A GENERALIZATION OF EULER'S FORMULA
AND ITS CONNECTION TO FIBONACCI NUMBERS

Jonathan F. Mason and Richard H. Hudson

1. INTRODUCTION

This paper began as a simple proof generalizing Euler's well-known formula for the
vertices, faces, and edges of a cube in 3 dimensions, to a tesseract, and to higher dimensions. Let
an $n$-cube with $n$-dimensional volume 1 consist of all $n$-tuples $(x_1, x_2, \ldots, x_n)$ where each $x_i$
$i = 1, \ldots, n$ satisfies $0 \leq x_i \leq 1$. The boundary points of the $n$-cube are the vertices, which we
will call 0-cubes to indicate that they are 0-dimensional. For each such vertex, we clearly have
$x_i$ fixed to be 0 or 1. A 1-cube will be an edge of the $n$-cube. For an edge, we have exactly
one of the $x_i$ free to take on values between 0 and 1 (inclusive) and the other $x_i$ fixed to be 0
or 1 for each $i = 1, \ldots, n$. Similarly, a $k$-cube, $k \leq n$, will have exactly $k$ of the $x_i$ free to take
on values between 0 and 1 (inclusive) and $n - k$ fixed to be 0 or 1.

By representing each vertex in this way, it is clear that there are $2^n$ vertices in an $n$-cube.
For a $k$-cube, since $n - k$ of the $x_i$ are fixed, and $k$ are not fixed, we must have exactly

$$\binom{n}{k} \cdot 2^{n-k} \quad (1.1)$$

$k$-cubes in an $n$-cube. In particular, there are $n \cdot (2^{n-1})$ edges and $\binom{n}{2} \cdot (2^{n-2})$ faces. Thus,

$$\text{Vertices} + \text{Faces} - \text{Edges} = 2^n - n \cdot (2^{n-1}) + \binom{n}{2} \cdot (2^{n-2}) = 2^n - 3(n^2 - 5n + 8), \quad (1.2)$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

F.S. Gordon (Ed.), Applications of Fibonacci Numbers, Volume 9. Proceedings of the Tenth International Research Conference on
Fibonacci Numbers and their Applications, 177–185.
© 2004 Kluwer Academic Publishers
which is a natural generalization of the well-known formula of Euler when \( n = 3 \), namely \( V + F = E + 2 \).

Note that it is an easy consequence of the binomial theorem (see, e.g., [1, p. 9]), that

\[
\sum_{k=0}^{n} \binom{n}{k} \cdot (2^{n-k}) = (1 + 2)^n = 3^n
\]

This gives the following table, which appears in [2, p. 89] when \( n \geq 5 \):

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 4 )</th>
<th>( 8 )</th>
<th>( 16 )</th>
<th>( 32 )</th>
<th>( 64 )</th>
<th>( 128 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \binom{n}{0} \cdot 2^n )</td>
<td>( \binom{n}{1} \cdot 2^{n-1} )</td>
<td>( \binom{n}{2} \cdot 2^{n-2} )</td>
<td>( \binom{n}{3} \cdot 2^{n-3} )</td>
<td>( \binom{n}{4} \cdot 2^{n-4} )</td>
<td>( \binom{n}{5} \cdot 2^{n-5} )</td>
<td>( \binom{n}{6} \cdot 2^{n-6} )</td>
<td>( \binom{n}{7} \cdot 2^{n-7} )</td>
<td>( \binom{n}{8} \cdot 2^{n-8} )</td>
</tr>
</tbody>
</table>

Table 1.1

Looking carefully at Table 1.1, we note that there is a one-to-one correspondence between entries in the table and the sequence of Fibonacci numbers. In Section 2, we will show how to prove this correspondence, but it is a somewhat more complicated derivation than the similar well-known correspondence between the Fibonacci numbers and the diagonal's of Pascal’s Triangle, so we will first illustrate it pictorially for small \( n \) in Tables 1.2 and 1.3 below:

Table 1.2

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_0 = 0 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( F_1 = 1 )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( F_2 = 1 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( F_3 = 2 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_4 = 3 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_5 = 5 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_6 = 8 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_7 = 13 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_8 = 21 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_9 = 34 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_{10} = 55 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_{11} = 89 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_{12} = 144 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 1.3

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_0 = 0 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( F_1 = 1 )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( F_2 = 1 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( F_3 = 2 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_4 = 3 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_5 = 5 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_6 = 8 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_7 = 13 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_8 = 21 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_9 = 34 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_{10} = 55 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_{11} = 89 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( F_{12} = 144 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
A GENERALIZATION OF FİLTER'S FORMULA AND ITS CONNECTION ...

