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FIBONOMIAL COEFFICIENTS AT MOST ONE AWAY FROM FIBONACCI NUMBERS

Abstract. Let $F_n$ be the $n$th Fibonacci number. For $1 \leq k \leq m$, let
\[
\left[ \begin{array}{c} m \\ k \end{array} \right]_F = \frac{F_m F_{m-1} \cdots F_{m-k+1}}{F_1 \cdots F_k}
\]
be the corresponding Fibonomial coefficient. It is known that $\left[ \begin{array}{c} m \\ k \end{array} \right]_F$ is a Fibonacci number if and only if either $k = 1$ or $m \in \{k, k + 1\}$. In this note, we find all solutions of the Diophantine equation
\[
\left[ \begin{array}{c} m \\ k \end{array} \right]_F \pm 1 = F_n.
\]

1. Introduction

Let $(F_n)_{n \geq 0}$ be the Fibonacci sequence given by $F_0 = 0$ and $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$, for $n \geq 1$. The first few terms of this sequence are $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$.

The problem of the existence of infinitely many prime numbers in the Fibonacci sequence remains open, however several results on the prime factors of a Fibonacci number are known. For instance, a primitive divisor $p$ of $F_n$ is a prime factor of $F_n$ which does not divide $\prod_{j=1}^{n-1} F_j$. It is known that a primitive divisor $p$ of $F_n$ exists whenever $n \geq 13$. The above statement is usually referred to the Primitive Divisor Theorem (see [1] for the most general version).

The Fibonomial coefficient $\left[ \begin{array}{c} m \\ k \end{array} \right]_F$ is defined, for $1 \leq k \leq m$, by replacing each integer appearing in the numerator and denominator of $\left( \begin{array}{c} m \\ k \end{array} \right) = \frac{m(m-1) \cdots (m-k+1)}{k(k-1) \cdots 1}$ with its respective Fibonacci number. That is
\[
\left[ \begin{array}{c} m \\ k \end{array} \right]_F = \frac{F_m F_{m-1} \cdots F_{m-k+1}}{F_1 \cdots F_k}.
\]

It is surprising that this quantity will always take integer values. This

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can be shown by an induction argument and the recursion formula
\[
\begin{bmatrix} m \\ k \end{bmatrix}_F = F_{k+1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_F + F_{m-k-1} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_F,
\]
which is a consequence of the formula \( F_m = F_{k+1}F_{m-k} + F_kF_{m-k-1} \).

As an application of the Primitive Divisor Theorem, it is immediate that if \( \begin{bmatrix} m \\ k \end{bmatrix}_F = F_n \), then \( \max\{m, n\} < 13 \). Hence, assuming that \( m-1 > k > 1 \) a quick computation reveals that there are no solutions for the previous Diophantine equation in that obtained range.

In this paper, we find all Fibonomial coefficients at most one away from a Fibonacci number. Our result is the following

**Theorem 1.** The solutions of the Diophantine equation
\[
\begin{bmatrix} m \\ k \end{bmatrix}_F \pm 1 = F_n
\]
with \( m-1 \geq k > 1 \), are \( (m, k, n) = (3, 2, 4) \) and \( (m, k, n) = (3, 2, 1), (3, 2, 2), (4, 2, 5), (4, 3, 3) \) according to whether the sign is + or −, respectively.

2. The proof of Theorem

2.1. Auxiliary results. The sequence of the Lucas numbers is defined by \( L_{n+1} = L_n + L_{n-1} \), with \( L_0 = 2 \) and \( L_1 = 1 \). Let us state some interesting and helpful facts which will be essential ingredients in what follows.

Let \( \alpha = (1 + \sqrt{5})/2 \) and \( \beta = (1 - \sqrt{5})/2 \). For all \( n \geq 1 \), we have
\begin{enumerate}
  \item [(L1)] \( F_{2n} = F_nL_n \);
  \item [(L2)] (Binet’s formulae) \( F_n = \alpha^n - \beta^n \) and \( L_n = \alpha^n + \beta^n \);
  \item [(L3)] \( \alpha^{n-2} \leq F_n \leq \alpha^{n-1} \).
\end{enumerate}

We may note that the Fibonacci and Lucas sequences can be extrapolated backwards using \( F_n = F_{n+2} - F_{n+1} \) and \( L_n = L_{n+2} - L_{n+1} \). Thus, for example, \( F_{-1} = 1, F_{-2} = -1 \), and so on. Furthermore, Binet’s formulae (L2) remain valid for Fibonacci and Lucas numbers with negative indices, and they allow us to show that

**Lemma 1.** For any integers \( a, b \), we have
\[
F_aL_b = F_{a+b} + (-1)^bF_{a-b}.
\]

**Proof.** Since \( \alpha = (-\beta)^{-1} \) and thus \( \beta = (-\alpha)^{-1} \), we have
\[
F_aL_b = \frac{\alpha^a - \beta^a}{\alpha - \beta} (\alpha^b + \beta^b) = F_{a+b} + \frac{\alpha^a\beta^b - \beta^a\alpha^b}{\alpha - \beta} = F_{a+b} + (-1)^bF_{a-b}. \]

As a consequence of the previous lemma, a straight calculation gives a different factorization for \( F_n \pm 1 \) depending on the class of \( n \) modulo 4:
2.2. The proof. For \(1 \leq n \leq 8\), a quick computation reveals that the only solutions are those in the statement of the theorem. So, let us assume that \(n > 8\).

