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A NOTE ON RIORDAN MATRICES

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ABSTRACT. A simple relation between semicirculant matrices and Riordan matrices is given. As a consequence of this relation, the Riordan group multiplication is explicitly defined in terms of the classical (usual) matrix multiplication. An important class of Riordan matrices is also given using an extension of the usual matrix similarity equivalence.

1. INTRODUCTION

A simple relation between infinite semicirculant matrices and Riordan matrices is given in this note. *Riordan matrices* are a certain subset of infinite lower-triangular matrices. We show that Riordan matrices can be factored as the product of a semicirculant matrix and an infinite triangular matrix called a *power matrix*.

The set of all Riordan matrices, under multiplication, forms a group structure called the *Riordan group*. Although the Riordan group is commonly used to solve classical combinatorial problems and problems related to special functions, its algebraic properties are also of interest. Moreover, understanding the algebraic properties of the Riordan group may lead to additional combinatorial and special function applications. In this note, Sections 2 and 3 are given as preliminary material. In Section 2, we give the definition of a Riordan matrix and a brief summary of the Riordan group. In Section 3, we define and give some properties of semicirculant and power matrices. The note results are given in Sections 4 and 5. In Section 4, we consider the Riordan group as a semidirect product of two Riordan subgroups. Using the semidirect product and properties of semicirculant and power matrices, we construct a matrix relation between semicirculant and Riordan matrices. As a consequence of the relation, the Riordan group multiplication is explicitly defined in terms of the classical (usual) matrix multiplication. Also by the relation, we can construct Riordan matrices without having to rely on finding the matrix formation rules. Finally, in Section 5, we consider another factorization of Riordan matrices which involves similar Riordan matrices. Similarity of matrices is well known for $n \times n$ matrices. However, there seems to be a gap between similarity of finite $n \times n$ matrices and similarity of infinite matrices. We introduce a similarity definition for Riordan matrices and use Riordan matrix multiplication to prove certain subsets of Riordan matrices are similar.

2. RIORDAN MATRICES, AND THE RIORDAN GROUP

The Riordan group is an algebraic structure whose elements are infinite matrices called *Riordan matrices* [12], [14]. The definition of a Riordan matrix and a brief summary of the Riordan group are outlined below.

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Related topics can be found in Merlini [5], Roman [10], and Sprugnoli [15].

Definition 2.1. *Consider an infinite matrix* $\mathbf{L} = (l_{ij})_{i,j\geq 0}$ *with entries that belong to the field of complex numbers and let* $\mathbf{L}_i(x) = \sum_{n\geq 0} l_{n,i}x^n$ *be the formal power series (fps) of the ith column of* $\mathbf{L}$. *Then,* $\mathbf{L}$ *is called a Riordan matrix if* $\mathbf{L}_i(x) = g(x)\cdot[f(x)]^i$ *where* $g(x) = 1+g_1x+g_2x^2+\cdots$, *and* $f(x) = f_1x+f_2x^2+f_3x^3+\cdots$ *such that* $g(x)$ *and* $f(x)$ *belong to the ring of formal power series* $\mathbf{C}[[X]]$ *and* $f_1 \neq 0$.

Therefore, formal power series (or generating functions) make up the columns of $\mathbf{L}$ such that the columns are of the form

$$\mathbf{L} = \begin{pmatrix} | & | & | & | & \\ g & gf & gf^2 & gf^3 & \cdots \\ | & | & | & | & \end{pmatrix}.$$

$\mathbf{L}$ is an infinite lower-triangular matrix which we denote as a pair $\mathbf{L} = (g(x), f(x))$. We also say the pair is a Riordan matrix. Pascal's triangle denoted by $\mathbf{P} = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ is a common example of a Riordan matrix. The first few entries of $\mathbf{P}$ are given in Section 4.

