CONGRESSUS NUMERANTIUM

A RIORDAN MATRIX APPROACH TO UNIFYING A SELECTED CLASS OF COMBINATORIAL ARRAYS

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1. Introduction

A selected class of combinatorial arrays are considered as Riordan matrices. What we mean by a *combinatorial array* is a lower-triangular array of numbers that count some combinatorial object. A Riordan matrix is a special type of infinite lower-triangular matrix and the set of all Riordan matrices forms a group called the *Riordan group* [23]. Using properties of the group, we construct an infinite two-dimensional array consisting of Riordan matrices. The Catalan, Motzkin, directed animal, and polyhex (Hex) arrays are included in the construction. Thus, as a consequence of the construction certain classes of combinatorial arrays, generating functions, and counting sequences are unified. Connections to Pascal's triangle and the RNA and Fibonacci numbers are also shown. Matrix multiplication is the main method used in the construction. In Section 2, we make the paper self contained by briefly outlining the Riordan group and defining multiplication of Riordan matrices. Readers familiar with the Riordan group may skip these details and go directly to Section 3. The results will begin here with the construction of an infinite array of Riordan matrices where the (i, j)th entry is derived as a generalized Catalan-type array. Also in Section 3, some new Catalan identities are derived and connections to admissible matrices as defined by Aigner [1] are given. In Section 4, we count lattice paths with certain restrictions and show they are counted by the alternate Fibonacci numbers. We conclude with giving some discussion of other approaches and a list of open problems in Section 5.

2. Riordan Group and Matrices

The Riordan group depends upon certain formal power series.

Definition 2.1. *An infinite matrix $L = (l_{n,k})_{n,k\geq 0}$ with complex entries $\mathbb{C}$ is called a Riordan matrix if the kth column satisfies*

$$\sum_{n\geq 0} l_{n,k} z^n = g(z)(f(z))^k$$

where $g(z) = 1 + g_1 z + g_2 z^2 + \cdots$ and $f(z) = f_1 z + f_2 z^2 + f_3 z^3 + \cdots$ belong to the ring of formal power series $\mathbb{C}[[z]]$ and $f_1 \neq 0$.

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The concept of representing columns of infinite matrices by formal power series is not new and goes back to Schur's paper on Faber polynomials [21]. A formal power series in auxiliary variable z of the form

$$b(z) = b_0 + b_1 z + b_2 z^2 + \cdots = \sum_{n \geq 0} b_n z^n$$

is called an *ordinary generating function* of the sequence $\{b_n\}$. A Riordan matrix is denoted by a pair of generating functions as $L = (g(z), f(z))$.

Example 2.1. *$P = (1/(1-z), z/(1-z))$ (the well-known Pascal's triangle) and $C = (c^2(z), zc^2(z))$ (see Figure 4) (Shapiro's Catalan array in [22]) are examples of Riordan matrices.*

The $c(z)$ denotes the *Catalan generating function*

$$c(z) := \left(1 - \sqrt{1-4z}\right)/2z = \sum_{n \geq 0} c_n z^n \tag{1}$$

and $c_n = (1/(1+n))\binom{2n}{n}$ denotes the *Catalan numbers* which occur in a wide variety of combinatorial problems and algebraic applications (see [28] pp. 219-229 and 231, respectively).

Multiplication of Riordan matrices is given by the following theorems.

Theorem 2.1. *If $L = (g(z), f(z))$ is a Riordan matrix and $h(z)$ is the generating function of the column vector $h = (h_0, h_1, \ldots)^T$, then the generating function for the column vector Lh is $g(z)h(f(z))$. If $\circledast$ denotes matrix multiplication, then we have the equivalent form*

$$L \circledast h(z) = g(z)h(f(z)).$$

Proof. See [23] (Equation 5) or [16] (Proof of Theorem 2.1). □

This theorem is called the *Fundamental Theorem of the Riordan Group* [16]. It leads to the next theorem by applying the fundamental theorem to an arbitrary Riordan matrix N, one column of N at a time.

Theorem 2.2. *If $L = (g(z), f(z))$ and $N = (h(z), l(z))$ are Riordan matrices, then*

$$L * N = (g(z)h(f(z)), l(f(z))),$$

and the set $\mathbf{R}$ of Riordan matrices is a group under matrix multiplication.

Proof. See [23] (Equation 6) and [26]. □

Example 2.2. *Consider the product of the Pascal and Catalan matrices. Then*

$$H = P * C \quad = \quad \left(\tfrac{1-3z-\sqrt{1-6z+5z^2}}{2z^2}, \tfrac{1-3z-\sqrt{1-6z+5z^2}}{2z}\right)$$

is the Hex matrix given in Section 3 (see Figure 4).

