RELATIONALITY BETWEEN GENERALIZED BERNOULLI NUMBERS AND POLYNOMIALS AND GENERALIZED EULER NUMBERS AND POLYNOMIALS

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Abstract. In this paper, concepts of the generalized Bernoulli and Euler numbers and polynomials are introduced, and some relationships between them are established.

1. Introduction

It is well-known that the Bernoulli and Euler polynomials are two classes of special functions. They play important roles and have made many unexpected appearances in Numbers Theory, Theory of Functions, Theoretical Physics, and the like. There has been much literature about Bernoulli and Euler polynomials, for some examples, please refer to references in this paper.

The Bernoulli numbers and polynomials and Euler numbers and polynomials are generalized to the generalized Bernoulli numbers and polynomials and to the generalized Euler numbers and polynomials in [2, 3, 4] in recent years.

In this article, we first re-state the definitions of the generalized Bernoulli and Euler numbers and polynomials, and then discuss the relationships between the generalized Bernoulli and Euler numbers and polynomials. These results generalize, reinforce, and deepen those in [1, 5, 8, 10, 11].

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2. Definitions of Bernoulli and Euler numbers and polynomials

In this section, we will restate definitions of (generalized) Bernoulli numbers, (generalized) Bernoulli polynomials, (generalized) Euler numbers, and (generalized) Euler polynomials as follows. For more details, please see [1, 2, 3, 4, 10].

Definition 2.1 ([1, 10]). The Bernoulli numbers \( B_k \) and Euler numbers \( E_k \) are defined respectively by

\[
\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k, \quad |t| < 2\pi;
\]

\[
\frac{2e^t}{e^{2t} + 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!} E_k, \quad |t| \leq \pi.
\]

Definition 2.2. [1, 10] The Bernoulli polynomials \( B_k(x) \) and Euler polynomials \( E_k(x) \) are defined respectively by

\[
\frac{t e^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k(x), \quad |t| < 2\pi, \quad x \in \mathbb{R};
\]

\[
\frac{2e^{xt}}{e^{2xt} + 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!} E_k(x), \quad |t| \leq \pi, \quad x \in \mathbb{R}.
\]

Note that \( B_k = B_k(0), \quad E_k = 2^k E_k\left(\frac{1}{2}\right) \).

Definition 2.3 ([2, 4]). Let \( a, b, c \) be positive numbers, the generalized Bernoulli numbers \( B_k(a, b) \) and the generalized Euler numbers \( E_k(a, b, c) \) are defined respectively by

\[
\frac{t}{b^t - a^t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k(a, b), \quad |t| < \frac{2\pi}{\ln b - \ln a};
\]

\[
\frac{2e^{xt}}{b^{2xt} + a^{2xt}} = \sum_{k=0}^{\infty} \frac{t^k}{k!} E_k(a, b, c).
\]

It is easy to see that \( B_k = B_k(1, c), \quad E_k = E_k(1, e, c) \).

Definition 2.4 ([2, 4]). The generalized Bernoulli polynomials \( B_k(x; a, b, c) \) and the generalized Euler polynomials \( E_k(x; a, b, c) \) are defined respectively by

\[
\frac{t e^{xt}}{b^t - a^t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k(x; a, b, c), \quad |t| < \frac{2\pi}{\ln b - \ln a}, \quad x \in \mathbb{R};
\]

\[
\frac{2e^{xt}}{b^{2xt} + a^{2xt}} = \sum_{k=0}^{\infty} \frac{t^k}{k!} E_k(x; a, b, c), \quad x \in \mathbb{R}.
\]

It is not difficult to see that \( B_k(x) = B_k(x; 1, e, c), \quad E_k(x) = E_k(x; 1, e, c), \quad B_k(a, b) = B_k(0; a, b, c), \) and \( E_k(a, b, c) = 2^k E_k\left(\frac{1}{2}; a, b, c\right), \) where \( x \in \mathbb{R} \).
3. Relationships between generalized Bernoulli and Euler polynomials

In this section, we will discuss some relationships between the generalized Bernoulli numbers and the generalized Euler numbers.

**Theorem 3.1.** Let \( k \in \mathbb{N} \) and \( r \in \mathbb{R} \), then we have

\[
k^{k-1} \sum_{j=0}^{k-1} \binom{k-1}{j} (r-1)^{k-j} \left( \ln c \right)^{k-j-1} B_j(a, b, c) \]

\[
= \sum_{j=0}^{k} \binom{k}{j} 2^{2j-1} \left[ (2 \ln b + r \ln c)^{k-j} - (2 \ln a + r \ln c)^{k-j} \right] B_j(a, b), \tag{3.1}
\]

where \( a, b, c \) are positive numbers.