Remark 2.1. *In Definition 2.1, the constant coefficient* $g_0 = 1$ *is primarily used with combinatorial applications. However,* $g_0 \neq 0$ *is sufficient for the definition.*

If we let $\mathbf{R}$ denote the set of all Riordan matrices, then $\mathbf{R}$ is a group under multiplication [12].

Definition 2.2. *Given two Riordan matrices* $\mathbf{L} = (g(x), f(x))$ *and* $\mathbf{N} = (h(x), k(x))$, *then*

$$\begin{aligned} \mathbf{L} * \mathbf{N} &= (g(x), f(x)) * (h(x), k(x)) \\ &= (g(x)\cdot h(f(x)), k(f(x))). \end{aligned}$$

With this multiplication $(\mathbf{R}, *)$ *is called the Riordan group where the symbol " $*$ " denotes Riordan matrix multiplication.*

The expression $g \cdot h(f)$ denotes the Cauchy (convolution) product of two fps. Furthermore, the composite functions $h \circ f$ and $k \circ f$ are well defined by properties of formal power series [4], [10]. The identity element of the group is $\mathbf{I} = (1, x)$. That is, $\mathbf{I}$ is an infinite unit diagonal matrix. The inverse of $\mathbf{L} = (g(x), f(x))$ is

$$\mathbf{L}^{-1} = \left(\frac{1}{g(\overline{f}(x))}, \overline{f}(x)\right)$$

where $\overline{f}(x)$ is the compositional inverse of $f(x)$. The existence of $\overline{f}(x)$ is also guaranteed by properties of formal power series.

Three important subgroups of the Riordan group are the associated, Appell, and Bell subgroups. The Bell subgroup consists of all pairs of the form $(g(x), xg(x))$. The Appell subgroup, which is normal, consists of all pairs of the form $(g(x), x)$. And, the associated subgroup consists of all pairs of the form $(1, f(x))$. The Riordan group is a semidirect product of the Appell and associated subgroups. That is,

$$(g(x), f(x)) = (g(x), x) * (1, f(x)).$$

The notion of semidirect product is used in Section 4 to help construct the relation between Riordan and semicirculant matrices.

3. Semicirculant and Power Matrices

Consider the formal power series

$$g(x) = g_0 + g_1x + g_2x^2 + g_3x^3 + \cdots = \sum_{n\geq 0} g_nx^n$$

associated with the sequence $\langle g_0, g_1, g_2, g_3, \ldots\rangle$. We associate the power series with the matrix

$$\mathbf{g}_s = \begin{pmatrix} g_0 & g_1 & g_2 & g_3 & \cdot \\ 0 & g_0 & g_1 & g_2 & \cdot \\ 0 & 0 & g_0 & g_1 & \cdot \\ 0 & 0 & 0 & g_0 & \\ \cdot & \cdot & \cdot & & \ddots \end{pmatrix}.$$

Matrices of this form are called infinite ordered semicirculant matrices [4]. The matrix $\mathbf{g}_s$ is defined by $\mathbf{g}_s = (g_{ij})$ where $g_{ij} = g_{j-i}$ for $j \geq i$ and $g_{ij} = 0$ for $j < i$. Let $\mathbf{H} = \{g(x) \mid g_0 = 1\}$ and $\mathbf{G}_s = \{\mathbf{g}_s \mid g_0 = 1\}$, then $\mathbf{G}_s$ is the set of all semicirculant matrices with unit diagonal entries. $\mathbf{G}_s$ is an abelian group under matrix multiplication. The map $\phi : \mathbf{H} \rightarrow \mathbf{G}_s$ defined by $g(x) \longmapsto \mathbf{g}_s$ is an isomorphism on $\mathbf{G}_s$ [4].

