Polynomials whose coefficients are $k$-Fibonacci numbers

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Abstract

Let $\{a_n\}_{n \geq 0}$ denote the linear recursive sequence of order $k$ $(k \geq 2)$ defined by the initial values $a_0 = a_1 = \cdots = a_{k-2} = 0$ and $a_{k-1} = 1$ and the recursion $a_n = a_{n-1} + a_{n-2} + \cdots + a_{n-k}$ if $n \geq k$. The $a_n$ are often called $k$-Fibonacci numbers and reduce to the usual Fibonacci numbers when $k = 2$. Let $P_{n,k}(x) = a_{k-1}x^n + a_kx^{n-1} + \cdots + a_{n+k-2}x + a_{n+k-1}$, which we will refer to as a $k$-Fibonacci coefficient polynomial. In this paper, we show for all $k$ that the polynomial $P_{n,k}(x)$ has no real zeros if $n$ is even and exactly one real zero if $n$ is odd. This generalizes the known result for the $k = 2$ and $k = 3$ cases corresponding to Fibonacci and Tribonacci coefficient polynomials, respectively. It also improves upon a previous upper bound of approximately $k$ for the number of real zeros of $P_{n,k}(x)$. Finally, we show for all $k$ that the sequence of real zeros of the polynomials $P_{n,k}(x)$ when $n$ is odd converges to the opposite of the positive zero of the characteristic polynomial associated with the sequence $a_n$. This generalizes a previous result for the case $k = 2$.

Keywords: $k$-Fibonacci sequence, zeros of polynomials, linear recurrences

MSC: 11C08, 13B25, 11B39, 05A20

1. Introduction

Let the recursive sequence $\{a_n\}_{n \geq 0}$ of order $k$ $(k \geq 2)$ be defined by the initial values $a_0 = a_1 = \cdots = a_{k-2} = 0$ and $a_{k-1} = 1$ and the linear recursion

$$a_n = a_{n-1} + a_{n-2} + \cdots + a_{n-k}, \quad n \geq k.$$  

(1.1)
The numbers $a_n$ are sometimes referred to as $k$-Fibonacci numbers (or generalized Fibonacci numbers) and reduce to the usual Fibonacci numbers $F_n$ when $k = 2$ and to the Tribonacci numbers $T_n$ when $k = 3$. (See, e.g., A000045 and A000073 in [11].) The sequence $a_n$ was first considered by Knuth [3] and has been a topic of study in enumerative combinatorics. See, for example, [1, Chapter 3] or [9] for interpretations of $a_n$ in terms of linear tilings or $k$-filtering linear partitions, respectively, and see [10] for a $q$-generalization of $a_n$.

Garth, Mills, and Mitchell [2] introduced the definition of the Fibonacci coefficient polynomials $p_n(x) = F_1x^n + F_2x^{n-1} + \cdots + F_nx + F_{n+1}$ and among other things determined the number of real zeros of $p_n(x)$. In particular, they showed that $p_n(x)$ has no real zeros if $n$ is even and exactly one real zero if $n$ is odd. Later, this result was extended by Mátyás [5, 6] to more general second order recurrences. The same result also holds for the Tribonacci coefficient polynomials $q_n(x) = T_2x^n + T_3x^{n-1} + \cdots + T_{n+1}x + T_{n+2}$, which was shown by Mátyás and Szalay [8].

If $k \geq 2$ and $n \geq 1$, then define the polynomial $P_{n,k}(x)$ by

$$P_{n,k}(x) = a_{k-1}x^n + a_kx^{n-1} + \cdots + a_{n+k-2}x + a_{n+k-1}. \quad (1.2)$$

We will refer to $P_{n,k}(x)$ as a $k$-Fibonacci coefficient polynomial. Note that when $k = 2$ and $k = 3$, the $P_{n,k}(x)$ reduce to the Fibonacci and Tribonacci coefficient polynomials $p_n(x)$ and $q_n(x)$ mentioned above. In [7], the following result was obtained concerning the number of real zeros of $P_{n,k}(x)$ as a corollary to a more general result involving sequences defined by linear recurrences with non-negative integral weights.

**Theorem 1.1.** Let $h$ denote the number of real zeros of the polynomial $P_{n,k}(x)$ defined by (1.2) above. Then we have

(i) $h = k - 2 - 2j$ for some $j = 0, 1, \ldots, (k - 2)/2$, if $k$ and $n$ are even,

(ii) $h = k - 1 - 2j$ for some $j = 0, 1, \ldots, (k - 2)/2$, if $k$ is even and $n$ is odd,

(iii) $h = k - 1 - 2j$ for some $j = 0, 1, \ldots, (k - 1)/2$, if $k$ is odd and $n$ is even,

(iv) $h = k - 2j$ for some $j = 0, 1, \ldots, (k - 1)/2$, if $k$ and $n$ are odd.

For example, Theorem 1.1 states when $k = 3$ that the number of real zeros of the polynomial $P_{n,3}(x)$ is either 0 or 2 if $n$ is even or 1 or 3 if $n$ is odd. As already mentioned, it was shown in [8] that $P_{n,3}(x)$ possesses no real zeros when $n$ is even and exactly one real zero when $n$ is odd.

In this paper, we show that the polynomial $P_{n,k}(x)$ possesses the smallest possible number of real zeros in every case and prove the following result.

**Theorem 1.2.** Let $k \geq 2$ be a positive integer and $P_{n,k}(x)$ be defined by (1.2) above. Then we have the following:

(i) If $n$ is even, then $P_{n,k}(x)$ has no real zeros.

(ii) If $n$ is odd, then $P_{n,k}(x)$ has exactly one real zero.

We prove Theorem 1.2 as a series of lemmas in the third and fourth sections below, and have considered separately the cases for even and odd $k$. Combining
Theorems 3.5 and 4.5 below gives Theorem 1.2. The crucial steps in our proofs of Theorems 3.5 and 4.5 are Lemmas 3.2 and 4.2, respectively, where we make a comparison of consecutive derivatives of a polynomial evaluated at the point $x = 1$. This allows us to show that there is exactly one zero when $x \leq -1$ in the case when $n$ is odd. We remark that our proof, when specialized to the cases $k = 2$ and $k = 3$, provides an alternative proof to the ones given in [2] and [8], respectively, in these cases. In the final section, we show for all $k$ that the sequence of real zeros of the polynomials $P_{n,k}(x)$ for $n$ odd converges to $-\lambda$, where $\lambda$ is the positive zero of the characteristic polynomial associated with the sequence $a_n$ (see Theorem 5.5 below). This generalizes the result for the $k = 2$ case, which was shown in [2].

2. Preliminaries

We seek to determine the number of real zeros of the polynomial $P_{n,k}(x)$. By the following lemma, we may restrict our attention to the case when $x \leq -1$.

**Lemma 2.1.** If $k \geq 2$ and $n \geq 1$, then the polynomial $P_{n,k}(x)$ has no zeros on the interval $(-1, \infty)$.

