) \mid_{\Omega_{\theta}}, \]
where \( \varphi \in L_{\Omega_{\theta}} \) and \( z^{-\alpha/2} \varphi \in L_{\Omega_{\theta}} \) \((\Omega_{\theta})\).

Let \( F, F', \ldots, F^{(m)} \in L_{\Omega_{\theta}} \geq 0 \). Then, \( f \in L_{\Omega_{\theta}}^{\alpha} \) for \( s \in D_{\Omega_{\theta}}^{\alpha/2} \).

\[ \varphi_{s+m,0} = \int_{\Omega_{\theta}} \frac{1}{\Gamma(s - m)} \Gamma(s - m + \alpha) \varphi_{s+m,0} \mid_{\Omega_{\theta}}. \]

We choose \( s \in D_{\Omega_{\theta}}^{\alpha/2} \).

\[ (M_s f)(t) = \langle f(t), \varphi_{s+m,0}(t) \rangle = \langle F^{(m)}, t^{-\alpha/2} \varphi_{s+m,0} \rangle = \]

\[ = \frac{(-1)^m}{\Gamma(s + \alpha)} \left( F(t), \sum_{j=0}^{m} \binom{m}{j} \left( -\frac{1}{2} \right)^{m-j} (s-1) \ldots (s-j) t^{-j-1} e^{-t/2} \right) = \]

\[ = \frac{(-1)^m}{\Gamma(s + \alpha)} \left( -2 \right)^{-m} \sum_{j=0}^{m} \binom{m}{j} \left( -2 \right)^{j} (s-1) \ldots (s-j) \int_{0}^{\infty} t^{-j-1} e^{-t/2} F(t) dt \]

Thus, we have

\[ (M_s f)(t) = 2^{-m} \frac{\Gamma(s)}{\Gamma(s + \alpha)} \sum_{j=0}^{m} \binom{m}{j} \left( -2 \right)^{j} \frac{\Delta_{0}^{m}(F^{e^{-x/2}})(s-j)}{\Gamma(s-j)}. \]

**Proposition 17.** Let \( f, \tilde{f} \in L_{\Omega_{\theta}}^{m} \) and \( (M_s f) \) be defined on \( D_{\Omega_{\theta}}^{\alpha/2} \), \( k \geq 0 \), then there is \( \sigma_0 > k + (1 - \alpha)/2 \) such that for all \( \sigma > \sigma_0 \),

\[ \left( \frac{2}{\pi i} \int_{-i\tau}^{+i\tau} \frac{d \sigma}{\Gamma(s + \alpha)} \{M_s f \}(s) z^{-\sigma} ds, x^{-\alpha/2} \phi(x) \} \rightarrow \]

\[ \rightarrow \langle f, \phi \rangle, \tau \rightarrow \infty, \phi \in \mathcal{D}(0, \infty). \]

**Proof.** Since \( f \in L_{\Omega_{\theta}}^{m} \) then \( f = x^{-\alpha/2} F^{(m)} \), where \( \text{supp} \ F \subset \mathcal{R}_{+} \) and \( F \) is slowly increasing.
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Take $a_0 \in R_+$ such that for some $k \geq 0$, Re$\alpha > a_0$, $\psi_{a_0,m,n} \in L_{b,n}, F, F^*$, $\psi_{a_0,m,n} = F_{b,n}$, if $k > 0$. If Re$\alpha > a_0$ then for $\varepsilon > 0$

$$\frac{1}{2\pi i} \int_{c-i\varepsilon}^{c+i\varepsilon} H_0(F(z)e^{-\varepsilon z}) \psi_{a_0,m,n} F(z)e^{-\varepsilon z} \, dz \rightarrow 0$$