Now, consider the set of all formal power series of the form

$$f(x) = f_1x + f_2x^2 + f_3x^3 + \cdots = \sum_{n\geq 1} f_nx^n$$

such that $f_1 \neq 0$. Under the operation of composition of formal power series, the set of all such power series also forms a group. There is also an association between the power series $f(x)$ and matrices of the form

$$\mathbf{F}_p = \begin{pmatrix} f_1^{(1)} & f_2^{(1)} & f_3^{(1)} & f_4^{(1)} & \cdot \\ 0 & f_2^{(2)} & f_3^{(2)} & f_4^{(2)} & \cdot \\ 0 & 0 & f_3^{(3)} & f_4^{(3)} & \cdot \\ 0 & 0 & 0 & f_4^{(4)} & \\ \cdot & \cdot & \cdot & \cdot & \ddots \end{pmatrix}.$$

Matrices of this form are called power matrices [4]. The matrix $\mathbf{F_p}$ is defined by $\mathbf{F_p} = (f_{ij})$ where $f_{ij} := f_j^{(i)}$. For $i, j = 1, 2, \ldots$, $f_{ij} = f_j^{(i)}$ when $j \geq i$ and $f_{ij} = 0$ when $j < i$. The row entries of $\mathbf{F_p}$ are given by the coefficients of the ith power of $f(x)$ and are obtained by the following sum

$$[f(x)]^i = \sum_{j\geq 1} f_j^{(i)}x^j.$$

For a discussion on finite semicirculant matrices and circulant matrices, see Davis [1].

To construct the factorization given in the next section, we introduce the following notation. The notation $\mathbf{M}^T$ denotes the transpose of a matrix $\mathbf{M}$. Let

$f(x) = \sum_{j\geq 1} f_j x^j$ such that $f_1 \neq 0$. Then, the following block matrix

$$\mathbf{f}_p = \left(\begin{array}{c|ccc} 1 & 0 & 0 & \cdots \\ \hline 0 & \mathbf{F}_p & & \\ 0 & & & \\ \vdots & & & \end{array}\right)$$

is defined as a special type of power matrix such that

$$(\mathbf{f}_p)^T = \left(\begin{array}{c|ccc} 1 & 0 & 0 & \cdots \\ \hline 0 & (\mathbf{F}_p)^T & & \\ 0 & & & \\ \vdots & & & \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 0 & f_1^{(1)} & 0 & 0 & 0 & . \\ 0 & f_2^{(1)} & f_2^{(2)} & 0 & 0 & . \\ 0 & f_3^{(1)} & f_3^{(2)} & f_3^{(3)} & 0 & . \\ 0 & f_4^{(1)} & f_4^{(2)} & f_4^{(3)} & f_4^{(4)} & \\ . & . & . & . & & \ddots \end{pmatrix}.$$

4. The Factorization

By properties given in Sections 2 and 3, we obtain the following lemma.

Lemma 4.1. $(g,x) * (1,f) = (\mathbf{g}_s)^T \cdot (\mathbf{f}_p)^T$ *where "$\cdot$" denotes the usual matrix multiplication.*

Proof. The proof easily follows by equating coefficients. ∎

This leads to the following proposition.

Proposition 4.2. *Let (g,f) be a Riordan matrix. Then $(g,f) = (\mathbf{g}_s)^T \cdot (\mathbf{f}_p)^T$.*

Proof. By Lemma 4.1 and equating coefficients we obtain

$$\begin{aligned} (\mathbf{g}_s)^T \cdot (\mathbf{f}_p)^T &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . \\ g_1 & 1 & 0 & 0 & 0 & . \\ g_2 & g_1 & 1 & 0 & 0 & . \\ g_3 & g_2 & g_1 & 1 & 0 & . \\ g_4 & g_3 & g_2 & g_1 & 1 & \\ . & . & . & . & & \ddots \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 0 & f_1^{(1)} & 0 & 0 & 0 & . \\ 0 & f_2^{(1)} & f_2^{(2)} & 0 & 0 & . \\ 0 & f_3^{(1)} & f_3^{(2)} & f_3^{(3)} & 0 & . \\ 0 & f_4^{(1)} & f_4^{(2)} & f_4^{(3)} & f_4^{(4)} & \\ . & . & . & . & & \ddots \end{pmatrix} \\ &= \begin{pmatrix} | & | & | & | & \\ g & gf & gf^2 & gf^3 & \cdots \\ | & | & | & | & \end{pmatrix} \\ &= (g,f). \end{aligned}$$

Thus, we obtain the result. ∎

Therefore, the elements of the Riordan group are explicitly given in terms of the classical matrix multiplication. To motivate Proposition 4.2, we give the following example.