Following the arrows and adding we obtain each Fibonacci number $F_i$ exactly once.

2. THE PROOF OF THE FIBONACCI CORRESPONDENCE
ILLUSTRATED IN TABLES 1.2 AND 1.3

We begin by defining $A_n$ to be the sum of the terms starting with $\left(\begin{array}{c} 0 \\ n \end{array}\right) \cdot 2^n$ in the first column plus $\left(\begin{array}{c} n-1 \\ 2 \end{array}\right) \cdot 2^{n-3}$ in the third column. We continue summing by moving up one row and over two columns each time. Note that we will encounter 0's when $2k > n - k$ or $3k > n$. Thus, there will only be $\left\lfloor \frac{n}{3} \right\rfloor + 1$ elements in the summation of $A_n$, and

$$A_n = \sum_{k=0}^{\left\lfloor \frac{n}{3} \right\rfloor} \left(\begin{array}{c} n-k \\ 2k \end{array}\right) \cdot 2^{n-3k}, \quad n \geq 0. \quad (2.1)$$

Similarly, we define $B_n$ to be the same sequence as $A_n$ but starting in the second column. Therefore,

$$B_n = \sum_{k=1}^{\left\lfloor \frac{n+2}{3} \right\rfloor} \left(\begin{array}{c} n-k \\ 2k-1 \end{array}\right) \cdot 2^{n-3k+1}, \quad n \geq 2 \text{ where } B_0 = B_1 = 0. \quad (2.2)$$

With these definitions now in place, we will show that $A_n + B_n = F_{2n+1}, \quad n \geq 0$, and $A_{n-1} + B_n = F_{2n}, \quad n \geq 1$. A bit more generally, $A_{\left\lfloor \frac{n}{2} \right\rfloor} + B_{\left\lfloor \frac{n}{2} \right\rfloor} = F_{n+1}, \quad n \geq 1$, and therefore,

$$\sum_{k=0}^{\left\lfloor \frac{n}{3} \right\rfloor} \left(\begin{array}{c} \frac{n}{2} - k \\ 2k \end{array}\right) \cdot 2^{\frac{n}{2} - 3k} + \sum_{k=1}^{\left\lfloor \frac{n+2}{3} \right\rfloor} \left(\begin{array}{c} \frac{n+1}{2} - k \\ 2k-1 \end{array}\right) \cdot 2^{\frac{n+1}{2} - 3k+1} = F_{n+1}. \quad (2.3)$$

To prove this we will argue by induction.

Initial Cases: $n = 0$ and $n = 1$: It is trivial to show by substitution that for $n = 0$ we get $F_1$ and for $n = 1$ we get $F_2$. Hence, the base cases both hold.

Now, we will assume that the result is true for both $n = m - 1$ and $n = m - 2$, and we will show that it is true for $n = m$. In other words, we will assume that $A_{\left\lfloor \frac{m-1}{2} \right\rfloor} + B_{\left\lfloor \frac{m-1}{2} \right\rfloor} = F_{m-1}$ and $A_{\left\lfloor \frac{m-2}{2} \right\rfloor} + B_{\left\lfloor \frac{m-2}{2} \right\rfloor} = F_{m-2}$, and we will show that $A_{\left\lfloor \frac{m}{2} \right\rfloor} + B_{\left\lfloor \frac{m}{2} \right\rfloor} = F_{m+1}$. By our assumptions, we know that

$$A_{\left\lfloor \frac{m}{2} \right\rfloor} + B_{\left\lfloor \frac{m}{2} \right\rfloor} + A_{\left\lfloor \frac{m-1}{2} \right\rfloor} + B_{\left\lfloor \frac{m-1}{2} \right\rfloor} = F_{m-1} + F_{m} = F_{m+1}. \quad (2.4)$$

We must now break this problem into two cases:

Case 1: $m = 0 \pmod{2}$
If \( m \equiv 0 \pmod{2} \), then \( \left\lfloor \frac{m-2}{2} \right\rfloor = \left\lfloor \frac{m-2}{2} \right\rfloor = \left\lfloor \frac{m}{2} \right\rfloor - 1 = \left\lfloor \frac{m+1}{2} \right\rfloor - 1 \). Therefore, by (2.4), we know that \( A_{\left\lfloor \frac{m-2}{2} \right\rfloor} + B_{\left\lfloor \frac{m-2}{2} \right\rfloor} + A_{\left\lfloor \frac{m-2}{2} \right\rfloor} + B_{\left\lfloor \frac{m}{2} \right\rfloor} = F_{m+1} \). Thus,

\[
2 \cdot \sum_{k=0}^{\left\lfloor \frac{m+1}{2} \right\rfloor} \left( \left\lfloor \frac{m}{2} \right\rfloor - k \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor - 2k} + \sum_{k=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \left\lfloor \frac{m}{2} \right\rfloor - k \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor - 2k - 1} \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor - 3k} + B_{\left\lfloor \frac{m}{2} \right\rfloor} = F_{m+1}. \tag{2.5}
\]

