Some properties of the generalized Apostol-type polynomials

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1 Introduction, definitions and motivation

The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$ of (real or complex) order $\alpha$, are usually defined by means of the following generating functions (see, for details, [1], pp.532-533 and [2], p.61 et seq.; see also [3] and the references cited therein):

\[
\left( \frac{z}{e^z - 1} \right)^\alpha \cdot e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{z^n}{n!} \quad (|z| < 2\pi), \tag{1.1}
\]

\[
\left( \frac{2}{e^z + 1} \right)^\alpha \cdot e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{z^n}{n!} \quad (|z| < \pi) \tag{1.2}
\]

and

\[
\left( \frac{2z}{e^z + 1} \right)^\alpha \cdot e^{xz} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x) \frac{z^n}{n!} \quad (|z| < \pi). \tag{1.3}
\]

So that, obviously, the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are given, respectively, by

\[
B_n(x) := B_n^{(1)}(x), \quad E_n(x) := E_n^{(1)}(x) \quad \text{and} \quad G_n(x) := G_n^{(1)}(x) \quad (n \in \mathbb{N}_0). \tag{1.4}
\]
For the classical Bernoulli numbers $B_n$, the classical Euler numbers $E_n$ and the classical Genocchi numbers $G_n$ of order $n$, we have

$$B_n := B_n(0) = B_n^{(1)}(0), \quad E_n := E_n(0) = E_n^{(1)}(0) \quad \text{and} \quad G_n := G_n(0) = G_n^{(1)}(0),$$

(1.5)

respectively.

Some interesting analogues of the classical Bernoulli polynomials and numbers were first investigated by Apostol (see [4], p.165, Eq. (3.1)) and (more recently) by Srivastava (see [5], pp.83-84). We begin by recalling here Apostol’s definitions as follows.

**Definition 1.1** (Apostol [4]; see also Srivastava [5]) The Apostol-Bernoulli polynomials $B_n(x;\lambda)$ ($\lambda \in \mathbb{C}$) are defined by means of the following generating function:

$$\frac{ze^{zx}}{\lambda e^z - 1} = \sum_{n=0}^{\infty} B_n(x;\lambda) \frac{z^n}{n!} \quad \left( |z| < 2\pi \text{ when } \lambda = 1; |z| < |\log \lambda| \text{ when } \lambda \neq 1 \right)$$

(1.6)

with, of course,

$$B_n(x) = B_n(x;1) \quad \text{and} \quad B_n(\lambda) := B_n(0;\lambda),$$

(1.7)

where $B_n(\lambda)$ denotes the so-called Apostol-Bernoulli numbers.

Recently, Luo and Srivastava [6] further extended the Apostol-Bernoulli polynomials as the so-called Apostol-Bernoulli polynomials of order $\alpha$.

**Definition 1.2** (Luo and Srivastava [6]) The Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ ($\lambda \in \mathbb{C}$) of order $\alpha \in \mathbb{N}_0$ are defined by means of the following generating function:

$$\left( \frac{e^{\alpha z}}{\lambda e^z - 1} \right)^\alpha \cdot e^{zx} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda) \frac{z^n}{n!} \quad \left( |z| < 2\pi \text{ when } \lambda = 1; |z| < |\log \lambda| \text{ when } \lambda \neq 1 \right)$$

(1.8)

with, of course,

$$B_n^{(\alpha)}(x) = B_n^{(\alpha)}(x;1) \quad \text{and} \quad B_n^{(\alpha)}(\lambda) := B_n^{(\alpha)}(0;\lambda),$$

(1.9)

where $B_n^{(\alpha)}(\lambda)$ denotes the so-called Apostol-Bernoulli numbers of order $\alpha$.

On the other hand, Luo [7], gave an analogous extension of the generalized Euler polynomials as the so-called Apostol-Euler polynomials of order $\alpha$.