Example 4.1. *Consider Pascal's matrix* $\mathbf{P} = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ *where* $g(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ *and* $f(x) = x + x^2 + x^3 + x^4 + \cdots = \frac{x}{1-x}$. *Then,* $\mathbf{P}$ *is decomposed into the following product:*

$$\mathbf{P} = (\mathbf{g}_s)^T \cdot (\mathbf{f}_p)^T = \left(\left(\frac{1}{1-x}\right)_s\right)^T \cdot \left(\left(\frac{x}{1-x}\right)_p\right)^T.$$

The equivalent matrix form is

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 1 & 1 & 0 & 0 & 0 & . \\ 1 & 1 & 1 & 0 & 0 & . \\ 1 & 1 & 1 & 1 & 0 & . \\ 1 & 1 & 1 & 1 & 1 & \\ . & . & . & . & & \ddots \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 0 & 1 & 0 & 0 & 0 & . \\ 0 & 1 & 1 & 0 & 0 & . \\ 0 & 1 & 2 & 1 & 0 & . \\ 0 & 1 & 3 & 3 & 1 & \\ . & . & . & . & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 1 & 1 & 0 & 0 & 0 & . \\ 1 & 2 & 1 & 0 & 0 & . \\ 1 & 3 & 3 & 1 & 0 & . \\ 1 & 4 & 6 & 4 & 1 & \\ . & . & . & . & & \ddots \end{pmatrix}.$$

As a result of Proposition 4.2, the following theorem arises.

Theorem 4.3. *Let* (g, f) *and* (h, k) *be Riordan matrices. Then*

$$(g, f) * (h, k) = (\mathbf{g}_s)^T \cdot ((\mathbf{h} \circ \mathbf{f})_s)^T \cdot \left((\mathbf{k} \circ \mathbf{f})_p\right)^T.$$

Proof. By the definition of Riordan matrix multiplication, properties of semicirculant and power matrices, and following Proposition 4.2 we obtain

$$\begin{aligned} (g, f) * (h, k) &= (g \cdot (h \circ f), (k \circ f)) \\ &= \{(\mathbf{g} \cdot (\mathbf{h} \circ \mathbf{f}))_s\}^T \cdot \left\{(\mathbf{k} \circ \mathbf{f})_p\right\}^T \\ &= \{(\mathbf{g}_s) \cdot (\mathbf{h} \circ \mathbf{f})_s\}^T \cdot \left((\mathbf{k} \circ \mathbf{f})_p\right)^T \\ &= (\mathbf{g}_s)^T \cdot ((\mathbf{h} \circ \mathbf{f})_s)^T \cdot \left((\mathbf{k} \circ \mathbf{f})_p\right)^T. \end{aligned}$$

Thus, we obtain the result. ∎

Therefore, the Riordan group multiplication is expressed as the classical matrix product made up of matrices associated with the individual components of each Riordan ordered pair. Theorem 4.3 which is the main result of this note also explains the change of order of composition of the Riordan product.