The group, we denote by $(\mathbf{R}, *)$, is called the Riordan group (in honor of John Riordan). This group is a generalization of the renewal array theory which was introduced by Rogers [19] to study generalizations of the ordinary Pascal, Catalan, and Motzkin arrays. For more details on the Riordan group see Shapiro, *et. al.* [23] and Sprugnoli [27], [26].

One key step when constructing Riordan matrices is to find the matrix formation rule. A *formation rule,* which we denote by $[\mathbf{Z}; \mathbf{A}]$, is a recurrence relation which defines the way entries of a Riordan matrix are computed. The notation $[\mathbf{Z}; \mathbf{A}]$ means "$\mathbf{Z}$" coincides with the formation of the zeroth column and "$\mathbf{A}$" coincides with the formation of the other columns.

Following Rogers [19] and Merlini, *et al.* [12], with minor adjustments, we give two useful characterizations of a given Riordan matrix. The formation rules which determine Riordan matrices are called **A**- and **Z**-sequences. The **Z**-sequence ($\mathbf{Z} = \{z_0, z_1, \ldots\}$) characterizes the zeroth column. This means every element $l_{n+1,0}$ can be expressed as a linear combination of all the elements in the preceding row, i.e.,

$$l_{n+1,0} = z_0 l_{n,0} + z_1 l_{n,1} + z_2 l_{n,2} + \cdots .$$

The **A**-sequence ($\mathbf{A} = \{a_0, a_1, \ldots\}$, $a_0 \neq 0$) characterizes the other columns. In this case every element $l_{n+1,k+1}$ can be expressed as a linear combination with coefficients in **A** of the elements in the preceding row, starting from the preceding column on, i.e.,

$$l_{n+1,k+1} = a_0 l_{n,k} + a_1 l_{n,k+1} + a_2 l_{n,k+2} + \cdots .$$

If $A(y)$ and $Z(y)$ are the generating functions (in auxiliary variable y) of the **A**- and **Z**-sequences, respectively, then g and f of $L = (g(z), f(z))$ are the solutions of

$$f(z) = zA(f(z)) \text{ and } g(z) = g_0 / (1 - z \cdot Z(f(z))). \tag{2}$$

Conversely, $A(y)$ and $Z(y)$ can be determined by letting $y = f(z)$ and eliminating z from

$$A(y) = y/z \text{ and } Z(y) = (g(z) - g_0) / zg(z). \tag{3}$$

Example 2.3. *The formation rule of C is $[\mathbf{2}, \mathbf{1}; \mathbf{1}, \mathbf{2}, \mathbf{1}]$ where $Z(y) = 2 + y$ and $A(y) = 1 + 2y + y^2$. In general, $l_{n+1,k+1}$ is computed as illustrated by Figure 1.*

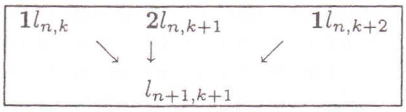

Fig. 1. A-sequence

3. The Infinite Array of Riordan Matrices

An infinite two-dimensional array of Riordan matrices denoted by $\widetilde{\mathcal{L}}$ in Figure 2 (below) is constructed. Most of the material related to the construction comes from Nkwanta [15]. The array $\widetilde{\mathcal{L}}$ is of combinatorial interest since matrices that have certain combinatorial applications (i.e., the Catalan, Motzkin, Hex, and directed animal arrays) appear in the entries of the array. Some combinatorial objects of interest that are counted by certain array entries are ballot sequences [22], random walks [3], lattice paths with various restrictions [5], [6], hexagonal graphs with certain restrictions [8], [24], and interval graphs [7]. Lattice path interpretations of the first two columns of $\widetilde{\mathcal{L}}$ are given in [15]. Therefore, a remarkable aspect of $\widetilde{\mathcal{L}}$ is it brings together various combinatorial objects and many well-known combinatorial arrays, generating functions, and counting sequences.

$$
\widetilde{\mathcal{L}} = \begin{array}{ccccccc}
C_0 & \to & C_0\widehat{E} & \to & C_0\widehat{E}^2 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
M & \to & M\widehat{E} & \to & M\widehat{E}^2 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
C & \to & C\widehat{E} & \to & C\widehat{E}^2 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
H & \to & H\widehat{E} & \to & H\widehat{E}^2 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

Fig. 2. Infinite Array I

3.1. **The Construction.** $\widetilde{\mathcal{L}}$ is constructed by starting with combinatorial arrays denoted by C_0, M, C, and H. These arrays are Riordan matrices and repeated right multiplication of each matrix by $\widehat{E} = (1/(1-z), z)$ is performed to move across the rows of the array. Matrix $\widehat{E}$, whose entries on and below the main diagonal are 1 and 0 everywhere else, computes the partial row sums of a matrix. The computation is defined by

$$L * \widehat{E} = (g(z) / (1 - f(z)), f(z)).$$

For examples see Figure 4 or references [1] and [9].