**Proof.** Clearly, the equation $P_{n,k}(x) = 0$ has no roots if $x \geq 0$ since it has positive coefficients. Suppose $-1 < x < 0$. If $n$ is odd, then

$$a_{k+2j-1}x^{n-2j} + a_{k+2j}x^{n-2j-1} > 0, \quad 0 \leq j \leq (n-1)/2,$$

since $x^{n-2j-1} > -x^{n-2j} > 0$ if $-1 < x < 0$ and $a_{k+2j} \geq a_{k+2j-1} > 0$. This implies

$$P_{n,k}(x) = \sum_{j=0}^{n-1} (a_{k+2j-1}x^{n-2j} + a_{k+2j}x^{n-2j-1}) > 0.$$ 

Similarly, if $n$ is even, then

$$P_{n,k}(x) = a_{k-1}x^n + \sum_{j=0}^{n-2} (a_{k+2j}x^{n-2j-1} + a_{k+2j+1}x^{n-2j-2}) > 0.$$

So we seek the zeros of $P_{n,k}(x)$ where $x \leq -1$, equivalently, the zeros of $P_{n,k}(-x)$ where $x \geq 1$. For this, it is more convenient to consider the zeros of $g_{n,k}(x)$ given by

$$g_{n,k}(x) := c_k(-x)P_{n,k}(-x), \tag{2.1}$$

see [7], where

$$c_k(x) := x^k - x^{k-1} - x^{k-2} - \ldots - x - 1 \tag{2.2}$$

denotes the characteristic polynomial associated with the sequence $a_n$. 

By [7, Lemma 2.1], we have
\[
g_{n,k}(x) = (-x)^{n+k} - a_{n+k}(-x)^{k-1} - (a_{n+1} + a_{n+2} + \cdots + a_{n+k-1})(-x)^{k-2}
- \cdots - (a_{n+k-2} + a_{n+k-1})(-x) - a_{n+k-1}
\]
\[
= (-x)^{n+k} - a_{n+k}(-x)^{k-1} - \sum_{r=1}^{k-1} \left( \sum_{j=r}^{k-1} a_{n+j} \right) (-x)^{k-r-1}.
\] (2.3)

We now wish to study the zeros of \( g_{n,k}(x) \), where \( x \geq 1 \). In the subsequent two sections, we undertake such a study, considering separately the even and odd cases for \( k \).

3. The case \( k \) even

Throughout this section, \( k \) will denote a positive even integer. We consider the zeros of the polynomial \( g_{n,k}(x) \) where \( x \geq 1 \), and for this, it is more convenient to consider the zeros of the polynomial
\[
f_{n,k}(x) := (1 + x)g_{n,k}(x),
\] (3.1)
where \( x \geq 1 \).

First suppose \( n \) is odd. Note that when \( k \) is even and \( n \) is odd, we have
\[
f_{n,k}(x) = -x^{n+k}(1 + x) + a_{n+k}x^k + a_{n}x^{k-1} - a_{n+1}x^{k-2} + a_{n+2}x^{k-3}
- \cdots + a_{n+k-2}x - a_{n+k-1}
\]
\[
= -x^{n+k}(1 + x) + a_{n+k}x^k + \sum_{r=0}^{k-1} \left( (-1)^r a_{n+r}x^{k-r-1} \right),
\] (3.2)

by (2.3) and the recurrence for \( a_n \). In the lemmas below, we ascertain the number of the zeros of the polynomial \( f_{n,k}(x) \) when \( x \geq 1 \). We will need the following combinatorial inequality.

**Lemma 3.1.** If \( k \geq 4 \) is even and \( n \geq 1 \), then
\[
a_{n+k+1} \geq \sum_{r=0}^{k-1} 2^{k-r} a_{n+2r+1}.
\] (3.3)

**Proof.** We have
\[
a_{n+k+1} = a_{n+k} + \sum_{r=1}^{k-1} a_{n+r} \geq 2 \sum_{r=1}^{k-1} a_{n+r}
= 2a_{n+k-1} + 2a_{n+k-2} + 2 \sum_{r=1}^{k-3} a_{n+r} \geq 2a_{n+k-1} + 4 \sum_{r=1}^{k-3} a_{n+r}
\]
\[= 2a_{n+k-1} + 4a_{n+k-3} + 4a_{n+k-4} + 4 \sum_{r=1}^{k-5} a_{n+r}\]
\[\geq 2a_{n+k-1} + 4a_{n+k-3} + 8 \sum_{r=1}^{k-5} a_{n+r}\]
\[= \cdots \geq \sum_{r=i-1}^{k-1} 2^{i-r} a_{n+2r+1} + 2^{k-i+1} \sum_{r=1}^{2i-1} a_{n+r}\]
\[\geq \sum_{r=i-1}^{k-1} 2^{i-r} a_{n+2r+1} + 2^{k-i+2} \sum_{r=1}^{2i-3} a_{n+r}\]
\[= \cdots \geq \sum_{r=0}^{k-1} 2^{r} a_{n+2r+1},\]

which gives (3.3). \square

The following lemma will allow us to determine the number of zeros of \(f_{n,k}(x)\) for \(x \geq 1\).

**Lemma 3.2.** Suppose \(k \geq 4\) is even and \(n\) is odd. If \(1 \leq i \leq k-1\), then \(f^{(i)}_{n,k}(1) < 0\) implies \(f^{(i+1)}_{n,k}(1) < 0\), where \(f^{(i)}_{n,k}\) denotes the \(i\)-th derivative of \(f_{n,k}\).

**Proof.** Let \(f = f_{n,k}\) and \(i = k - j\) for some \(1 \leq j \leq k - 1\). Then the assumption \(f^{(k-j)}(1) < 0\) is equivalent to

\[
k! \frac{1}{j!} a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} < \prod_{s=1}^{k-j} (n+j+s) + \prod_{s=1}^{k-j} (n+j+s+1). \quad (3.4)
\]

We will show that inequality (3.4) implies

\[
k! \frac{1}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} < \prod_{s=0}^{k-j} (n+j+s) + \prod_{s=0}^{k-j} (n+j+s+1). \quad (3.5)
\]

Observe first that the left-hand side of both inequalities (3.4) and (3.5) is positive as

\[
k! \frac{1}{j!} a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r}\]
\[= \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} \left( \frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} > 0,
\]
since \( a_{n+k} = \sum_{r=0}^{k-1} a_{n+r} \) and \( \frac{k!}{j!} > \frac{(k-r-1)!}{(j-r-1)!} \). Note also that
\[
\prod_{s=0}^{k-j}(n+j+s) + \prod_{s=1}^{k-j}(n+j+s+1) > n+j,
\]
so to show (3.5), it suffices to show
\[
\frac{k!}{(j-1)!}a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} \\
\leq (n+j) \left( \frac{k!}{j!}a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} \right). \tag{3.6}
\]
For (3.6), it is enough to show
\[
\frac{k!}{(j-1)!}a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} \\
\leq (j+1) \left( \frac{k!}{j!}a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} \right). \tag{3.7}
\]
Starting with the left-hand side of (3.7), we have
\[
\frac{k!}{(j-1)!}a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} \\
= \frac{k!}{(j-1)!} \sum_{r=j-1}^{k-1} a_{n+r} + \sum_{r=0}^{j-2} \left( \frac{k!}{(j-1)!} + (-1)^r \frac{(k-r-1)!}{(j-r-2)!} \right) a_{n+r} \\
= \frac{k!}{(j-1)!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} \left( \frac{k!}{j!} + (-1)^r (j-r-1) \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\
= \frac{k!}{(j-1)!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} j \left( \frac{k!}{j!} + (-1)^r (k-r-1)! \right) a_{n+r} \\
+ \sum_{r=0}^{j-1} (-1)^{r+1}(r+1) \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} \\
\leq \frac{k!}{(j-1)!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} j \left( \frac{k!}{j!} + (-1)^r (k-r-1)! \right) a_{n+r} \\
+ \sum_{r=0}^{\lfloor \frac{j-2}{2} \rfloor} (2r+2) \frac{(k-2r-2)!}{(j-2r-2)!} a_{n+2r+1}
\[
= (j + 1) \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} j \cdot \left( \frac{k!}{j!} + (-1)^r \frac{(k - r - 1)!}{(j - r - 1)!} \right) a_{n+r}
- \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{\lfloor \frac{j-2}{2} \rfloor} (2r + 2) \frac{(k - 2r - 2)!}{(j - 2r - 2)!} a_{n+2r+1}.
\]