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$$\mathbf{P} = (\mathbf{g}_s)^T \cdot (\mathbf{f}_p)^T = \left(\left(\frac{1}{1-x}\right)_s\right)^T \cdot \left(\left(\frac{x}{1-x}\right)_p\right)^T.$$

The equivalent matrix form is

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 1 & 1 & 0 & 0 & 0 & . \\ 1 & 1 & 1 & 0 & 0 & . \\ 1 & 1 & 1 & 1 & 0 & . \\ 1 & 1 & 1 & 1 & 1 & \\ . & . & . & . & & \ddots \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 0 & 1 & 0 & 0 & 0 & . \\ 0 & 1 & 1 & 0 & 0 & . \\ 0 & 1 & 2 & 1 & 0 & . \\ 0 & 1 & 3 & 3 & 1 & \\ . & . & . & . & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 1 & 1 & 0 & 0 & 0 & . \\ 1 & 2 & 1 & 0 & 0 & . \\ 1 & 3 & 3 & 1 & 0 & . \\ 1 & 4 & 6 & 4 & 1 & \\ . & . & . & . & & \ddots \end{pmatrix}.$$

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Proof. By the definition of Riordan matrix multiplication, properties of semicirculant and power matrices, and following Proposition 4.2 we obtain

$$\begin{aligned} (g, f) * (h, k) &= (g \cdot (h \circ f), (k \circ f)) \\ &= \{(\mathbf{g} \cdot (\mathbf{h} \circ \mathbf{f}))_s\}^T \cdot \left\{(\mathbf{k} \circ \mathbf{f})_p\right\}^T \\ &= \{(\mathbf{g}_s) \cdot (\mathbf{h} \circ \mathbf{f})_s\}^T \cdot \left((\mathbf{k} \circ \mathbf{f})_p\right)^T \\ &= (\mathbf{g}_s)^T \cdot ((\mathbf{h} \circ \mathbf{f})_s)^T \cdot \left((\mathbf{k} \circ \mathbf{f})_p\right)^T. \end{aligned}$$

Thus, we obtain the result. ∎

Therefore, the Riordan group multiplication is expressed as the classical matrix product made up of matrices associated with the individual components of each Riordan ordered pair. Theorem 4.3 which is the main result of this note also explains the change of order of composition of the Riordan product.

Example 4.2. *Consider* $\mathbf{P} * \mathbf{A} = \left(\frac{1}{1-x}, \frac{x}{1-x}\right) * \left(\frac{1}{1-x}, x\right)$. *Then, we obtain*

$$\begin{aligned}
\mathbf{P} * \mathbf{A} &= \left(\left(\frac{1}{1-x}\right) \cdot \left(\frac{1-x}{1-2x}\right), \left(\frac{x}{1-x}\right)\right) \\
&= \left(\left(\frac{1}{1-x}\right)_s\right)^T \cdot \left(\left(\frac{1-x}{1-2x}\right)_s\right)^T \cdot \left(\left(\frac{x}{1-x}\right)_p\right)^T .
\end{aligned}$$

The equivalent matrix form in this case is

$$\begin{aligned}
\mathbf{P} * \mathbf{A} &= \begin{pmatrix} 1 & 0 & 0 & 0 & . \\ 1 & 1 & 0 & 0 & . \\ 1 & 1 & 1 & 0 & . \\ 1 & 1 & 1 & 1 & \\ . & . & . & & \ddots \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & . \\ 1 & 1 & 0 & 0 & . \\ 2 & 1 & 1 & 0 & . \\ 4 & 2 & 1 & 1 & \\ . & . & . & & \ddots \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & . \\ 0 & 1 & 0 & 0 & . \\ 0 & 1 & 1 & 0 & . \\ 0 & 1 & 1 & 1 & \\ . & . & . & & \ddots \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & . \\ 2 & 1 & 0 & 0 & . \\ 4 & 3 & 1 & 0 & . \\ 8 & 7 & 4 & 1 & \\ . & . & . & & \ddots \end{pmatrix}.
\end{aligned}$$

The Riordan product of $\mathbf{P} * \mathbf{A}$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & . \\ 2 & 1 & 0 & 0 & . \\ 4 & 3 & 1 & 0 & . \\ 8 & 7 & 4 & 1 & \\ . & . & . & & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & . \\ 1 & 1 & 0 & 0 & . \\ 1 & 2 & 1 & 0 & . \\ 1 & 3 & 3 & 1 & \\ . & . & . & & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & . \\ 1 & 1 & 0 & 0 & . \\ 1 & 1 & 1 & 0 & . \\ 1 & 1 & 1 & 1 & \\ . & . & . & & \ddots \end{pmatrix}.$$

In the Riordan product case, usual matrix multiplication follows by the definition of a Riordan matrix.