To move down the columns of the array, we multiply repeatedly on the right by D_j. For $j = 0$, the matrix D_0 (see Figure 3) is defined by $D_0 = (d(z), zd(z))$ where

$$d(z) = \left((1 - z + z^2) - \sqrt{1 - 2z - z^2 - 2z^3 + z^4}\right) / 2z^2. \tag{4}$$

This matrix is an element of the Bell subgroup of the Riordan group [15]. These are elements of the form $(g(z), zg(z))$. D_0 arises in connection with counting certain lattice paths that are related to RNA secondary structures of length n from molecular biology [14]. The leftmost column of D_0 contains the RNA numbers $\{1, 1, 1, 2, 4, 8, 17, \ldots\}$, [24]. For $j > 0$ the matrix D_j, which acts as a *column transition matrix*, is defined by

$$D_j = \left(d(z)\left((1 - z) / (1 - zd(z))\right)^j, zd(z)\right).$$

See Remark 3.2 for more details on D_j. The first few entries for $j = 0, 1, 2$ are given in Figure 3. Simplified forms for D_1 and D_2 are given by

$$
\begin{aligned}
D_1 = &\ \left(\left(\frac{\sqrt{1+z+z^2}}{\sqrt{1-3z+z^2}} - 1\right) \frac{1-z}{2z}, zd(z)\right) \text{ and} \\
D_2 = &\ \left((1 - 2z + z^2) / (1 - 3z + z^2), zd(z)\right).
\end{aligned}
$$

These matrices are related to the Fibonacci numbers. Computing the row sums of D_1 gives the even Fibonacci numbers $\{f_{2n}\} = \{1, 2, 5, 13, 34, \ldots\}$. The generating function of the leftmost column of D_2 gives the odd Fibonacci numbers plus a leading one, $\{1, f_{2n+1}\} = \{1, 1, 3, 8, 21, \ldots\}$. A lattice path interpretation of D_1 is subsequently given in Section 4.

$$D_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdot \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdot \\ 2 & 3 & 3 & 1 & 0 & 0 & \cdot \\ 4 & 6 & 6 & 4 & 1 & 0 & \cdot \\ 8 & 13 & 13 & 10 & 5 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \end{pmatrix} \qquad D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdot \\ 2 & 2 & 1 & 0 & 0 & 0 & \cdot \\ 5 & 4 & 3 & 1 & 0 & 0 & \cdot \\ 12 & 10 & 7 & 4 & 1 & 0 & \cdot \\ 29 & 25 & 18 & 11 & 5 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \end{pmatrix}$$

$$D_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdot \\ 3 & 2 & 1 & 0 & 0 & 0 & \cdot \\ 8 & 5 & 3 & 1 & 0 & 0 & \cdot \\ 21 & 14 & 8 & 4 & 1 & 0 & \cdot \\ 55 & 38 & 23 & 12 & 5 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \end{pmatrix}$$

Fig. 3. Riordan Matrices Connected to $\widetilde{\mathcal{L}}$

$\widetilde{\mathcal{L}}$ is now constructed. The first few entries are explicitly given in Figure 4.