The equation (1) can be rewritten as \([m/k]_F = F_n + 1\). By the relations in (2), we have eight possibilities for this Diophantine equation (again depending on the class of \(n\) modulo 4): For the (+) case

\[
\begin{align*}
[m]_F &= F_{2\ell+1}L_{2\ell-1} ; \\
[k]_F &= F_{2\ell}L_{2\ell+1} \\
[m]_F &= F_{2\ell+2}L_{2\ell} ; \\
[k]_F &= F_{2\ell+1}L_{2\ell+2}
\end{align*}
\]

and the (-) case

\[
\begin{align*}
[m]_F &= F_{2\ell-1}L_{2\ell+1} ; \\
[k]_F &= F_{2\ell+1}L_{2\ell} \\
[m]_F &= F_{2\ell+2}L_{2\ell} ; \\
[k]_F &= F_{2\ell+1}L_{2\ell+2}.
\end{align*}
\]

We shall work only on the first equation in the left-hand side of (3) (the other ones can be handled in much the same way). Let us assume that \(m \geq \max\{14, k+1\}\). Thus, we have

\[
F_m \cdots F_{m-k+1} = F_{2\ell+1}L_{2\ell-1}F_1 \cdots F_k.
\]

Since \(L_{2\ell-1} = F_{4\ell-2}/F_{2\ell-1}\) (see (L1)), we get

\[
F_m \cdots F_{m-k+1}F_{2\ell-1} = F_{2\ell+1}F_{4\ell-2}F_1 \cdots F_k.
\]

However \(4\ell - 2 > 2\ell + 1\), since \(\ell = \lfloor n/4 \rfloor > 2\), and then the Primitive Divisor Theorem yields \(m = 4\ell - 2\). Thus, the identity (6) becomes

\[
F_{m-1} \cdots F_{m-k+1}F_{2\ell-1} = F_{2\ell+1}F_1 \cdots F_k.
\]

Since \(m-1 \geq 13\), we can use again the Primitive Divisor Theorem to get \(m - 1 = \max\{2\ell + 1, k\}\). However \(m - 1 = 4\ell - 3 > 2\ell + 1\) and therefore \(m - 1 = k\) and (7) becomes \(F_{2\ell-1} = F_{2\ell+1}\) which is an absurd.

So, we only need to consider the range \(2 \leq k \leq 10\) and \(k + 2 \leq m \leq 12\). By using (L3) we get

\[
\left(\frac{F_m}{F_1}\right) < \alpha^{m-1} \text{ and } \left(\frac{F_{m-t}}{F_{t+1}}\right) < \alpha^{m-2t}, \text{ for } 1 \leq t \leq k - 1.
\]
Therefore, we have

\[
\binom{m}{k}_F \leq \alpha^{m-1+m-2+\cdots+m-2(k-1)} = \alpha^{m-1+(m-k)(k-1)} \\
\leq \alpha^{43} < 9.7 \times 10^8 - 1.
\]

Thus \( F_n \leq \binom{m}{k}_F + 1 < 9.7 \times 10^8 \) and then \( n < 40 \).

We have written a simple program in Mathematica to see that in the obtained range \( 2 \leq k \leq m - 1 \leq 10 \) and \( 9 \leq n \leq 39 \) there is no further solution. Thus we have our desired result. Explicitly, \( \binom{3}{2}_F + 1 = F_4, \) \( \binom{3}{2}_F - 1 = F_1 = F_2, \) \( \binom{4}{2}_F - 1 = F_5 \) and \( \binom{4}{3}_F - 1 = F_3. \) \( \blacksquare \)

We finish by pointing out that the same method can be applied to provide all solutions of \( \binom{m}{k}_F + 1 = F_n^2. \) In fact, if \( n > 2 \), we have \( \binom{4}{2}_F + 1 = F_3^2, \) \( \binom{6}{5}_F + 1 = F_4^2, \) as the only such solutions. The useful fact here is that \( F_n^2 - 1 \) can be factored as \((F_n - 1)(F_n + 1)\) and thus we can use the formulas in (2).

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**References**