Below we show
\[
\sum_{r=0}^{\lfloor \frac{j-2}{2} \rfloor} (2r + 2) \frac{(k - 2r - 2)!}{(j - 2r - 2)!} a_{n+2r+1} \leq \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r}.
\]  

(3.8)

Then from (3.8), we have
\[
\frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k - r - 1)!}{(j - r - 2)!} a_{n+r}
\leq (j + 1) \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} j \cdot \left( \frac{k!}{j!} + (-1)^r \frac{(k - r - 1)!}{(j - r - 1)!} \right) a_{n+r}
\leq (j + 1) \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} (j + 1) \left( \frac{k!}{j!} + (-1)^r \frac{(k - r - 1)!}{(j - r - 1)!} \right) a_{n+r}
\]
= (j + 1) \frac{k!}{j!} a_{n+k} + (j + 1) \sum_{r=0}^{j-1} (-1)^r \frac{(k - r - 1)!}{(j - r - 1)!} a_{n+r},
\]

which gives (3.7), as desired.

To finish the proof, we need to show (3.8). We may assume \( j \geq 2 \), since the inequality is trivial when \( j = 1 \). By Lemma 3.1 and the fact that \( 2^m \geq 2m \) if \( m \geq 1 \), we have
\[
\sum_{r=j}^{k-1} a_{n+r} \geq a_{n+k-1}
\geq \sum_{r=0}^{\frac{k-2}{2}} 2^{\frac{k-2}{2}} a_{n+2r+1} \geq \sum_{r=0}^{\lfloor \frac{j-2}{2} \rfloor} (k - 2r - 2) a_{n+2r+1} \geq \sum_{r=0}^{\lfloor \frac{j-2}{2} \rfloor} (k - 2r - 2) a_{n+2r+1},
\]
the last inequality holding since \( j \leq k - 1 \), with \( k \) even. So to show (3.8), it is enough to show
\[
(k - 2r - 2) \frac{k!}{j!} \geq (2r + 2) \frac{(k - 2r - 2)!}{(j - 2r - 2)!}, \quad 0 \leq r \leq \lfloor (j - 2)/2 \rfloor,
\]
(3.9)

where \( 2 \leq j \leq k - 1 \). Since the ratio \( \frac{k!}{j!} / (k - 2r - 2)!/(j - 2r - 2)! \) is decreasing in \( j \) for fixed \( k \) and \( r \), one needs to verify (3.9) only when \( j = k - 1 \), and it holds in this case since \( 2r + 2 \leq j < k \). This completes the proof. \( \square \)
We now determine the number of zeros of \( f_{n,k}(x) \) on the interval \([1, \infty)\).

**Lemma 3.3.** Suppose \( k \geq 4 \) is even and \( n \) is odd. Then the polynomial \( f_{n,k}(x) \) has exactly one zero on the interval \([1, \infty)\). Furthermore, this zero is simple.

**Proof.** Let \( f = f_{n,k} \), where we first assume \( n \geq 3 \). Then

\[
f(1) = -2 + a_{n+k} + \sum_{r=0}^{k-1} (-1)^r a_{n+r} = -2 + 2 \sum_{r=0}^{k-1} a_{n+2r} > 0,
\]

since \( a_{n+k-2} \geq a_{k+1} = 2 \). Let \( \ell \) be the smallest positive integer \( i \) such that \( f^{(i)}(1) < 0 \); note that \( 1 \leq \ell \leq k+1 \) since \( f^{(k+1)}(1) < 0 \). Then

\[
f^{(\ell+1)}(1), f^{(\ell+2)}(1), \ldots, f^{(k+1)}(1)
\]

are all negative, by Lemma 3.2. Since \( f^{(k+1)}(x) < 0 \) for all \( x \geq 1 \), it follows that \( f^{(\ell)}(x) < 0 \) for all \( x \geq 1 \). To see this, note that if \( \ell \leq k \), then \( f^{(k)}(1) < 0 \) implies \( f^{(k)}(x) < 0 \) for all \( x \geq 1 \), which in turn implies each of \( f^{(k)}(x), f^{(k-1)}(x), \ldots, f^{(1)}(x) \) is negative for all \( x \geq 1 \).

If \( \ell \geq 2 \), then \( f^{(\ell-1)}(1) \geq 0 \) and \( f^{(\ell)}(x) < 0 \) for all \( x \geq 1 \). Since \( f^{(\ell-1)}(1) \geq 0 \) and \( \lim_{x \to \infty} f^{(\ell-1)}(x) = -\infty \), we have either (i) \( f^{(\ell-1)}(1) = 0 \) and \( f^{(\ell-1)}(x) \) has no zeros on the interval \((1, \infty)\) or (ii) \( f^{(\ell-1)}(1) > 0 \) and \( f^{(\ell-1)}(x) \) has exactly one zero on the interval \((1, \infty)\). If \( \ell \geq 3 \), then \( f^{(\ell-2)}(x) \) would also have at most one zero on \((1, \infty)\) since \( f^{(\ell-2)}(1) \geq 0 \), with \( f^{(\ell-2)}(x) \) initially increasing up to some point \( s \geq 1 \) before it decreases monotonically to \(-\infty\) (where \( s = 1 \) if \( f^{(\ell-1)}(1) = 0 \) and \( s > 1 \) if \( f^{(\ell-1)}(1) > 0 \)). Note that each derivative of \( f \) of order \( \ell \) or less is eventually negative. Continuing in this fashion, we then see that if \( \ell \geq 2 \), then \( f'(x) \) has at most one zero on the interval \((1, \infty)\), with \( f'(1) \geq 0 \) and \( f'(x) \) eventually negative.

If \( \ell = 1 \), then \( f'(x) < 0 \) for all \( x \geq 1 \). Since \( f(1) > 0 \) and \( \lim_{x \to \infty} f(x) = -\infty \), it follows in either case that \( f \) has exactly one zero on the interval \([1, \infty)\), which finishes the case when \( n \geq 3 \).