Remark 4.1. *When* $h(x)$ *is a binomial series, the J. C. P. Miller formula given by Henrici* [4] *can be used to compute* $(\mathbf{h} \circ \mathbf{f})_s$.

5. Similarity of Riordan Matrices

Definition 5.1. *Two Riordan matrices* $\mathbf{M}$ *and* $\mathbf{N}$ *are said to be similar if there exists a Riordan matrix* $\mathbf{L}$ *such that* $\mathbf{N} = \mathbf{L}^{-1} * \mathbf{M} * \mathbf{L}$.

Consider the Riordan matrices denoted $\mathbf{A}$, $\mathbf{P}$, $\mathbf{S} = (\mathbf{C}_0 * \mathbf{A})$, $\mathbf{T}$, $\mathbf{C}_0$, and $\mathbf{D}_0$. All of these matrices arise in combinatorial applications. In Riordan pair form we have:

$$\mathbf{C}_0 = \left(\frac{1-\sqrt{1-4x^2}}{2x^2}, \frac{1-\sqrt{1-4x^2}}{2x}\right)$$

$$\mathbf{S} = \left(\left(\frac{\sqrt{1+2x}}{\sqrt{1-2x}} - 1\right)\frac{1}{2x}, \frac{1-\sqrt{1-4x^2}}{2x}\right)$$

$$\mathbf{T} = \left(\frac{1}{\sqrt{1-2x-3x^2}}, \left(1 - \frac{\sqrt{1-3x}}{\sqrt{1+x}}\right)\frac{1}{2}\right)$$

$$\mathbf{D}_0 = \left(\frac{(1-x+x^2)-\sqrt{1-2x-x^2-2x^3+x^4}}{2x^2}, \frac{(1-x+x^2)-\sqrt{1-2x-x^2-2x^3+x^4}}{2x}\right).$$

Matrices $\mathbf{A}$ and $\mathbf{P}$ are given in Section 4. The first few entries of $\mathbf{T}$, $\mathbf{S}$, $\mathbf{C}_0$, and $\mathbf{D}_0$ are as follows:

$$\mathbf{C}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 0 & 1 & 0 & 0 & 0 & . \\ 1 & 0 & 1 & 0 & 0 & . \\ 0 & 2 & 0 & 1 & 0 & . \\ 2 & 0 & 3 & 0 & 1 & \\ . & . & . & . & & \ddots \end{pmatrix}, \qquad \mathbf{D}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 1 & 1 & 0 & 0 & 0 & . \\ 1 & 2 & 1 & 0 & 0 & . \\ 2 & 3 & 3 & 1 & 0 & . \\ 4 & 6 & 6 & 4 & 1 & \\ . & . & . & . & & \ddots \end{pmatrix},$$

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 1 & 1 & 0 & 0 & 0 & . \\ 2 & 1 & 1 & 0 & 0 & . \\ 3 & 3 & 1 & 1 & 0 & . \\ 6 & 4 & 4 & 1 & 1 & \\ . & . & . & . & & \ddots \end{pmatrix}, \quad \text{and} \quad \mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 1 & 1 & 0 & 0 & 0 & . \\ 3 & 1 & 1 & 0 & 0 & . \\ 7 & 4 & 1 & 1 & 0 & . \\ 19 & 9 & 5 & 1 & 1 & \\ . & . & . & . & & \ddots \end{pmatrix}.$$

Then, the following equivalence propositions arise.

Proposition 5.1. *The matrices* $\mathbf{D}_0$ *and* $\mathbf{P}$ *are similar since* $\mathbf{D}_0 = (\mathbf{C}_0)^{-1} * \mathbf{P} * \mathbf{C}_0$.