**Proof.** From (2.5), Cauchy multiplication, and the power series identity theorem, we have

\[
\frac{2te^{rt}}{b^{2t} + a^{2t}} = \frac{2t}{b^{2t} - a^{2t}} \left[ (b^2 e^r)^t - (a^2 e^r)^t \right]
\]

\[
= \left[ \sum_{k=0}^{\infty} 2^{2k-1} B_k(a, b) \frac{t^k}{k!} \right] \left[ \sum_{k=0}^{\infty} \left[ \left( \ln(b^2 e^r) \right)^k - \left( \ln(a^2 e^r) \right)^k \right] \frac{t^k}{k!} \right]
\]

\[
= \sum_{k=0}^{\infty} \left[ \sum_{j=0}^{k} \binom{k}{j} 2^{2j-1} \left[ (2 \ln b + r \ln c)^{k-j} - (2 \ln a + r \ln c)^{k-j} \right] B_j(a, b) \right] \frac{t^k}{k!}. \tag{3.2}
\]

From (2.6), Cauchy multiplication, and the power series identity theorem, we have

\[
\frac{2te^{rt}}{b^{2t} + a^{2t}} = \frac{2te^t}{b^{2t} + a^{2t}} \cdot c^{(r-1)t}
\]

\[
= \left[ \sum_{k=0}^{\infty} E_k(a, b, c) \frac{k+1}{k!} \right] \left[ \sum_{k=0}^{\infty} \left[ (r-1)^k \left( \ln c \right)^k \right] \frac{t^k}{k!} \right]
\]

\[
= \sum_{k=0}^{\infty} \left[ \sum_{j=0}^{k} \binom{k}{j} (r-1)^{k-j} \left( \ln c \right)^{k-j-1} B_j(a, b, c) \right] \frac{t^k}{k!}. \tag{3.3}
\]

Equating coefficients of the terms \( \frac{t^k}{k!} \) in (3.2) and (3.3) leads to (3.1). \( \square \)

Taking \( r = 1 \) in (3.1) and defining \( \beta = 1 \), we have

**Corollary 3.1.1.** For nonnegative integer \( k \) and positive numbers \( a, b, c \), we have

\[
kE_{k-1}(a, b, c)
\]

\[
= \sum_{j=0}^{k} \binom{k}{j} 2^{2j-1} \left[ (2 \ln b + \ln c)^{k-j} - (2 \ln a + \ln c)^{k-j} \right] B_j(a, b). \tag{3.4}
\]
Furthermore, setting \( a = 1 \) and \( b = c = e \) in (3.4), we have

**Corollary 3.1.2.** For nonnegative integer \( k \), we have

\[
k E_{k-1} = \sum_{j=0}^{k} \binom{k}{j} 2^{2j-1}(3^{k-j} - 1)B_j. \tag{3.5}
\]

**Remark 3.1.** In [7, p. 943] and [9], the following formulae were given respectively:

\[
(1 + 2^{2k})(1 - 2^{2k-1})B_{2k} = \sum_{j=0}^{k} \binom{2k}{2j} B_{2j} E_{2k-2j}, \tag{3.6}
\]

\[
(2 - 2^{2k})B_{2k} = \sum_{j=0}^{k} \binom{2k}{2j} 2^{2j} B_{2j} E_{2k-2j}. \tag{3.7}
\]

Replacing \( k \) by \( 2k \) in (3.5), we have

**Corollary 3.1.3.** For nonnegative integer \( k \), we have

\[
\sum_{j=0}^{2k} \binom{2k}{j} 2^{2j-1}(3^{2k-j} - 1)B_j = 0. \tag{3.8}
\]

Taking \( r = 2 \) in (3.1) leads to the following

**Corollary 3.1.4.** For positive integer \( k \) and positive numbers \( a, b, c \), we have

\[
k \sum_{j=0}^{k-1} \binom{k-1}{j} (\ln c)^{k-j-1} E_j(a, b, c)
= \sum_{j=0}^{k} \binom{k}{j} 2^{k+j-1} \left[ (\ln b + \ln c)^{k-j} - (\ln a + \ln c)^{k-j} \right] B_j(a, b). \tag{3.9}
\]