If \( n = 1 \), then \( f_{1,k}(x) = -x^{k+1} + x + 2x^k + x - 1 \) so that \( f_{1,k}(1) = 0 \), with

\[
f'_{1,k}(x) = -(k+1)x^k - (k+2)x^{k+1} + 2kx^{k-1} + 1
\]

\[
\leq -(k+1)x^{k-1} - (k+2)x^{k+1} + 2kx^{k-1} + 1 = -3x^{k-1} + 1 < 0
\]

for \( x \geq 1 \). Thus, there is exactly one zero on the interval \([1, \infty)\) in this case as well.

Let \( t \) be the root of the equation \( f_{n,k}(x) = 0 \) on \([1, \infty)\). We now show that \( t \) has multiplicity one. First assume \( n \geq 3 \). Then \( t > 1 \). We consider cases depending on the value of \( f'(1) \). If \( f'(1) < 0 \), then \( f'(x) < 0 \) for all \( x \geq 1 \) and thus \( f'(t) < 0 \) is non-zero, implying \( t \) is a simple root. If \( f'(1) > 0 \), then \( f'(t) < 0 \) due to \( f(1) > 0 \) and the fact that \( f'(x) \) would then have one root \( v \) on \((1, \infty)\) with \( v < t \). Finally, if \( f'(1) = 0 \), then the proof of Lemma 3.2 above shows that \( f''(1) < 0 \) and thus \( f''(x) < 0 \) for all \( x \geq 1 \), which implies \( f'(t) < 0 \). If \( n = 1 \), then \( t = 1 \) and \( f'_{1,k}(1) < 0 \). Thus, \( t \) is a simple root in all cases, as desired, which completes the proof. \( \square \)
We next consider the case when $n$ is even.

**Lemma 3.4.** Suppose $k \geq 4$ and $n$ are even. Then $f_{n,k}(x)$ has no zeros on $[1, \infty)$.

**Proof.** In this case, we have

$$f_{n,k}(x) = x^{n+k}(1+x) + a_{n+k}x^k + \sum_{r=0}^{k-1} (-1)^r a_{n+r}x^{k-r-1},$$

by (2.3) and (3.1). If $x \geq 1$, then $f_{n,k}(x) > 0$ since $a_{n+k} = \sum_{r=0}^{k-1} a_{n+r}$ and $x^k \geq x^{k-r-1}$ for $0 \leq r \leq k-1$. \qed

The main result of this section now follows rather quickly.

**Theorem 3.5.** (i) If $k$ is even and $n$ is odd, then the polynomial $P_{n,k}(x)$ has one real zero $q$, and it is simple with $q \leq -1$.

(ii) If $k$ and $n$ are even, then the polynomial $P_{n,k}(x)$ has no real zeros.

**Proof.** Note first that the preceding lemmas, where we assumed $k \geq 4$ is even, may be adjusted slightly and are also seen to hold in the case $k = 2$. First suppose $n$ is odd. By Lemma 3.3, the polynomial $f_{n,k}(x)$, and hence $g_{n,k}(x)$, has one zero for $x \geq 1$, and it is simple. By [7, Lemma 2.3], the characteristic polynomial $c_k(x) = x^k - x^{k-1} - x^{k-2} - \cdots - 1$ has one negative real zero when $k$ is even, and it is seen to lie in the interval $(-1, 0)$. Since $g_{n,k}(x) = c_k(-x)P_{n,k}(-x)$, it follows that $P_{n,k}(-x)$ has one zero for $x \geq 1$. Thus, $P_{n,k}(x)$ has one zero for $x \leq -1$, and it is simple. By Lemma 2.1, the polynomial $P_{n,k}(x)$ has exactly one real zero.

If $n$ is even, then the polynomial $f_{n,k}(x)$, and hence $g_{n,k}(x)$, has no zeros for $x \geq 1$, by Lemma 3.4. By (2.1), it follows that $P_{n,k}(x)$ has no zeros for $x \leq -1$. By Lemma 2.1, $P_{n,k}(x)$ has no real zeros. \qed

**4. The case $k$ odd**

Throughout this section, $k \geq 3$ will denote a positive odd integer. We study the zeros of the polynomial $g_{n,k}(x)$ when $x \geq 1$, and for this, it is again more convenient to consider the polynomial $f_{n,k}(x) := (1+x)g_{n,k}(x)$. First suppose $n$ is odd. When $k$ and $n$ are both odd, note that

$$f_{n,k}(x) = x^{n+k}(1+x) - a_{n+k}x^k - a_n x^{k-1} + a_{n+1}x^{k-2} - \cdots + a_{n+k-2}x - a_{n+k-1}$$

$$= x^{n+k}(1+x) - a_{n+k}x^k + \sum_{r=0}^{k-1} (-1)^r a_{n+r}x^{k-r-1},$$

by (2.3) and the recurrence for $a_n$. In the lemmas below, we ascertain the number of zeros of the polynomial $f_{n,k}(x)$ when $x \geq 1$. We start with the following inequality.
Lemma 4.1. Suppose \( k \geq 3 \) is odd and \( n \geq 1 \). If \( 1 \leq j \leq k - 1 \), then

\[
\frac{k!}{j!} a_{n+k-1} \geq \sum_{r=1}^{\left\lfloor \frac{j}{2} \right\rfloor} 2^r \frac{(k-2r)!}{(j-2r)!} a_{n+2r-1}.
\] (4.1)

Proof. First note that we have the inequality

\[
a_{n+k-1} \geq \sum_{r=1}^{\frac{k-3}{2}} 2^{\frac{k-1}{2} - r} a_{n+2r}.
\] (4.2)

To show (4.2), proceed as in the proof of Lemma 3.1 above and write

\[
a_{n+k-1} \geq a_{n+k-2} + \sum_{r=2}^{k-3} a_{n+r}
\]
\[
\geq 2a_{n+k-3} + 2 \sum_{r=2}^{k-4} a_{n+r}
\]
\[
= 2a_{n+k-3} + 2a_{n+k-4} + 2 \sum_{r=2}^{k-5} a_{n+r}
\]
\[
\geq 2a_{2n+k-3} + 4a_{n+k-5} + 4 \sum_{r=2}^{k-6} a_{n+r}
\]
\[
= \cdots \geq \sum_{r=1}^{\frac{k-3}{2}} 2^{\frac{k-1}{2} - r} a_{n+2r}.
\]

Since \( 2^m \geq 2m \) if \( m \geq 1 \), we have

\[
a_{n+k-1} \geq \sum_{r=1}^{\frac{k-3}{2}} 2^{\frac{k-1}{2} - r} a_{n+2r} \geq \sum_{r=1}^{\frac{k-3}{2}} (k-2r-1)a_{n+2r}.
\] (4.3)

First suppose \( j \leq k - 2 \). In this case, we show

\[
\frac{k!}{j!} a_{n+k-1} \geq \sum_{r=1}^{\left\lfloor \frac{j}{2} \right\rfloor} r \frac{(k-2r)!}{(j-2r)!} a_{n+2r-1},
\] (4.4)

which implies (4.1). And (4.4) is seen to hold since by (4.3),

\[
\frac{k!}{j!} a_{n+k-1} \geq \sum_{r=1}^{\frac{k-3}{2}} \frac{(k-2r-1)k!}{j!} a_{n+2r} \geq \sum_{r=1}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{(k-2r-1)k!}{j!} a_{n+2r},
\]

with \( a_{n+2r} \geq a_{n+2r-1} \) and

\[
\frac{(k-2r-1)k!}{r(k-2r)!} \geq \frac{(k-2)!}{(k-2r-2)!} \geq \frac{j!}{(j-2r)!}.
\]
The $j = k - 1$ case of (4.1) follows from noting
\[
3k a_{n+k-1} \geq k a_{n+k-1} + \sum_{r=1}^{k-3} 2k(k-2r-1)a_{n+2r}
\]
\[
\geq (k-1)a_{n+k-2} + \sum_{r=1}^{k-3} 2r(k-2r)a_{n+2r-1} = \sum_{r=1}^{k-1} 2r(k-2r)a_{n+2r-1},
\]
since $k(k-2r-1) \geq r(k-2r)$ if $1 \leq r \leq \frac{k-3}{2}$.