Proof. Use the definition of Riordan similarity, and perform Riordan matrix multiplication. See Nkwanta for more details [7]. ∎

Recall, matrix $\mathbf{P}$ is Pascal's matrix. The matrix $\mathbf{C}_0$ is an aerated Catalan matrix. Also, by induction, it can be shown that $(\mathbf{D}_0)^k$ is similar to $\mathbf{P}^k$. To see this, since $\mathbf{D}_0 = (\mathbf{C}_0)^{-1} * \mathbf{P} * \mathbf{C}_0$, successively multiplying equation $\mathbf{C}_0 * \mathbf{D}_0 = \mathbf{P} * \mathbf{C}_0$ on the right by $\mathbf{D}_0$ gives

$$\begin{aligned} \mathbf{C}_0 * (\mathbf{D}_0)^2 &= \mathbf{P}^2 * \mathbf{C}_0 \\ \mathbf{C}_0 * (\mathbf{D}_0)^3 &= \mathbf{P}^3 * \mathbf{C}_0 \\ &\vdots \\ \mathbf{C}_0 * (\mathbf{D}_0)^k &= \mathbf{P}^k * \mathbf{C}_0 \\ &\vdots \end{aligned}$$

The matrix $\mathbf{P}^k = \left(\frac{1}{1-kx}, \frac{x}{1-kx}\right)$ is a generalized version of the Pascal matrix $\mathbf{P}$ given in Section 4.

Consider for $n \geq 0$ the following Riordan matrix

$$\mathbf{D}_n = \left(g(x) \cdot \left(\frac{1-x}{1-xg(x)} \right)^n, xg(x) \right)$$

where $g(x) = \frac{(1-x+x^2)-\sqrt{1-2x-x^2-2x^3+x^4}}{2x^2}$. Then, the following proposition arises.

Proposition 5.2. *The matrices* $\mathbf{D}_0$ *and* $\mathbf{D}_n$ *are similar since*

$$\mathbf{D}_n = \mathbf{A}^{-n} * \mathbf{D}_0 * \mathbf{A}^n \quad (n \geq 1).$$

Proof. Use the definition of Riordan similarity, and perform Riordan matrix multiplication. ∎

The matrices $\mathbf{A}^n = \left(\frac{1}{(1-x)^n}, x \right)$ and $\mathbf{A}^{-n} = \left((1-x)^n, x \right)$ are generalized versions of matrix $\mathbf{A}$ given in Section 4. The matrix $\mathbf{D}_n$ also arises in combinatorial applications. The matrix $\mathbf{D}_0$ has a lattice path interpretation and arises in connection with RNA secondary structures [6]. Likewise $\mathbf{D}_1$ and $\mathbf{D}_2$ have lattice path interpretations which arise in connection with the Fibonacci numbers [8]. Also note that similarity of Riordan matrices is an equivalence relation. Thus, it easily follows that $\mathbf{P}$ and $\mathbf{D}_n$ are similar.

Proposition 5.3. *The matrices* $\mathbf{S}$ *and* $\mathbf{T}$ *are similar since* $\mathbf{S} = \mathbf{P}^{-1} * \mathbf{T} * \mathbf{P}$.

Proof. Use the definition of Riordan similarity, and perform Riordan matrix multiplication. ∎

Matrix $\mathbf{T}$ is of interest since its left most column is sequence $\langle 1, 1, 3, 7, 19 \ldots \rangle$. This sequence essentially goes back to Euler and is commonly called the central trinomial coefficients.

It is important to note that all the propositions discussed in this section can be proved using Theorem 4.3. In addition, the theorem can also serve as a procedure for computing and constructing Riordan matrices. For related topics on factorizations of Riordan matrices, see Getu and Shapiro [2], and Peart and Woodson [9] for triple factorization involving Stieltjes matrices, and Getu [3] for LDU factorization involving Hankel matrices.

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