Taking \( a = 1 \) and \( b = c = e \) in (3.9) yields

**Corollary 3.1.5.** For positive integer \( k \), we have

\[
k \sum_{j=0}^{k-1} \binom{k-1}{j} E_j = 2^{k-1} \sum_{j=0}^{k} \binom{k}{j} (2^k - 2^j)B_j. \tag{3.10}
\]

**Remark 3.2.** The result in (3.10) is equivalent to Lemma 2 in [5, p. 6].

Setting \( a = 1 \) and \( b = c = e \) in (3.1) gives us

**Corollary 3.1.6.** Let \( r \in \mathbb{R} \), then we have

\[
k \sum_{j=0}^{k-1} \binom{k-1}{j} (r-1)^{k-j-1} E_j = \sum_{j=0}^{k} \binom{k}{j} 2^{2j-1} \left[ (2+r)^{k-j} - r^{k-j} \right] B_j. \tag{3.11}
\]
Theorem 3.2. For positive numbers $a, b, c$ and nonnegative integer $k$, we have
\[
\sum_{j=0}^{k} \binom{k}{j} B_j(a,b)E_{k-j}(a,b,c) = \sum_{j=0}^{k} \binom{k}{j} 2^{2j-1} \left[ (\ln b + \ln c)^{k-j} + (\ln a + \ln c)^{k-j} \right] B_j(a,b). \tag{3.12}
\]

Proof. By (2.5), Cauchy multiplication, and the power series identity theorem, we have
\[
\frac{2te^t}{(b^t - a^t)(b^{2t} + a^{2t})} = \frac{2t}{b^{2t} - a^{2t}} \left[ (be^t + (ac)^t) \right]
\]
\[
= \left[ \sum_{k=0}^{\infty} 2^{k-1} B_k(a,b) \frac{t^k}{k!} \right] \left[ \sum_{k=0}^{\infty} [\ln(bc)]^k + (\ln(ac))^k \right] \frac{t^k}{k!}
\]
\[
= \sum_{k=0}^{\infty} \left[ \sum_{j=0}^{k} \binom{k}{j} 2^{2j-1} \left[ (\ln b + \ln c)^{k-j} - (\ln a + \ln c)^{k-j} \right] B_j(a,b) \right] \frac{t^k}{k!}.
\tag{3.13}
\]

By (2.6), Cauchy multiplication, and the power series identity theorem, we have
\[
\frac{2te^t}{(b^t - a^t)(b^{2t} + a^{2t})} = \left[ \sum_{k=0}^{\infty} B_k(a,b) \frac{t^{k+1}}{k!} \right] \left[ \sum_{k=0}^{\infty} E_k(a,b,c) \frac{t^k}{k!} \right]
\]
\[
= \sum_{k=0}^{\infty} \left[ \sum_{j=0}^{k} \binom{k}{j} B_j(a,b) E_{k-j}(a,b,c) \right] \frac{t^k}{k!}.
\tag{3.14}
\]

Equating coefficients of $\frac{t^k}{k!}$ in (3.13) and (3.14) leads to (3.12). \hfill \Box

Taking $a = 1$, $b = c = e$ in (3.12), we have

Corollary 3.2.1. For nonnegative integer $k$, we have
\[
\sum_{j=0}^{k} \binom{k}{j} B_j E_{k-j} = \sum_{j=0}^{k} \binom{k}{j} (2^{k-j-1} + 2^{2j-1}) B_j. \tag{3.15}
\]

4. Relations between generalized Bernoulli and Euler polynomials

In this section, we will discuss some relationships between the generalized Bernoulli polynomials and the generalized Euler polynomials.
Theorem 4.1. For positive numbers \(a, b, c\), nonnegative integer \(k\), and \(x \in \mathbb{R}\), we have
\[
kE_{k-1}(x; a, b, c) = \sum_{j=0}^{k} \binom{k}{j} 2^j \left[ (\ln b)^{k-j} - (\ln a)^{k-j} \right] B_j \left( \frac{x}{2}; a, b, c \right). \tag{4.1}\]