$$C_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & \cdot \\ 1 & 0 & 1 & 0 & \cdot \\ 0 & 2 & 0 & 1 & \\ \cdot & \cdot & \cdot & & \cdot \end{pmatrix} \longrightarrow C_0\widehat{E} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot \\ 1 & 1 & 0 & 0 & \cdot \\ 2 & 1 & 1 & 0 & \cdot \\ 3 & 3 & 1 & 1 & \\ \cdot & \cdot & \cdot & & \cdot \end{pmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot \\ 1 & 1 & 0 & 0 & \cdot \\ 2 & 2 & 1 & 0 & \cdot \\ 4 & 5 & 3 & 1 & \\ \cdot & \cdot & \cdot & & \cdot \end{pmatrix} \longrightarrow M\widehat{E} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot \\ 2 & 1 & 0 & 0 & \cdot \\ 5 & 3 & 1 & 0 & \cdot \\ 13 & 9 & 4 & 1 & \\ \cdot & \cdot & \cdot & & \cdot \end{pmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot \\ 2 & 1 & 0 & 0 & \cdot \\ 5 & 4 & 1 & 0 & \cdot \\ 14 & 14 & 6 & 1 & \\ \cdot & \cdot & \cdot & & \cdot \end{pmatrix} \longrightarrow C\widehat{E} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot \\ 3 & 1 & 0 & 0 & \cdot \\ 10 & 5 & 1 & 0 & \cdot \\ 35 & 21 & 7 & 1 & \\ \cdot & \cdot & \cdot & & \cdot \end{pmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot \\ 3 & 1 & 0 & 0 & \cdot \\ 10 & 6 & 1 & 0 & \cdot \\ 36 & 29 & 9 & 1 & \\ \cdot & \cdot & \cdot & & \cdot \end{pmatrix} \longrightarrow H\widehat{E} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot \\ 4 & 1 & 0 & 0 & \cdot \\ 17 & 7 & 1 & 0 & \cdot \\ 75 & 39 & 10 & 1 & \\ \cdot & \cdot & \cdot & & \cdot \end{pmatrix}$$

Fig. 4. Array Entries

Observing the leftmost columns of $C_0, M, C, H, M\widehat{E}$, and $C\widehat{E}$ and looking at more entries we notice the following sequences:

$$\begin{aligned}
\{(c_0)_n\} &= \{1,0,1,0,2,0,5,0,14,0,42,\ldots\} \text{ (aerated Catalan)}\\
\{m_n\} &= \{1,1,2,4,9,21,51,127,323,\ldots\} \text{ (Motzkin)}\\
\{c_n\} &= \{1,2,5,14,42,132,429,1430\ldots\} \text{ (Catalan)}\\
\{h_n\} &= \{1,3,10,36,137,543,2219,\ldots\} \text{ (hexagonal)}\\
\{(me)_n\} &= \{1,2,5,13,35,96,267,750,\ldots\} \text{ (directed animal)}\\
\{(ce)_n\} &= \{1,3,10,35,126,462,1716,\ldots\}.
\end{aligned}$$

These counting sequences are known and can be found in [24] and [25].

Continuing to focus on the zeroth column of $\widetilde{\mathcal{L}}$, we observe a nice pattern given by

$$\begin{aligned}
C_0 &= \left(c\left(z^2\right), zc\left(z^2\right)\right) = \left(\tfrac{(1-0z)-\sqrt{(1-0z)^2-4z^2}}{2z^2}, zc\left(z^2\right)\right)\\
M &= \left(m\left(z\right), zm\left(z\right)\right) = \left(\tfrac{(1-1z)-\sqrt{(1-1z)^2-4z^2}}{2z^2}, zm\left(z\right)\right)\\
C &= \left(c^2\left(z\right), zc^2\left(z\right)\right) = \left(\tfrac{(1-2z)-\sqrt{(1-2z)^2-4z^2}}{2z^2}, zc^2\left(z\right)\right)\\
H &= \left(h\left(z\right), zh\left(z\right)\right) = \left(\tfrac{(1-3z)-\sqrt{(1-3z)^2-4z^2}}{2z^2}, zh\left(z\right)\right).
\end{aligned}$$

The generating functions from these matrices prove the related corresponding sequences given above (see Proposition 3.1). Proposition 3.5 proves the last two sequences. Moreover, the pattern of generating functions subsequently leads to Equation 10.

Following Peart and Woodson [17], each of the above matrices can be triple factored into three simpler Riordan matrices yielding

$$\begin{aligned}
C_0 &= P^0 C_0 F_{0,0} = P^0 C_0\\
M &= P^1 C_0 F_{0,0} = P^1 C_0\\
C &= P^2 C_0 F_{0,0} = P^2 C_0\\
H &= P^3 C_0 F_{0,0} = P^3 C_0
\end{aligned}$$

where P is Pascal's matrix and C_0 is the aerated Catalan matrix given above. The matrix $F_{0,0} = (1, z)$, which is the identity element of $(\mathbf{R}, *)$, comes from the Fibonacci matrix $F_{\varepsilon,\delta} = \left(1/\left(1-\varepsilon z-\delta z^2\right), z\right)$ given in [17]. The ith triple factor is $P^i C_0$ where $P^i = \left(1/\left(1-iz\right), z/\left(1-iz\right)\right)$ is a generalized Pascal matrix. Therefore by triple factorization and construction, $\widetilde{\mathcal{L}} = \mathcal{L}$ where $\mathcal{L}$ (Figure 5) is subsequently deduced as $\mathcal{L}_{i,j} = P^i * C_0 * \widehat{E}^j$ where $\widehat{E}^j = \left(1/\left(1-z\right)^j, z\right)$. This triple product of matrices is a generalized Catalan-type matrix (see Theorem 3.8). An interesting point is the Pascal matrix P now appears in the entries of the array.