(see [6]). We have

$$\lim_{r \to \infty} \frac{e^{\pi r^{2/3}}}{2\pi} \int_{c-i\varepsilon}^{c+i\varepsilon} H_0(F(z)e^{-\varepsilon z}) \psi_{a_0,m,n} F(z)e^{-\varepsilon z} \, dz = 0$$

$$= \lim_{r \to \infty} \frac{e^{\pi r^{2/3}}}{2\pi} \int_{c-i\varepsilon}^{c+i\varepsilon} H_0(F(z)e^{-\varepsilon z}) \psi_{a_0,m,n} F(z)e^{-\varepsilon z} \, dz = 0$$

$$= \lim_{r \to \infty} \frac{e^{\pi r^{2/3}}}{2\pi} \int_{c-i\varepsilon}^{c+i\varepsilon} H_0(F(z)e^{-\varepsilon z}) \psi_{a_0,m,n} F(z)e^{-\varepsilon z} \, dz = 0$$

and so the Proposition is proved. □

Proposition 18 Let $s \in L_{b,n}, k \in R_+$, and $f = \sum_{m=0}^{\infty} \kappa_{a,m,n} \kappa_{a,n} \in L_{b,n}$. Then the inverse of MTT is given by

$$f(t) = \sum_{m=0}^{\infty} \kappa_{a,m,n} \kappa_{a,n}$$

where $\Delta^k$ are the finite differences, $n \in N_0$, where, for $s \in N_0, s \leq k + (1 - \alpha)/2$ the values of $(\mathcal{M}f)(x)$ are determined in the Remark at the end of Section 3.
Proof. From Proposition 6 (ii) we have
\[ (-1)^n b_{n, a} \tau n = \Delta^n (M_x f)(1) = \]
\[ = \sum_{n=0}^{n} (-1)^{n-m} \binom{n}{m} (M_x f)(m+1), n \in \mathbb{N}_0, ([4]; p.325) \]
and
\[ b_{n, a} = (-1)^n (1/n) \Delta^n (M_x f)(1). \]

\[ \blacksquare \]

6. The space of \( N_{\alpha} \)-transforms, \( \alpha > -1 \).

Denote by \( N_{\alpha} \) the space of all the Newton series of the form
\[ F_y \in N_{\alpha} \Leftrightarrow F_y = \sum_{n=0}^{\infty} (-1)^n b_{n, a} \tau n \left( \frac{s-1}{a} \right), \]
where \( \sum_{n=0}^{\infty} |b_{n, a}|^2 n^{-2k} < \infty \) for some \( k \in \mathbb{R} \). The abscissa of the common and absolute convergence of \( F \) are denoted by \( \lambda_F \) and \( \mu_F \) respectively.

Proposition 12. \( M_x \) transform is an algebraic isomorphism between the spaces \( LG'_{\alpha} \) and \( N_{\alpha} \).

Proof. It follows from Proposition 6. \( \blacksquare \)

In particular, when \( a = 0 \) we have the mapping \( LG'_{\alpha} \leftrightarrow N_{\alpha} \). (See [6]).

By using the connections between Newton's and Dirichlet's series we obtain:
\[ \mu_{F_0} = \lim_{n \to \infty} \frac{\sum_{k=0}^{n} |b_{k, a} \tau k|}{ln n}, \mu_{F_0} \geq 0, \]
and
\[ \mu_{F_0} = \lim_{n \to \infty} \frac{\sum_{k=0}^{n} |b_{k, a} \tau k|}{ln n}, \mu_{F_0} < 0. \]

All of the above enables us to inspect the spaces \( LG'_{\alpha} \) through the spaces of all the Newton series.
Proposition 20. If \( f \in L^2_{\alpha} \) and \( F = (M_\omega f) \) is defined by (1), then the following holds:

(i) \( f \in L^2_{\alpha} \Rightarrow k \geq \mu_F + (\alpha - 1)/2; \)

(ii) \( f \notin L^2_{\alpha} \Rightarrow k \leq \mu_F + \alpha/2; \)

(iii) \( k > \mu_F + \alpha/2 \Rightarrow f \in L^2_{\alpha}; \)

(iv) \( k < \mu_F + (\alpha - 1)/2 \Rightarrow f \notin L^2_{\alpha}. \)

Proof. All the above follows from the fact that if \( f \in L^2_{\alpha} \) then \( s \in D_{\alpha/2} \) and \( (M_\omega f)(s) \) converges when \( Re s > k + (1 - \alpha)/2. \)

For more details see [6].