It will help us later to move the 2 into the first summation and then bring out the first term of that summation. We are then left with

\[
\left( \left\lfloor \frac{m}{2} \right\rfloor + 1 \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor} + \left( \left\lfloor \frac{m}{2} \right\rfloor - 1 \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor - 2k} + \sum_{k=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \left\lfloor \frac{m}{2} \right\rfloor - k \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor - 3k} + B_{\left\lfloor \frac{m}{2} \right\rfloor} = F_{m+1}. \tag{2.6}
\]

We can make the substitution \( \left( \left\lfloor \frac{m}{2} \right\rfloor + 1 \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor} = \left( \left\lfloor \frac{m}{2} \right\rfloor \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor} \), and obtain

\[
\left( \left\lfloor \frac{m}{2} \right\rfloor + 1 \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor} + \left( \left\lfloor \frac{m}{2} \right\rfloor - 1 \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor - 2k} + \sum_{k=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \left\lfloor \frac{m}{2} \right\rfloor - k \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor - 3k} + B_{\left\lfloor \frac{m}{2} \right\rfloor} = F_{m+1}. \tag{2.7}
\]

To progress further, we have three more cases to consider.

Case 1(a): \( m \equiv 0 \pmod{6} \), so that \( \left\lfloor \frac{m-2}{2} \right\rfloor = \left\lfloor \frac{m}{2} \right\rfloor - 1 \). From (2.7), we have that

\[
\left( \left\lfloor \frac{m}{2} \right\rfloor + 1 \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor} + \left( \left\lfloor \frac{m}{2} \right\rfloor - 1 \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor - 2k} + \sum_{k=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \left\lfloor \frac{m}{2} \right\rfloor - k \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor - 3k} + B_{\left\lfloor \frac{m}{2} \right\rfloor} = F_{m+1}. \tag{2.8}
\]
A GENERALIZATION OF EULER'S FORMULA AND ITS CONNECTION...

Then, after we pull out the last term of the second sum, our two sums have the same indices and we are free to combine them as follows

\[
\left( \binom{\frac{m}{6}}{0} \right) 2^{\frac{m}{6}} + \sum_{k=1}^{\frac{m}{6} - 1} \left[ \left( \binom{\frac{m}{6}}{2k} - k - 1 \right) + \left( \binom{\frac{m}{6}}{2k - 1} - 1 \right) \right] \cdot 2^{\frac{m}{6} - 2k} + B_{\frac{m}{6}} = F_{m+1}.
\]  

(2.9)

Again, using a result of Paal, ecc [1, μ, δ], we can simplify this to

\[
\left( \binom{\frac{m}{6}}{0} \right) 2^{\frac{m}{6}} + \sum_{k=1}^{\frac{m}{6} - 1} \left( \binom{\frac{m}{6}}{2k} - k \right) \cdot 2^{\frac{m}{6} - 2k} + \left( \binom{\frac{m}{6}}{2k - 1} - 1 \right) + B_{\frac{m}{6}} = F_{m+1}.
\]  

(2.10)

Now, because

\[
\left( \binom{m}{2} \right) - \frac{m}{6} - 1 = 2^{\frac{m}{6}} \left( \binom{m}{6} - 1 \right), \quad \left( \binom{m}{2} - \frac{m}{6} - 1 \right) = 1 = \left( \binom{m}{2} - \frac{m}{6} \right),
\]

(2.11)

it becomes evident that we can add the two terms on each side of the summation to the ends of the summation. Then (2.10) becomes

\[
\sum_{k=1}^{\frac{m}{6} - 1} \left( \binom{\frac{m}{6}}{2k} - k \right) \cdot 2^{\frac{m}{6} - 2k} + B_{\frac{m}{6}} = F_{m+1}.
\]  

(2.12)

Dut, \( \sum_{k=0}^{\frac{m}{6} - 1} \binom{\frac{m}{6}}{2k} \cdot 2^{\frac{m}{6} - 2k} = A_{\frac{m}{6}} \). Therefore, \( A_{\frac{m}{6}} + B_{\frac{m}{6} - 1} = F_{m+1} \), and our theorem is proven when \( m \equiv 0 \pmod{6} \).