**Lemma 4.2.** Suppose $k, n \geq 3$ are odd. If $1 \leq i \leq k-1$, then $f^{(i)}_{n,k}(1) > 0$ implies $f^{(i+1)}_{n,k}(1) > 0$.

**Proof.** Let $f = f_{n,k}$ and $i = k - j$ for some $1 \leq j \leq k - 1$. Then the assumption $f^{(k-j)}(1) > 0$ is equivalent to
\[
\frac{(n+k)!}{(n+j)!} + \frac{(n+k+1)!}{(n+j+1)!} > \frac{k!}{j!} a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r}.
\]

Using (4.5), we will show $f^{(k-j+1)}(1) > 0$, i.e.,
\[
\frac{(n+k)!}{(n+j-1)!} + \frac{(n+k+1)!}{(n+j)!} > \frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r}.
\]

Note that the right-hand side of both inequalities (4.5) and (4.6) is positive since $a_{n+k} = \sum_{r=0}^{k-1} a_{n+r}$. Since the left-hand side of (4.6) divided by the left-hand side of (4.5) is greater than $n + j$, it suffices to show
\[
\frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r}
\]
\[
\leq (n+j) \left( \frac{k!}{j!} a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} \right).
\]

For (4.7), it is enough to show
\[
\frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r}
\]
\[
\leq (j+3) \left( \frac{k!}{j!} a_{n+k} + \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r} \right),
\]
since $n \geq 3$. 

**Polynomials whose coefficients are $k$-Fibonacci numbers**
Starting with the left-hand-side of (4.8), and proceeding at this stage as in the proof of Lemma 3.2 above, we have

\[
\frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} \leq \frac{k!}{(j-1)!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} \left( \frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\
+ \sum_{r=1}^{\left\lfloor \frac{j}{2} \right\rfloor} 2r \frac{(k-2r)!}{(j-2r)!} a_{n+2r-1} \\
= (j+3) \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} \left( \frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\
- 3 \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=1}^{\left\lfloor \frac{j}{2} \right\rfloor} 2r \frac{(k-2r)!}{(j-2r)!} a_{n+2r-1} \\
\leq (j+3) \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} \left( \frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r},
\]

where the last inequality follows from Lemma 4.1. Thus,

\[
\frac{k!}{(j-1)!} a_{n+k} + \sum_{r=0}^{j-2} (-1)^r \frac{(k-r-1)!}{(j-r-2)!} a_{n+r} \leq (j+3) \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} \left( \frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\
\leq (j+3) \frac{k!}{j!} \sum_{r=j}^{k-1} a_{n+r} + \sum_{r=0}^{j-1} \left( j+3 \right) \left( \frac{k!}{j!} + (-1)^r \frac{(k-r-1)!}{(j-r-1)!} \right) a_{n+r} \\
= (j+3) \frac{k!}{j!} a_{n+k} + (j+3) \sum_{r=0}^{j-1} (-1)^r \frac{(k-r-1)!}{(j-r-1)!} a_{n+r},
\]

which gives (4.8) and completes the proof. \( \square \)

We can now determine the number of zeros of \( f_{n,k}(x) \) on the interval \([1, \infty)\).

**Lemma 4.3.** Suppose \( k \geq 3 \) and \( n \) are odd. Then \( f_{n,k}(x) \) has exactly one zero on the interval \([1, \infty)\) and it is simple.

**Proof.** If \( n \geq 3 \), then use Lemma 4.2 and the same reasoning as in the proof of Lemma 3.3 above. Note that in this case we have

\[
f_{n,k}(1) = 2 - a_{n+k} + \sum_{r=0}^{k-1} (-1)^{r+1} a_{n+r} = 2 - 2 \sum_{r=0}^{\frac{k-1}{2}} a_{n+2r} < 0,
\]
as $a_{n+k-1}, a_{n+k-3} > 0$. If $n = 1$, then $f_{1,k}(x) = x^{k+1}(1 + x) - 2x^k + x - 1$ and the result also holds as $f_{1,k}(1) = 0$ with $f'_{1,k}(x) > 0$ if $x \geq 1$.

We next consider the case when $n$ is even.

**Lemma 4.4.** If $k \geq 3$ is odd and $n$ is even, then $f_{n,k}(x)$ has no zeros on $[1, \infty)$.

**Proof.** In this case, we have

$$f_{n,k} = -x^{n+k}(1 + x) - a_{n+k}x^k + \sum_{r=0}^{k-1} (-1)^{r+1} a_{n+r}x^{k-r-1}.$$ 

If $x \geq 1$, then $f_{n,k}(x) < 0$ since $a_{n+k} = \sum_{r=0}^{k-1} a_{n+r}$ and $-x^k \leq -x^{k-r-1}$ for $0 \leq r \leq k-1$.

We now prove the main result of this section.

**Theorem 4.5.** (i) If $k \geq 3$ and $n$ are odd, then the polynomial $P_{n,k}(x)$ has one real zero $q$, and it is simple with $q \leq -1$.

(ii) If $k \geq 3$ is odd and $n$ is even, then the polynomial $P_{n,k}(x)$ has no real zeros.

**Proof.** First suppose $n$ is odd. By Lemma 4.3, the polynomial $f_{n,k}(x)$, and hence $g_{n,k}(x)$, has one zero on $[1, \infty)$, and it is simple. By [7, Lemma 2.3], the characteristic polynomial $c_k(x) = x^k - x^{k-1} - x^{k-2} - \cdots - 1$ has no negative real zeros when $k$ is odd. Since $g_{n,k}(x) = c_k(-x)P_{n,k}(-x)$, it follows that $P_{n,k}(x)$ has one zero for $x \leq -1$, and hence one real zero, by Lemma 2.1.

If $n$ is even, then the polynomial $f_{n,k}(x)$, and hence $g_{n,k}(x)$, has no zeros for $x \geq 1$, by Lemma 4.4. Thus, neither does $P_{n,k}(-x)$, which implies it has no real zeros.

### 5. Convergence of zeros

In this section, we show that for each fixed $k \geq 2$, the sequence of real zeros of $P_{n,k}(x)$ for $n$ odd is convergent. Before proving this, we remind the reader of the following version of Rouché’s Theorem which can be found in [4].