Proof. By (2.7), Cauchy multiplication, and the power series identity theorem, we have
\[
\frac{2tc^x}{b^t + a^t} = \frac{2tc^x(b' - a')}{b^{2t} - a^{2t}} = \left[ \sum_{k=0}^{\infty} \left( \frac{k}{k!} \right)^2 \frac{1}{k!} \left[ \sum_{j=0}^{k} \binom{k}{j} 2^j \left[ (\ln b)^{k-j} - (\ln a)^{k-j} \right] B_j \left( \frac{x}{2}; a, b, c \right) \right] \right] \frac{1}{k!}. \tag{4.2}
\]
By (2.8), Cauchy multiplication, and the power series identity theorem, we have
\[
\frac{2tc^x}{b^t + a^t} = t \sum_{k=0}^{\infty} \frac{t^k k!}{k!} E_k(x; a, b, c) = \sum_{k=0}^{\infty} \left[ kE_{k-1}(x; a, b, c) \right] \frac{t^k}{k!}. \tag{4.3}
\]
Equating coefficients of \(\frac{t^k}{k!}\) in (4.2) and (4.3) leads to (4.1). \(\square\)

Letting \(a = 1\) and \(b = c = e^{1/2}\) in (4.1) and defining \(1^0 = 1\), we have

Corollary 4.1.1. For \(k \in \mathbb{N}\) and \(x \in \mathbb{R}\), we have
\[
kE_{k-1}(x) = \sum_{j=0}^{k-1} \binom{k}{j} 2^j B_j \left( \frac{x}{2} \right). \tag{4.4}\]

From (2.3), Cauchy multiplication, and the power series identity theorem, it follows that

Corollary 4.1.2. For \(k \in \mathbb{N}\) and \(x \in \mathbb{R}\), we have
\[
2^k \left[ B_k \left( \frac{x+1}{2} \right) - B_k \left( \frac{x}{2} \right) \right] = \sum_{j=0}^{k-1} \binom{k}{j} 2^j B_j \left( \frac{x}{2} \right). \tag{4.5}\]

Combining (4.4) with (4.5), we have

Corollary 4.1.3. For \(k \in \mathbb{N}\) and \(x \in \mathbb{R}\), we have
\[
kE_{k-1}(x) = 2^k \left[ B_k \left( \frac{x+1}{2} \right) - B_k \left( \frac{x}{2} \right) \right]. \tag{4.6}\]

Remark 4.1. The formula (4.6) is the same as Lemma 3 in [5, p. 6].

Using (2.7), (2.8), Cauchy multiplication, and the power series identity theorem, we obtain
Theorem 4.2. For positive numbers \(a, b, c\), nonnegative integer \(k\), and \(x \in \mathbb{R}\),

\[
2^k B_k(x; a, b, c) = \sum_{j=0}^{k} \binom{k}{j} B_j(x; a, b, c) E_{k-j}(x; a, b, c).
\]

Taking \(a = 1\) and \(b = c = e\) in (4.7), we have

**Corollary 4.2.1.** Let \(x \in \mathbb{R}\) and \(k\) be nonnegative integer, then

\[
2^k B_k(x) = \sum_{j=0}^{k} \binom{k}{j} B_j(x) E_{k-j}(x).
\]

Theorem 4.3. For positive numbers \(a, b, c\), nonnegative integer \(k\), and \(x \in \mathbb{R}\), we have

\[
kE_{k-1}(x; a, b, c) = 2B_k(x; a, b, c) - 2 \sum_{j=0}^{k} \binom{k}{j} 2^j (\ln a)^{k-j} B_j \left( \frac{x}{2}; a, b, c \right).
\]

**Proof.** By (2.7), Cauchy multiplication, and the power series identity theorem, we obtain

\[
\frac{2te^{xt}}{b^t + a^t} = \frac{2te^{xt}}{b^t - a^t} - \frac{4te^{xt}a^t}{b^t - a^t} = 2\sum_{k=0}^{\infty} \frac{t^k}{k!} B_k(x; a, b, c) - \sum_{k=0}^{\infty} \frac{t^k}{k!} (\ln a)^k B_k \left( \frac{x}{2}; a, b, c \right).
\]

By (2.8), Cauchy multiplication, and the power series identity theorem, we have

\[
\frac{2te^{xt}}{b^t + a^t} = \int \sum_{k=0}^{\infty} \frac{t^k}{k!} E_k(x; a, b, c) = \sum_{k=0}^{\infty} \frac{t^k}{k!} kE_{k-1}(x; a, b, c).
\]

Equating coefficients of \(\frac{t^k}{k!}\) in (4.10) and (4.11) leads to (4.9).

If having \(a = 1\) and \(b = c = e\) in (4.9), then

**Corollary 4.3.1.** For \(k \in \mathbb{N}\) and \(x \in \mathbb{R}\), we have

\[
kE_{k-1} = 2B_k(x) - 2^k B_k \left( \frac{x}{2} \right).
\]

**Remark 4.2.** The formula (4.12) is the same as that in [10, p. 48].
REFERENCES


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