$$
\mathcal{L} = \begin{array}{ccccccc}
C_0 & \rightarrow & C_0\widehat{E} & \rightarrow & C_0\widehat{E}^2 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
PC_0 & \rightarrow & PC_0\widehat{E} & \rightarrow & PC_0\widehat{E}^2 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
P^2C_0 & \rightarrow & P^2C_0\widehat{E} & \rightarrow & P^2C_0\widehat{E}^2 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
P^3C_0 & \rightarrow & P^3C_0\widehat{E} & \rightarrow & P^3C_0\widehat{E}^2 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

Fig. 5. Infinite Array II

Moving down the jth column of the array, left multiplication by P gives the same matrix relations as right multiplication by D_j since P and D_j are similar Riordan matrices [15]. This leads to the following interesting remarks.

Remark 3.1. *Left multiplication by P is an Euler (binomial series) transform on the first Riordan pair component since each component is derived by*

$$k_{i+1}(z) = (1/(1-z))\, k_i(z/(1-z)) \tag{5}$$

where $k_i(z)$ is the generating function given by Equation 10. Likewise right multiplication by D_0 gives an equivalent transform and each component in this case is derived by

$$k_{i+1}(z) = k_i(z)\, d(zk_i(z)) \tag{6}$$

where $d(z)$ is the first component generating function (Equation 4) of D_0. Both relations can be proved by induction and they can be shown to be equivalent by direct computation. Thus, the first components of C_0, M, C, and H are obtained by Euler transforms. Moreover, the notion of Euler transform holds for all of $\mathcal{L}$.

Remark 3.2.

$$\mathcal{L}_{i+1,j} = P * \mathcal{L}_{i,j} = \mathcal{L}_{i,j} * D_j \tag{7}$$

where

$$D_j = \widehat{E}^{-j} * D_0 * \widehat{E}^j = \left((1-z)^j, z\right) * D_0 * \left(1/(1-z)^j, z\right).$$

The matrix D_0 is also defined by $D_0 = (C_0)^{-1} * P * C_0$ where

$$(C_0)^{-1} = \left(1/\left(1+z^2\right), z/\left(1+z^2\right)\right).$$

Although the construction for moving down the columns of $\mathcal{L}$ started with D_j, the triple factorization involving P simplifies the process. Still focusing on the zeroth column of $\mathcal{L}$, we make the strong assumption that the formation rule of $P^i * C_0$ is $[\mathbf{i}, \mathbf{1}; \mathbf{1}, \mathbf{i}, \mathbf{1}]$. By the assumption and definition of Riordan matrices we obtain

$$g = 1 + z(\mathbf{i}g + \mathbf{1}gf) \text{ and } gf^k = z\left(\mathbf{1}gf^{k-1} + \mathbf{i}gf^k + \mathbf{1}gf^{k+1}\right). \tag{8}$$

From these equations we obtain

$$g(z) = 1/(1 - \mathbf{i}z - zf(z)) \text{ and } f(z) = z\left(1 + \mathbf{i}f(z) + f^2(z)\right). \tag{9}$$

Solving for g and f leads to the following proposition.

Proposition 3.1. *$\pounds_{i,0} = P^i * C_0 = (k_i(z), zk_i(z))$ where*

$$k_i(z) = \left((1-iz) - \sqrt{(1-iz)^2 - 4z^2}\right)/2z^2 \tag{10}$$

and $[\mathbf{i}, \mathbf{1}; \mathbf{1}, \mathbf{i}, \mathbf{1}]$ *is the formation rule.*

Proof. (Sketch) Show that the formation rule coincides with the matrix. Find the Riordan pair and verify the pair by using Theorem 2.2 and computing $P^i * C_0$ and simplifying. □

$P^i * C_0$ is also an element of the Bell subgroup [15]. A nice characterization of $k_i(z)$ is it unifies the Catalan, Motzkin, and Hex counting numbers. The next corollary uses the same generating function and unifies the Motzkin, Catalan, and Hex counting numbers with alternating signs.

Corollary 3.2. $k_{-i}(z) = \left((1+iz) - \sqrt{(1+iz)^2 - 4z^2}\right)/2z^2$.

Proof. Use Theorem 2.1 and $P^{-i} = (1/(1+iz), z/(1+iz))$ and compute $P^{-i} \circledast \left(1 - \sqrt{1-4z^2}\right)/2z^2$. □

Solving f of Equation 9 for f^2 leads to the following lemma.