**Theorem 5.1** (Rouché). If $p(z)$ and $q(z)$ are analytic interior to a simple closed Jordan curve $C$, and are continuous on $C$, with

$$|p(z) - q(z)| < |q(z)|, \quad z \in C,$$

then the functions $p(z)$ and $q(z)$ have the same number of zeros interior to $C$.

We now give three preliminary lemmas.
Lemma 5.2. (i) If \( k \geq 2 \), then the polynomial \( c_k(x) = x^k - x^{k-1} - \cdots - x - 1 \) has one positive real zero \( \lambda \), with \( \lambda > 1 \). All of its other zeros have modulus strictly less than one.

(ii) The zeros of \( c_k(x) \), which we will denote by \( \alpha_1 = \lambda, \alpha_2, \ldots, \alpha_k \), are distinct and thus
\[
a_n = c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_n\alpha_k^n, \quad n \geq 0,
\]
where \( c_1, c_2, \ldots, c_k \) are constants.

(iii) The constant \( c_1 \) is a positive real number.

Proof. (i) It is more convenient to consider the polynomial \( d_k(x) := (1-x)c_k(x) \). Note that
\[
d_k(x) = (1-x) \left( x^k - \frac{1-x^k}{1-x} \right) = 2x^k - x^{k+1} - 1.
\]
We regard \( d_k(z) \) as a complex function. Since on the circle \( |z| = 3 \) in the complex plane holds
\[
|2z^k| = 2 \cdot 3^k < 3^{k+1} - 1 = |z^{k+1}| - 1 \leq |z^k + 1 - 1|,
\]
it follows from Rouché’s Theorem that \( d_k(z) \) has \( k + 1 \) zeros in the disc \( |z| < 3 \) since the function \(-z^{k+1} - 1\) has all of its zeros there. On the other hand, on the circle \( |z| = 1 + \epsilon \), we have
\[
|z^{k+1}| = (1 + \epsilon)^{k+1} < 2(1 + \epsilon)^k - 1 \leq |2z^k - 1|,
\]
which implies that the polynomial \( d_k(z) \) has exactly \( k \) zeros in the disc \( |z| < 1 + \epsilon \), for all \( \epsilon > 0 \) sufficiently small such that \( \frac{\ln(1-\epsilon)}{\ln(1+\epsilon)} < 2 \leq k \). Letting \( \epsilon \to 0 \), we see that there are \( k \) zeros for the polynomial \( d_k(z) \) in the disc \( |z| \leq 1 \). But \( z = 1 \) is a zero of the polynomial \( d_k(z) = (1-z)c_k(z) \) on the circle \( |z| = 1 \), and it is the only such zero since \( d_k(1) = 0 \) implies \( |z|^k \cdot |2 - z| = 1 \), or \( |2 - z| = 1 \), which is clearly satisfied by only \( z = 1 \). Hence, the polynomial \( c_k(z) \) has \( k - 1 \) zeros in the disc \( |z| < 1 \) and exactly one zero in the domain \( 1 < |z| < 3 \). Finally, by Descartes’ rule of signs and since \( c_k(1) < 0 \), we see that \( c_k(x) \) has exactly one positive real zero \( \lambda \), with \( 1 < \lambda < 3 \).

(ii) We’ll prove only the first statement, as the second one follows from the first and the theory of linear recurrences. For this, first note that \( d_k'(x) = 0 \) implies \( x = 0, \frac{2k}{k+1} \). Now the only possible rational roots of the equation \( d_k(x) = 0 \) are \( \pm 1 \), by the rational root theorem. Thus \( d_k \left( \frac{2k}{k+1} \right) = 0 \) is impossible as \( k \geq 2 \), which implies \( d_k(x) \) and \( d_k'(x) \) cannot share a zero. Therefore, the zeros of \( d_k(x) \), and hence of \( c_k(x) \), are distinct.

(iii) Substitute \( n = 0, 1, \ldots, k - 1 \) into (5.1), and recall that \( a_0 = a_1 = \cdots = a_{k-2} = 0 \) with \( a_{k-1} = 1 \), to obtain a system of linear equations in the variables \( c_1, c_2, \ldots, c_k \). Let \( A \) be the coefficient matrix for this system (where the equations are understood to have been written in the natural order) and let \( A' \) be the matrix obtained from \( A \) by replacing the first column of \( A \) with the vector \((0, \ldots, 0, 1)\) of
length $k$. Now the transpose of $A$ and of the $(k-1) \times (k-1)$ matrix obtained from $A'$ by deleting the first column and the last row are seen to be Vandermonde matrices. Therefore, by Cramer’s rule, we have

$$c_1 = \frac{\det A'}{\det A} = \frac{(-1)^{k+1} \prod_{2 \leq i < j \leq k} (\alpha_j - \alpha_i)}{\prod_{1 \leq i < j \leq k} (\alpha_j - \alpha_i)} = \frac{1}{(-1)^{k-1} \prod_{j=2}^{k} (\alpha_j - \alpha_1)} = \frac{1}{\prod_{j=2}^{k} (\alpha_1 - \alpha_j)}.$$ 

If $j \geq 2$, then either $\alpha_j < 0$ or $\alpha_j$ and $\alpha_\ell$ are complex conjugates for some $\ell$. Note that $\alpha_1 - \alpha_j > 0$ in the first case and $$(\alpha_1 - \alpha_j)(\alpha_1 - \alpha_\ell) = (\alpha_1 - a)^2 + b^2 > 0$$ in the second, where $\alpha_j = a + bi$. Since all of the complex zeros of $c_k(x)$ which aren’t real come in conjugate pairs, it follows that $c_1$ is a positive real number.

We give the zeros of $c_k(z)$ for $2 \leq k \leq 5$ as well as the value of the constant $c_1$ in Table 1 below, where $\overline{z}$ denotes the complex conjugate of $z$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>The zeros of $c_k(z)$</th>
<th>The constant $c_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.61803, -0.61803</td>
<td>0.44721</td>
</tr>
<tr>
<td>3</td>
<td>1.83928, $r_1 = -0.41964 + 0.60629i$, $\overline{r_1}$</td>
<td>0.18280</td>
</tr>
<tr>
<td>4</td>
<td>1.92756, -0.77480, $r_1 = -0.07637 + 0.81470i$, $\overline{r_1}$</td>
<td>0.07907</td>
</tr>
<tr>
<td>5</td>
<td>1.96594, $r_1 = 0.19537 + 0.84885i$, $\overline{r_1}$, $\overline{r_2}$, $r_2 = -0.67835 + 0.45853i$, $\overline{r_2}$</td>
<td>0.03601</td>
</tr>
</tbody>
</table>

Table 1: The zeros of $c_k(z)$ and the constant $c_1$.

The next lemma concerns the location of the positive zero of the $k$-th derivative of $f_{n,k}(x)$.

**Lemma 5.3.** Suppose $k \geq 2$ is fixed and $n$ is odd. Let $s_n (= s_{n,k})$ be the zero of $f_{n,k}(x)$ on $[1, \infty)$, where $f_{n,k}(x)$ is given by (3.1), and let $t_n (= t_{n,k})$ be the positive zero of the $k$-th derivative of $f_{n,k}(x)$. Let $\lambda$ be the positive zero of $c_k(x)$. Then we have

(i) $t_n < s_n$ for all odd $n$, and

(ii) $t_n \to \lambda$ as $n$ odd increases without bound.