7. Remarks on Dirichlet's series

Let

\[ f(z) = \sum_{p=0}^{\infty} \frac{a_p}{(p + 1)^z}, \text{Re} z > \mu, \]

be Dirichlet's series whose abscissa of absolute convergence is \( \mu \in \mathbb{R} \) and let \( \sigma > \mu. \)

Consider the formal series

\[ g(z) = \sum_{p=0}^{\infty} \frac{a_p e^{(p+1/3) \sqrt{z}}} {(p + 1)^{k-1/3}}, z > 0. \]

Put \( b_{n,\alpha} = (g(z), e_{n,\alpha}(z)). \) Then by using ([1], p.9) we obtain

\[ b_{n,\alpha} = r_n \sum_{p=0}^{\infty} \frac{a_p}{(p + 1)^{k-1/3}} \int_0^{+\infty} e^{(p+1/3) \sqrt{z}} f_n(z) \, dz = \]

\[ = \sum_{p=0}^{\infty} \frac{a_p}{(p + 1)^{(1/r_n)}} \left( \frac{p}{p + 1} \right)^n. \]

Since, \( i/r_n \sim n^{3/2}; n \to \infty, \) we have

\[ |b_{n,\alpha}| < C n^{3/2} \text{ and } g(z) \in L^2_{\alpha}, k > (1 + \alpha)/2. \]
8. Solving integro-differential equations via MMC

In this Section we shall give an application of the inversion formula given in Proposition 18, and the operational calculus from Sections 3. and 4. in order to show its possibilities from the numerical point of view.

Consider the equation

\[ zf'(z) + \frac{1}{2} \int_{\mathbb{R}} e^{-\frac{t^2}{2}} f(t) dt = h(z) \]  

along the initial condition \( \int_{\mathbb{R}} f(t)e^{-\frac{t^2}{2}} dt = A \), where \( h \in L^2(0) \) is known, and \( A \) is a given constant.

For appropriate \( h \) the solution of (12) belongs to \( L^2(\mathbb{R}_+) \).

Assume that this holds. In solving this equation we use the following Proposition 21. If \( f, g \in L^2(\mathbb{R}_+) \) and \( \text{supp} g \) is a compact subset of \( \mathbb{R}_+ \), then

\[ (f \ast g)(z) = \int\limits_0^{\infty} f(t) e^{-\frac{z^2}{2} - \frac{t^2}{2}} g(z \sqrt{t}) dt, z \in \mathbb{R}_+ \]

Proof. (13) follows from Lebesque's theorem and Proposition 9.

Setting \( g(t) = \frac{e^{-t^2}}{t} H(t - 1/2) H(1 - t), t \in \mathbb{R} \),

\( (H \) is Heaviside's function) equation (12) gives the equivalent convolution form
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(14) \[ z f'(x) + e^{\alpha x} f(x) = h(x), x \in \mathbb{R}_+. \]

Applying MMT($\mathcal{M}_0$) from Proposition 10 and 6, we obtain the difference equation

(15) \[-x F(s) + F(s + 1) + \Gamma(s) F(s) G(s) = H(s), \]

where

(\mathcal{M}_0 f)(s) = F(s), (\mathcal{M}_0 g)(s) = G(s), (\mathcal{M}_0 h)(s) = H(s). \]

Since (\mathcal{M}_0 f)(1) = F(1) = A, one can find all the coefficients of f(x), n ≥ 2, in the simplest case of the numerical inversion formula (8):

(16) \[ f(x) = \sum_{n=0}^{\infty} (-1)^{n} \Delta^{n} F(1) a_{n} \]

Then, (16) is the solution of our equation (12).

Note, if \[ f \approx \sum_{n=0}^{\infty} a_{n} \phi_{n} \]

then

\[ \left\| f - \sum_{n=n_0}^{\infty} a_{n} \phi_{n} \right\|_{L_2} \approx \sum_{n=n_0+1}^{\infty} |a_{n}|^2. \]

References


REZUME

MODIFIKOVANA MELINOVA TRANSFORMACIJA I KONVOLUCIJA

Uvedeni su pojemovi modifikovane Melinove transformacije i konvolucije na prostorima $L_\alpha^*$, $\alpha > -1$. Osebina ovih prostora u oznaku na navedene pojmove omogućavaju odgovarajući operacioni račun koji je na kraju rača i primenjen na rastvaranje jedne integro-diferencijalne jednačine.

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