Case 1(b): \( m \not\equiv 0 \pmod{6} \), but \( m \equiv 0 \pmod{2} \), so that \( m \equiv 2 \pmod{6} \) or \( m \equiv 4 \pmod{6} \).

Equation (2.7) will still hold, so

\[
\left( \binom{\frac{m}{6}}{0} \right) 2^{\frac{m}{6}} + \sum_{k=1}^{\frac{m}{6} - 1} \left( \binom{\frac{m}{6}}{2k} - k \right) \cdot 2^{\frac{m}{6} - 2k} + B_{\frac{m}{6}} = F_{m+1}.
\]  

(2.13)
However, this time \( \left\lfloor \frac{m-8}{6} \right\rfloor = \left\lfloor \frac{m}{6} \right\rfloor \), so our first step will be to combine the summations.

\[
\left( \left\lfloor \frac{m}{2} \right\rfloor \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor} \cdot \sum_{k=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \binom{\left\lfloor \frac{m}{2} \right\rfloor}{k} - \binom{\left\lfloor \frac{m}{2} \right\rfloor - 1}{k} \right) \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor - 3k} + B_{\left\lfloor \frac{m}{2} \right\rfloor} = F_{m+1}. \quad (2.14)
\]

As before, we may combine the two terms of the summand and add the first term of the summand to the sum. This leaves us with

\[
\sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \binom{\left\lfloor \frac{m}{2} \right\rfloor}{k} \cdot 2^{\left\lfloor \frac{m}{2} \right\rfloor - 3k} + B_{\left\lfloor \frac{m}{2} \right\rfloor} = F_{m+1}. \quad (2.15)
\]

Again, this just means \( A_{\left\lfloor \frac{m}{2} \right\rfloor} + B_{\left\lfloor \frac{m}{2} \right\rfloor} = F_{m+1} \), and the formula is proven for \( m \equiv 0 \pmod{2} \).

**Case 2:** \( m \equiv 1 \pmod{2} \)

If \( m \equiv 1 \pmod{2} \), then \( \left\lfloor \frac{m-2}{2} \right\rfloor + 1 = \left\lfloor \frac{m-1}{2} \right\rfloor = \left\lfloor \frac{m}{2} \right\rfloor - 1 \). Therefore, by (2.4), we know that \( A_{\left\lfloor \frac{m-2}{2} \right\rfloor} + B_{\left\lfloor \frac{m-2}{2} \right\rfloor} = F_{m+1} \). Therefore,

\[
\sum_{k=0}^{\left\lfloor \frac{m-2}{2} \right\rfloor} \binom{\left\lfloor \frac{m-2}{2} \right\rfloor}{k} \cdot 2^{\left\lfloor \frac{m-2}{2} \right\rfloor - 3k} + 2 \sum_{k=1}^{\left\lfloor \frac{m-2}{2} \right\rfloor} \binom{\left\lfloor \frac{m-2}{2} \right\rfloor}{k} \cdot 2^{\left\lfloor \frac{m-2}{2} \right\rfloor - 3k + 1} + A_{\left\lfloor \frac{m}{2} \right\rfloor} = F_{m+1}. \quad (2.16)
\]

Now, we will move the 2 inside the second summation. Then,

\[
\sum_{k=0}^{\left\lfloor \frac{m-2}{2} \right\rfloor} \binom{\left\lfloor \frac{m-2}{2} \right\rfloor}{k} \cdot 2^{\left\lfloor \frac{m-2}{2} \right\rfloor - 3k} + \sum_{k=1}^{\left\lfloor \frac{m-2}{2} \right\rfloor} \binom{\left\lfloor \frac{m-2}{2} \right\rfloor}{k} \cdot 2^{\left\lfloor \frac{m-2}{2} \right\rfloor - 3k + 1} + A_{\left\lfloor \frac{m}{2} \right\rfloor} = F_{m+1}, \quad (2.17)
\]

which becomes

\[
\sum_{k=0}^{\left\lfloor \frac{m-2}{2} \right\rfloor} \binom{\left\lfloor \frac{m-2}{2} \right\rfloor}{k} \cdot 2^{\left\lfloor \frac{m-2}{2} \right\rfloor - 3k} + \sum_{k=0}^{\left\lfloor \frac{m-2}{2} \right\rfloor} \binom{\left\lfloor \frac{m-3}{2} \right\rfloor}{k} \cdot 2^{\left\lfloor \frac{m-3}{2} \right\rfloor - 3k + 1} + A_{\left\lfloor \frac{m}{2} \right\rfloor} = F_{m+1}. \quad (2.18)
\]