**Proof.** Suppose $k$ is even, the proof when $k$ is odd being similar. Then $f_{n,k}$ is given by (3.2) above. Throughout the following proof, $n$ will always represent an odd integer and $f = f_{n,k}$. Recall from Lemma 3.3 that $f$ has exactly one zero on the interval $[1, \infty)$.

(i) By Descartes’ rule of signs, the polynomial $f^{(k)}(x)$ has one positive real zero $t_n$. If $t_n < 1 \leq s_n$, then we are done, so let us assume $t_n \geq 1$. The condition $t_n \geq 1$,
or equivalently $f^{(k)}(1) \geq 0$, then implies $n \geq 3$, and thus $f(1) > 0$. (Indeed, $t_n \geq 1$ for all $n$ sufficiently large since $a_{n+k} \sim c_1 \lambda^{n+k}$, with $\lambda > 1$.)

Now observe that $f^{(k)}(1) \geq 0$ implies $f^{(i)}(1) > 0$ for $1 \leq i \leq k-1$, as the proof of Lemma 3.2 above shows in fact that $f^{(i)}(1) \leq 0$ implies $f^{(i+1)}(1) < 0$. Since $f^{(i)}(1) > 0$ for $0 \leq i \leq k-1$ and $f^{(k)}(1) \geq 0$, it follows that each of the polynomials $f(x), f'(x), \ldots, f^{(k)}(x)$ has exactly one zero on $[1, \infty)$ since $f^{(k+1)}(x) < 0$ for all $x \geq 1$. Furthermore, the zero of $f^{(i)}(x)$ on $[1, \infty)$ is strictly larger than the zero of $f^{(i+1)}(x)$ on $[1, \infty)$ for $0 \leq i \leq k-1$. In particular, the zero of $f(x)$ is strictly larger than the zero of $f^{(k)}(x)$, which establishes the first statement.

(ii) Let us assume $n$ is large enough to ensure $t_n \geq 1$. Note that

$$\frac{f^{(k)}(x)}{k!} = -\binom{n+k}{k}x^n - \binom{n+k+1}{k}x^{n+1} + a_{n,k}$$

so that

$$-2\binom{n+k+1}{k}x^{n+1} + a_{n,k} \leq \frac{f^{(k)}(x)}{k!} \leq -2\binom{n+k}{k}x^n + a_{n,k}, \quad x \geq 1. \quad (5.2)$$

Setting $x = t_n$ in (5.2), and rearranging, then gives

$$\left(\frac{a_{n+k}}{2(n+k+1)}\right)^{1/(n+1)} \leq t_n \leq \left(\frac{a_{n+k}}{2(n+k)}\right)^{1/n}. \quad (5.3)$$

The second statement then follows from letting $n$ tend to infinity in (5.3) and noting $\lim_{n \to \infty} (a_{n+k})^{1/n} = \lambda$ (as $a_{n+k} \sim c_1 \lambda^{n+k}$, by Lemma 5.2).

We will also need the following formula for an expression involving the zeros of $c_k(x)$.

**Lemma 5.4.** If $\alpha_1 = \lambda, \alpha_2, \ldots, \alpha_k$ are the zeros of $c_k(x)$, then

$$\sum_{j=0}^{k-1} (-1)^j \lambda^{k-j-1} S_j(\alpha_2, \alpha_3, \ldots, \alpha_k)$$

$$= \frac{k\lambda^{k+2} - (2k-1)\lambda^{k+1} - (k-1)\lambda^k + 2k\lambda^{k-1} - \lambda - 1}{(\lambda - 1)^2(\lambda + 1)}, \quad (5.4)$$

where $S_j(\alpha_2, \alpha_3, \ldots, \alpha_k)$ denotes the $j$-th symmetric function in the quantities $\alpha_2, \alpha_3, \ldots, \alpha_k$ if $1 \leq j \leq k-1$, with $S_0(\alpha_2, \alpha_3, \ldots, \alpha_k) := 1$.

**Proof.** Let us assume $k$ is even, the proof in the odd case being similar. First note that

$$(-1)^{i+1} = S_i(\alpha_1, \alpha_2, \ldots, \alpha_k) = S_i(\alpha_2, \ldots, \alpha_k) + \lambda S_{i-1}(\alpha_2, \ldots, \alpha_k), \quad 1 \leq i \leq k,$$

which gives the recurrences

$$S_{2r}(\alpha_2, \ldots, \alpha_k) = -1 - \lambda S_{2r-1}(\alpha_2, \ldots, \alpha_k), \quad 1 \leq r \leq (k-2)/2, \quad (5.5)$$
and
\[ S_{2r+1}\{\alpha_2, \ldots, \alpha_k\} = 1 - \lambda S_{2r}\{\alpha_2, \ldots, \alpha_k\}, \quad 0 \leq r \leq (k-2)/2. \quad (5.6) \]
Iterating (5.5) and (5.6) yields
\[ S_{2r}\{\alpha_2, \ldots, \alpha_k\} = -\left(1 + \lambda + \cdots + \lambda^{2r-1}\right) + \lambda^{2r} \]
\[ = -\frac{1 - 2\lambda^{2r} + \lambda^{2r+1}}{1 - \lambda}, \quad 1 \leq r \leq (k-2)/2, \quad (5.7) \]
and
\[ S_{2r+1}\{\alpha_2, \ldots, \alpha_k\} = (1 + \lambda + \cdots + \lambda^{2r}) - \lambda^{2r+1} \]
\[ = \frac{1 - 2\lambda^{2r+1} + \lambda^{2r+2}}{1 - \lambda}, \quad 0 \leq r \leq (k-2)/2. \quad (5.8) \]
Note that (5.7) also holds in the case when \( r = 0 \).

By (5.7) and (5.8), we then have
\[
\sum_{j=0}^{k-1} (-1)^j \lambda^{k-j-1} S_j\{\alpha_2, \alpha_3, \ldots, \alpha_k\}
\]
\[ = -\sum_{r=0}^{k-1} \lambda^{k-2r-1} \left(1 - \frac{2\lambda^{2r} + \lambda^{2r+1}}{1 - \lambda}\right) - \sum_{r=0}^{k-1} \lambda^{k-2r-2} \left(1 - \frac{2\lambda^{2r+1} + \lambda^{2r+2}}{1 - \lambda}\right) \]
\[ = \frac{1}{\lambda - 1} \sum_{r=0}^{k-1} (\lambda^{k-2r-1} - 2\lambda^{k-1} + \lambda^k) + \frac{1}{\lambda - 1} \sum_{r=0}^{k-1} (\lambda^{k-2r-2} - 2\lambda^{k-1} + \lambda^k) \]
\[ = \frac{\lambda}{\lambda - 1} \left(\lambda^k - 1\right) + \frac{1}{\lambda - 1} \left(\lambda^k - 1\right) - \frac{2k\lambda^{k-1}}{\lambda - 1} + \frac{k\lambda^k}{\lambda - 1}, \]
which gives (5.4).

We now can prove the main result of this section.