**Definition 1.3** (Luo [7]) The Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ ($\lambda \in \mathbb{C}$) of order $\alpha \in \mathbb{N}_0$ are defined by means of the following generating function:

$$\left( \frac{2}{\lambda e^z + 1} \right)^\alpha \cdot e^{zx} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda) \frac{z^n}{n!} \quad (|z| < |\log (-\lambda)|)$$

(1.10)
with, of course,
\[ E_n^{(\alpha)}(x) = E_n^{(\alpha)}(x; 1) \quad \text{and} \quad E_n^{(\alpha)}(\lambda) := E_n^{(\alpha)}(0; \lambda), \] (1.11)

where \( E_n^{(\alpha)}(\lambda) \) denotes the so-called Apostol-Euler numbers of order \( \alpha \).

On the subject of the Genocchi polynomials \( G_n(x) \) and their various extensions, a remarkably large number of investigations have appeared in the literature (see, for example, [8–14]). Moreover, Luo (see [12–14]) introduced and investigated the Apostol-Genocchi polynomials of (real or complex) order \( \alpha \), which are defined as follows:

**Definition 1.4** The Apostol-Genocchi polynomials \( G_n^{(\alpha)}(x; \lambda) \) (\( \lambda \in \mathbb{C} \)) of order \( \alpha \in \mathbb{N}_0 \) are defined by means of the following generating function:
\[
\left( \frac{2z}{\lambda e^z + 1} \right)^\alpha \cdot e^{xz} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x; \lambda) \frac{z^n}{n!} \quad (|z| < |\log(-\lambda)|), \] (1.12)

with, of course,
\[ G_n^{(\alpha)}(x) = G_n^{(\alpha)}(x; 1), \quad G_n^{(\alpha)}(\lambda) := G_n^{(\alpha)}(0; \lambda), \]
\[ G_n(x; \lambda) := G_n^{(1)}(x; \lambda) \quad \text{and} \quad G_n(\lambda) := G_n^{(1)}(\lambda), \] (1.13)

where \( G_n(\lambda), G_n^{(\alpha)}(\lambda) \) and \( G_n(x; \lambda) \) denote the so-called Apostol-Genocchi numbers, the Apostol-Genocchi numbers of order \( \alpha \) and the Apostol-Genocchi polynomials, respectively.

Recently, Luo and Srivastava [15] introduced a unification (and generalization) of the above-mentioned three families of the generalized Apostol type polynomials.

**Definition 1.5** (Luo and Srivastava [15]) The generalized Apostol type polynomials \( F_n^{(\alpha)}(x; \lambda; u, v) \) (\( \alpha \in \mathbb{N}_0, \lambda, u, v \in \mathbb{C} \)) of order \( \alpha \) are defined by means of the following generating function:
\[
\left( \frac{2^n u^x}{\lambda e^z + 1} \right)^\alpha \cdot e^{xz} = \sum_{n=0}^{\infty} F_n^{(\alpha)}(x; \lambda; u, v) \frac{z^n}{n!} \quad (|z| < |\log(-\lambda)|), \] (1.14)

where
\[ F_n^{(\alpha)}(\lambda; u, v) := F_n^{(\alpha)}(0; \lambda; u, v) \] (1.15)

denote the so-called Apostol type numbers of order \( \alpha \).

So that, by comparing Definition 1.5 with Definitions 1.2, 1.3 and 1.4, we have
\[ B_n^{(\alpha)}(x; \lambda) = (-1)^\alpha F_n^{(\alpha)}(x; -\lambda; 0, 1), \] (1.16)
\[ E_n^{(\alpha)}(x; \lambda) = F_n^{(\alpha)}(x; \lambda; 1, 0), \] (1.17)
\[ G_n^{(\alpha)}(x; \lambda) = F_n^{(\alpha)}(x; \lambda; 1, 1). \] (1.18)
A polynomial $p_n(x)$ ($n \in \mathbb{N}, x \in \mathbb{C}$) is said to be a quasi-monomial [16], whenever two operators $\hat{M}, \hat{P}$, called multiplicative and derivative (or lowering) operators respectively, can be defined in such a way that

$$\hat{P}p_n(x) = np_{n-1}(x), \quad (1.19)$$

$$\hat{M}p_n(x) = p_{n+1}(x), \quad (1.20)$$

which can be combined to get the identity

$$\hat{M}\hat{P}p_n(x) = np_n(x). \quad (1.21)$$

The Appell polynomials [17] can be defined by considering the following generating function:

$$A(t)e^{xt} = \sum_{n=0}^{\infty} \frac{R_n(x)}{n!} t^n, \quad (1.22)$$

where

$$A(t) = \sum_{k=0}^{\infty} \frac{R_k}{k!} t^k \quad (A(0) \neq 0) \quad (1.23)$$

is analytic function at $t = 0$.