**Case 2(a):** \( m \equiv 3 \pmod{6} \).
A GENERALIZATION OF EULER'S FORMULA AND ITS CONNECTION

Then, \( \left\lfloor \frac{m-r}{k} \right\rfloor = \left\lfloor \frac{m-r}{k} \right\rfloor - 1 \), so our first summation can be rewritten to produce the equation

\[
\left( \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m-r}{6} \right\rfloor - 1 \right) \cdot 2^l \varpi^{l-3^l \left\lfloor \frac{m-r}{6} \right\rfloor - 1} + \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \left\lfloor \frac{m}{2} \right\rfloor - k - 1 \right) 2^l \varpi^{l-3^k} = 0.
\]

\[\sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \left\lfloor \frac{m}{2} \right\rfloor - k - 1 \right) 2^l \varpi^{l-3^k} + A_l \varpi = F_{m+1}.
\]

(2.19)

However, \( \left\lfloor \frac{m}{2} \right\rfloor = \left\lfloor \frac{m-r}{k} \right\rfloor \), and since \( \left( \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m-r}{6} \right\rfloor - 1 \right) \cdot 2^l \varpi^{l-3^l \left\lfloor \frac{m-r}{6} \right\rfloor - 1} = 1 \), we may now write (2.10) as

\[
1 + \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \left\lfloor \frac{m}{2} \right\rfloor - k - 1 \right) 2^l \varpi^{l-3^k} + A_l \varpi = F_{m+1}.
\]

(2.20)

We may also combine the two summations to produce

\[
1 + \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \left\lfloor \frac{m}{2} \right\rfloor - k - 1 \right) 2^l \varpi^{l-3^k} + A_l \varpi = F_{m+1}.
\]

(2.21)

Again, we may combine the two combinations as follows

\[
1 + \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \left\lfloor \frac{m}{2} \right\rfloor - k - 1 \right) 2^l \varpi^{l-3^k} + A_l \varpi = F_{m+1}.
\]

(2.22)

Finally, we can add the one into the summation because

\[
\left( \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m-r}{6} \right\rfloor \right) 2^l \varpi^{l-3^l \left\lfloor \frac{m-r}{6} \right\rfloor - 1} = 1
\]

(2.23)
when \( m \equiv 3 \mod 6 \). Therefore,

\[
\sum_{k=0}^{[m/3]} \left( \left( \frac{3}{2k+1} \right) - k \right) \cdot 2^{[m/3]-3k-1} + A_{[m/3]} = F_{m+1}. \tag{2.24}
\]

Now, we must rewrite the summation as

\[
\sum_{k=1}^{[m/3]+1} \left( \left( \frac{m}{2k+1} \right) - k \right) \cdot 2^{[m/3]-3k+3} + A_{[m/3]} = F_{m+1}. \tag{2.25}
\]

However, \([m/6] = [m/2] + 1\) and \([m+1/2] = [m/2] + 1\), so

\[
\sum_{k=1}^{[m/3]} \left( \left( \frac{m+1}{2k+1} \right) - k \right) \cdot 2^{[m+1/2]-3k+1} + A_{[m/3]} = F_{m+1}. \tag{2.26}
\]

Therefore,

\[
R_{[m/3]} + A_{[m/3]} = F_{m+1} \tag{2.27}
\]

and our theorem is proven for \( m \equiv 3 \mod 6 \).

Case 2(b): \( m \equiv 1, 5 \mod 6 \)

Then, \([m/6] = [m/3] \), so, by (2.18),

\[
\sum_{k=0}^{[m/3]} \left[ \left( \frac{m}{2k} \right) + \left( \frac{m}{2k+1} \right) \right] \cdot 2^{[m/3]-3k-1} + A_{[m/3]} = F_{m+1}. \tag{2.28}
\]

The two terms of this summation can be combined into

\[
\sum_{k=0}^{[m/3]} \left( \left( \frac{m}{2k+1} \right) - k \right) \cdot 2^{[m/3]-3k-1} + A_{[m/3]} = F_{m+1}. \tag{2.29}
\]

This equation can be easily transformed into (2.25). Therefore, our equation holds when \( m \equiv 1 \mod 6 \) and when \( m \equiv 5 \mod 6 \). Thus it is true for \( m \equiv 1 \mod 2 \), and this completes the proof.
REFERENCES


AMS Classification Numbers: 11R39, 11A99