**Theorem 5.5.** Suppose \( k \geq 2 \) and \( n \) is odd. Let \( r_n (= r_{n,k}) \) denote the real zero of the polynomial \( P_{n,k}(x) \) defined by (1.2) above. Then \( r_n \to -\lambda \) as \( n \to \infty \).

**Proof.** Let \( n \) denote an odd integer throughout. We first consider the case when \( k \) is even. Equivalently, we show that \( s_n \to \lambda \) as \( n \to \infty \), where \( s_n \) denotes the zero of \( f_{n,k}(x) \) on the interval \([1, \infty)\). By Lemma 5.3, we have \( t_n < s_n \) for all \( n \) with \( t_n \to \lambda \) as \( n \to \infty \), where \( t_n \) is the positive zero of the \( k \)-th derivative of \( f_{n,k}(x) \). So it is enough to show \( s_n \to \lambda \) for all \( n \) sufficiently large, i.e., \( f_{n,k}(\lambda) < 0 \).

By Lemma 5.2, we have
\[ f_{n,k}(\lambda) = -\lambda^{n+k}(1 + \lambda) + a_{n,k}\lambda^k + \sum_{r=0}^{k-1} (-1)^r a_{n+r}\lambda^{k-r-1} \]
\[ \sim -\lambda^{n+k}(1 + \lambda) + c_1\lambda^{n+2k} + \sum_{r=0}^{k-1} (-1)^r c_1\lambda^{n+k-1} \]

\[ = \lambda^{n+k}(-1 - \lambda + c_1\lambda^k), \]

so that \( f_{n,k}(\lambda) < 0 \) for large \( n \) if \(-1 - \lambda + c_1\lambda^k < 0\), i.e.,

\[ \lambda^k < \frac{1 + \lambda}{c_1}. \] (5.9)

So to complete the proof, we must show (5.9). By Lemmas 5.2 and 5.4, we have

\[ \frac{1}{c_1} = \prod_{j=2}^{k} (\lambda - \alpha_j) = \sum_{j=0}^{k-1} (-1)^j \lambda^{k-j-1} S_j\{\alpha_2, \alpha_3, \ldots, \alpha_k\} \]

\[ = \frac{k\lambda^{k+2} - (2k - 1)\lambda^{k+1} - (k - 1)\lambda^k + 2k\lambda^{k-1} - \lambda - 1}{(\lambda - 1)^2(\lambda + 1)}, \]

so that (5.9) holds if and only

\[ \lambda^k(\lambda - 1)^2 < k\lambda^{k+2} - (2k - 1)\lambda^{k+1} - (k - 1)\lambda^k + 2k\lambda^{k-1} - \lambda - 1, \]

i.e.,

\[ 1 + \lambda + k\lambda^k + (2k - 3)\lambda^{k+1} < 2k\lambda^{k-1} + (k - 1)\lambda^{k+2}. \] (5.10)

Recall from the proof of Lemma 5.2 that \( 2\lambda^k = 1 + \lambda^{k+1} \). Substituting \( \lambda^{k+1} = \frac{\lambda + \lambda^{k+2}}{2}, \)

\[ \lambda^k = \frac{1 + \lambda + \lambda^{k+2}}{2} = \frac{2 + \lambda + \lambda^{k+2}}{4}, \]

and

\[ \lambda^{k-1} = \frac{\lambda^k}{\lambda} = \frac{2 + \lambda + \lambda^{k+2}}{4\lambda} \]

into (5.10), and rearranging, then gives

\[ \left(1 - \frac{\lambda}{2} - \frac{k}{\lambda}\right) + \frac{5k\lambda}{4} < \lambda^{k+2} \left(\frac{k}{2\lambda} - \frac{k}{4} + \frac{1}{2}\right). \] (5.11)

For (5.11), note first that \( c_k(2) > 0 \) as \( 2^k > 2^{k-1} + \cdots + 1 \), which implies \( \lambda < 2 \leq k \) and thus \( 1 - \frac{\lambda}{2} - \frac{k}{\lambda} < 0 \). So to show (5.11), it is enough to show

\[ \frac{5k}{4} < \lambda^{k+1} \left(\frac{k}{2\lambda} - \frac{k}{4} + \frac{1}{2}\right). \] (5.12)

For (5.12), we’ll consider the cases \( k = 2 \) and \( k \geq 4 \). If \( k = 2 \), then \( \lambda = \theta = \frac{1 + \sqrt{5}}{2} \), so that (5.12) reduces in this case to \( \frac{5}{2} < \theta^2 = \theta + 1 \), which is true. Now suppose
$k \geq 4$ is even. First observe that $c_k \left( \frac{5}{3} \right) < 0$, whence $\lambda > \frac{5}{3}$, as $d_k \left( \frac{5}{3} \right) > 0$ since $\left( \frac{5}{3} \right)^k (2 - \frac{5}{3}) > 1$ for all $k \geq 3$. Thus, we have

$$\lambda^k = (\lambda^{k-1} + 1) + \lambda^{k-2} + \lambda^{k-3} + \cdots + \lambda$$

$$> 2\lambda^{k-1} + \lambda^{k-2} + \lambda^{k-3} + \cdots + \lambda > 2 \cdot \frac{5}{3} + \frac{5(k - 2)}{3} = \frac{5k}{3}.$$ 

So to show (5.12) when $k \geq 4$, it suffices to show

$$0 < \lambda \left( \frac{k}{2\lambda} - \frac{k}{4} + \frac{1}{2} \right) - \frac{3}{4} = \frac{k(2 - \lambda)}{4} + \frac{2\lambda - 3}{4},$$

which is true as $\frac{5}{3} < \lambda < 2$. This completes the proof in the even case.

If $k$ is odd, then we proceed in a similar manner. Instead of inequality (5.9), we get

$$\lambda^k + \frac{1}{\lambda} < \frac{1 + \lambda}{c_1}, \quad (5.13)$$

which is equivalent to

$$\left( 1 - \frac{\lambda}{2} - \frac{k}{\lambda} + \frac{(\lambda - 1)^2}{\lambda} \right) + \frac{5k\lambda}{4} < \lambda^{k+2} \left( \frac{k}{2\lambda} - \frac{k}{4} + \frac{1}{2} \right). \quad (5.14)$$

Note that the sum of the first four terms on the left-hand side of (5.14) is negative since $1 - \frac{k}{\lambda} < 0$ and $-\frac{\lambda}{2} + \frac{(\lambda - 1)^2}{\lambda} < 0$ as $\frac{5}{3} < \lambda < 2$ for $k \geq 3$. Thus, it suffices to show (5.12) in the case when $k \geq 3$ is odd, which has already been done since the proof given above for it applies to all $k \geq 3$.

$$\lambda$$

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Table 2: Some real zeros of $P_{n,k}(-x)$, where $\lambda$ is the positive zero of $c_k(x)$.

Perhaps the proofs presented here of Theorems 1.2 and 5.5 could be generalized to show comparable results for polynomials associated with linear recurrent sequences having various non-negative real weights, though the results are not true for all linear recurrences having such weights, as can be seen numerically in the case $k = 3$. Furthermore, numerical evidence (see Table 2 below) suggests that the sequence of zeros in Theorem 5.5 decreases monotonically for all $k$, as is true in the $k = 2$ case (see [2, Theorem 3.1]).
References