Lemma 3.3. $(1 - zk_i(z))^2 = k_i(z)(1-(i+2)z)$.

Proof. See [15], pp. 38. □

Corollary 3.4. *(Catalan Identities) For $\Delta = z/(1-iz)$ we have the following:*

(1) $c(\Delta^2) = (1-iz)k_i(z)$
(2) $c(\Delta^2) = 1 + z^2k_i^2(z)$.

Moving down the next column of $\pounds$ leads to the following proposition.

Proposition 3.5. *$\pounds_{i,1} = P^i * C_0 * \widehat{E} = (k_i^*(z), zk_i(z))$ where*

$$k_i^*(z) = k_i(z)/(1-zk_i(z)) = \left(\sqrt{1-(i-2)z}/\sqrt{1-(i+2)z} - 1\right)/2z \tag{11}$$

and $[(\mathbf{i}+\mathbf{1}), \mathbf{1}; \mathbf{1}, \mathbf{i}, \mathbf{1}]$ *is the formation rule.*

Proof. Follow the proof of Proposition 3.1. □

$k_i^*(z)$ unifies counting numbers related to directed animals [4], nonisomorphic trees [11], and certain coefficients of Chebyshev polynomials [10] (pp. 178-180, and 515). This generating function leads to another Catalan identity.

Lemma 3.6. $k_i^*(z) = (1/(1-(i+2)z))\, c(-z/(1-(i+2)z))$.

Proof. Use Equation 1 and simplify. □

Remark 3.3. *Lemma 3.6 and Corollary 3.4* (1) *can also be proved by using the Touchard identities given by Riordan* [18] *(pp. 156).*

Now moving down the third column of $\pounds$ leads to the next proposition.

Proposition 3.7. *$\pounds_{i,2} = P^i * C_0 * \widehat{E}^2 = (k_i^{**}(z), zk_i(z))$ where*

$$k_i^{**}(z) = k_i(z)/(1-zk_i(z))^2 = 1/(1-(i+2)z) \tag{12}$$

and $[(\mathbf{i}+\mathbf{2}), \mathbf{0}; \mathbf{1}, \mathbf{i}, \mathbf{1}]$ *is the formation rule.*

Proof. Follow the proof of Proposition 3.1. □

As a consequence of Propositions 3.1, 3.5, and 3.7, the following theorems arise.

Theorem 3.8. *For* $i, j \geq 0$,

$$\mathcal{L}_{i,j} = P^i * C_0 * \widehat{E}^j = \left(\frac{k_i(z)}{(1-zk_i(z))^j}, zk_i(z)\right).$$

Proof. By the construction, Theorem 2.2, Corollary 3.4, and simplifying

$$\begin{aligned} P^i * C_0 * \widehat{E}^j &= \left(\frac{(1-iz)^{j-1}c(\Delta^2)}{(1-iz-zc(\Delta^2))^j}, \frac{zc(\Delta^2)}{(1-iz)}\right) \\ &= \left(\frac{k_i(z)}{(1-zk_i(z))^j}, zk_i(z)\right). \end{aligned}$$

This proves the theorem. □

We now completely determine $\mathcal{L}_{i,j}$ and find the **A**- and **Z**-sequences. We start with the following lemma.

Lemma 3.9. *If* $L = (g(z), f(z))$ *is a Riordan matrix and* $f(z) = zb(z)$ *such that* α *is a real number and* $b(z)$ *satisfies*

$$b(z) = 1 + \alpha zb(z) + z^2b^2(z),$$

then the generating function for the **A**-*sequence of* L *is* $A(y) = 1 + \alpha y + y^2$.

Proof. Use Equation 3. □

Theorem 3.10. $A_{i,j}(y) = 1 + iy + y^2$ *and* $Z_{i,j}(y) = i + y + \left(1 - (1-y)^j\right)/y$ *are the generating functions for the* **A**- *and* **Z**-*sequences of* $\mathcal{L}_{i,j}$.

Proof. Using Corollary 3.4, $k_i(z)$ satisfies

$$k_i(z) = 1 + izk_i(z) + z^2k_i^2(z). \tag{13}$$

$A_{i,j}(y)$ is obtained by Lemma 3.9. Applying Equation 3 gives $Z_{i,j}(y)$. □

Remark 3.4. *Given* $A(y) = 1 + \alpha y + y^2$ *yields another Catalan identity,*

$$A\left(zc^2(z)\right) = (1 + (\alpha - 2)z)c^2(z).$$

Introducing the concepts of the **A**- and **Z**-sequences proved to be most useful in helping to find $Z_{i,j}(y)$. This completes the construction of $\mathcal{L}$. A surprising connection to $\mathcal{L}$ is mentioned in the following remark.

Remark 3.5. *The matrix*

$$B = \left(1/\left(1-z^2\right), z\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & 0 & \cdot \\ 1 & 0 & 1 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 1 & 0 & \cdot \\ 1 & 0 & 1 & 0 & 1 & \\ \cdot & \cdot & \cdot & \cdot & & \cdot \end{pmatrix}$$

gives $\left(P^i * C_0\right) * B = P^i * B_0$ *where* B_0 *is the aerated central binomial array* $B_0 = \left(1/\left(\sqrt{1-4z^2}\right), \left(1-\sqrt{1-4z^2}\right)/2z\right)$ *and the formation rule is* $[\mathbf{i}, \mathbf{2}; \mathbf{1}, \mathbf{i}, \mathbf{1}]$. *This array connects* $\mathcal{L}$ *to the infinite two-dimensional array* $\mathfrak{B} = P^i * B_0 * \widehat{E}^j$. *Generating functions and counting sequences involving the central binomial and trinomial arrays are included in this construction. Thus further extending* $\mathcal{L}$ *and unifying more combinatorial arrays, generating functions, and counting sequences.*

3.2. **The Connection to Admissible Matrices.** Related topics on combinatorial quantities represented by infinite matrices are given by Aigner [1] and Kemeny [9]. Aigner's paper introduces a certain class of infinite matrices called *admissible matrices* where Catalan, Motzkin, central trinomial, and polyhex arrays are unified. We conclude this section with giving some connections of $\mathcal{L}$ to admissible matrices.

Definition 3.1. *(Aigner [1]) An infinite matrix $A = (a_{n,k})$ with the mth row denoted $r_m = (a_{m,0}, a_{m,1}, \ldots)$ is called admissible if it is lower triangular with main diagonal equal 1 and $r_m \cdot r_n = a_{m+n,0}$ $\forall$ m, n where $r_m \cdot r_n = \sum_k a_{m,k}a_{n,k}$ is the usual inner product.*

Admissible matrices are also described by the following proposition.

Proposition 3.11. *(Aigner [1]) Let $A = (a_{n,k})$ be admissible with $a_{n,k} = b_n$ for all n. Set*

$$s_0 = b_0, s_1 = b_1 - b_0, \ldots, s_n = b_n - b_{n-1}, \ldots.$$

Then,

$$\begin{aligned} a_{0,0} &= 1, \\ a_{0,k} &= 0 \text{ for } k > 0, \text{ and} \\ a_{n,k} &= a_{n-1,k-1} + s_k \cdot a_{n-1,k} + a_{n-1,k+1} \ (n \geq 1). \end{aligned} \tag{14}$$

Conversely, if $a_{n,k}$ is given by recurrence 14, then $A = (a_{n,k})$ is admissible with $a_{n+1,n} = s_0 + s_1 + \cdots + s_n$.

Proof. See Aigner [1] (Proposition 1). □

Admissible matrices as given by the sequence $\sigma = \{s_0, s_1, s_3, \ldots\}$ via recurrence (14) are considered. Then for the special case $\sigma = \{a, s, \ldots\}$ we denote $A = A^{(a,s)}$ and observe

$$C_0 = A^{(0,0)} \qquad C_0\widehat{E} = A^{(1,0)}$$

$$M = A^{(1,1)} \qquad M\widehat{E} = A^{(2,1)}$$

$$C = A^{(2,2)} \qquad C\widehat{E} = A^{(3,2)}$$

$$H = A^{(3,3)} \qquad H\widehat{E} = A^{(4,3)}$$

are admissible. The leftmost entries of $\mathcal{L}_{i,0}$ and $\mathcal{L}_{i,1}$ are called Catalan-like numbers of type $\sigma = (a, s)$. Thus, we generalize and obtain Catalan-like arrays of type σ.

Proposition 3.12. *$\mathcal{L}_{i,0} = A^{(i,i)}$ and $\mathcal{L}_{i,1} = A^{(i+1,i)}$ are admissible.*

Proof. Use Propositions 3.1, 3.5, and 3.11 . Also see Aigner [1] (Proposition 2(ii) and Equations 2, 5, and 13 of Section 3). □

4. Lattice Paths

The matrix D_j was derived as a consequence of constructing $\mathcal{L}$. A lattice path interpretation of D_1 is given in this section.

We consider counting paths which start at the origin $(0, 0)$ and take unit steps, $(0, 1) =$ N (north), $(0, -1) =$ S (south), $(1, 0) =$ E (east), and $(-1, 0) =$ W (west) with the following restrictions: 1) no paths pass below the x-axis, 2) no paths begin with a W step, 3) all W steps remain on the *x-axis*, and 4) no S step immediately follows an N step, i.e there are no NS steps. We call this class of paths NESW* paths. A typical example is denoted by the steps EWEEWNESWN.

Let $d(n,k)$ denote the number of NESW* paths where n is the number of steps and k is the final height. Computing the first few paths gives the first few entries of D_1. By convention $d(0,0) = 1$ and it is obvious $d(n,n) = 1$ since the only way to be at height n after n steps is to always go north. By Riordan group methods, it is easy to show the formation rule of D_1 is given by the following recurrence relations $d(1,0) = 1$, $d(2,0) = 2$, and

$$d(n+1,0) = 2d(n,0) + d(n-1,0), \tag{15}$$

for the leftmost column, and for $n > j$ $(n \geq 2)$ and $k \geq 1$ we have

$$d(n+1,k) = d(n,k-1) + d(n,k) + \sum_{j\geq 1} d(n-j,k+j), \tag{16}$$

for the other columns.

We want to find the total number of NESW* paths of length n and connect D_1 to the paths by showing the paths satisfy (15) and (16). To do this we consider the following combinatorial arguments. Let $d(n,k)$ denote the number of NESW* paths of length n and height k. To form such a path, we consider the following cases. First, if the last step is N, then there are $d(n,k-1)$ possibilities to move up to height k. If the last step is E, then there are $d(n,k)$ possibilities to remain at height k. If the last step is S, then there are $d(n-j,k+j)$ possibilities to move down to height k. Summing over the cases gives (16). For $k > 0$ there are no paths with last step W since the W steps remain on the *x-axis*. For the zeroth column, if the last step is S then there are $d(n-1,0)$ possibilities. If the last step is E, then there are $d(n,0)$ possibilities. Also if the last step is W, there are $d(n,0)$ possibilities. Combining these cases gives (15). There are no paths with last step N for height $k = 0$. The boundary condition $d(n,k) = 0$ $(k > n)$ is trivial since there are no paths of this form. This proves the formation rule and gives D_1 a NESW* lattice path interpretation.

To find the total number of NESW* paths of length n, we multiply by $(1,1,\ldots)^T$ which has $1/(1-z)$ as its generating function. Then by the fundamental theorem and simplifying, the generating function for the total number of all NESW* paths is

$$\begin{aligned} D_1 \circledast (1/(1-z)) &= \left(\left(\frac{\sqrt{1+z+z^2}}{\sqrt{1-3z+z^2}} - 1\right)\frac{1-z}{2z}, zd(z)\right) \circledast (1/(1-z)) \\ &= ((1-z)/(1-3z+z^2)) = \sum_{n\geq 0} f_{2n} z^n. \end{aligned}$$

Thus the NESW* paths are counted by the even Fibonacci numbers.

5. Other Approaches and Open Problems

$\mathcal{L}$ is the focus of this paper. Some remarkable aspects of this array are it compresses a wealth of combinatorial information into the matrix $\mathcal{L}_{i,j}$ and it unifies a large collection of combinatorial arrays, generating functions, and counting sequences.

In addition to the combinatorial arrays mentioned in this paper, unifying combinatorial arrays whose entries are associated with Fibonacci, Pell, and Schröder arrays would be of interest. This paper is only concerned with Riordan matrices generated by ordinary generating functions. Considering Riordan matrices generated by exponential generating functions would be of interest. This could lead to finding unifying arrays involving Bell, Bernoulli, Eulerian, Stirling, and secant/tangent arrays. Also, finding unifying arrays involving special functions like Bessel functions, and Legéndre, Laguerre, and Chebyshev polynomials would be of interest.

The remarks given in the paper provide interesting exercises for the reader. Some open problems are:

(1) Find connections of $\mathcal{L}_{i,j}$ to the umbral calculus, see [2] and [20] (pp. 6-31).
(2) Determine whether $\mathcal{L}_{i,j}$ is connected to Hankel or Stieltjes matrices, or generating trees and AGT matrices (see [13] and [29]).
(3) Determine whether additional group properties can be obtained by studying properties of $\mathcal{L}_{i,j}$.
(4) Find other combinatorial interpretations for D_j for $j = 0, 1$.
(5) Generalize and find a lattice path interpretation for the sequence of matrices $\{D_j\}_{j\geq 0}$.

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