From [18], we know that the multiplicative and derivative operators of $R_n(x)$ are

$$\hat{M} = (x + \alpha_0) + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_x^{n-k}, \quad (1.24)$$

$$\hat{P} = D_x \quad (1.25)$$

where

$$\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} t^n \quad \text{(1.26)}$$

By using (1.21), we have the following lemma.

**Lemma 1.6** ([18]) *The Appell polynomials $R_n(x)$ defined by (1.22) satisfy the differential equation:*

$$\frac{\alpha_{n-1}}{(n-1)!} y^{(n)} + \frac{\alpha_{n-2}}{(n-2)!} y^{(n-1)} + \cdots + \frac{\alpha_1}{1!} y' + (x + \alpha_0) y' - ny = 0, \quad (1.27)$$

*where the numerical coefficients $\alpha_k, k = 1, 2, \ldots, n-1$ are defined in (1.26), and are linked to the values $R_k$ by the following relations:*

$$R_{k+1} = \sum_{h=0}^{k} \binom{k}{h} R_h \alpha_{k-h}. \quad (1.28)$$
Let \( \mathcal{P} \) be the vector space of polynomials with coefficients in \( \mathbb{C} \). A polynomial sequence \( \{P_n\}_{n \geq 0} \) be a polynomial set. \( \{P_n\}_{n \geq 0} \) is called a \( \sigma \)-Appell polynomial set of transfer power series \( A \) is generated by

\[
G(x, t) = A(t)G_0(x, t) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n,
\]

where \( G_0(x, t) \) is a solution of the system:

\[
\sigma G_0(x, t) = tG_0(x, t),
\]

\( G_0(x, 0) = 1. \)

In [19], the authors investigated the connection coefficients between two polynomials. And there is a result about connection coefficients between two \( \sigma \)-Appell polynomial sets.

**Lemma 1.7** ([19]) Let \( \sigma \in \Lambda^{(-1)} \). Let \( \{P_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \) be two \( \sigma \)-Appell polynomial sets of transfer power series, respectively, \( A_1 \) and \( A_2 \). Then

\[
Q_n(x) = \sum_{m=0}^{n} \frac{n!}{m!} \alpha_{n-m} P_m(x),
\]

where

\[
\frac{A_2(t)}{A_1(t)} = \sum_{k=0}^{\infty} \alpha_k t^k.
\]

In recent years, several authors obtained many interesting results involving the related Bernoulli polynomials and Euler polynomials [5, 20–40]. And in [29], the authors studied some series identities involving the generalized Apostol type and related polynomials.

In this paper, we study some other properties of the generalized Apostol type polynomials \( F_n^{(\alpha)}(x; \lambda; u, v) \), including the recurrence relations, the differential equations and some connection problems, which extend some known results. As special, we obtain some properties of the generalized Apostol-Euler polynomials, the generalized Apostol-Bernoulli polynomials and Apostol-Genocchi polynomials of high order.

### 2 Recursion formulas and differential equations

From the generating function (1.14), we have

\[
\frac{\partial}{\partial x} F_n^{(\alpha)}(x; \lambda; u, v) = n F_{n-1}^{(\alpha)}(x; \lambda; u, v).
\]

A recurrence relation for the generalized Apostol type polynomials is given by the following theorem.

**Theorem 2.1** For any integral \( n \geq 1 \), \( \lambda \in \mathbb{C} \) and \( \alpha \in \mathbb{N} \), the following recurrence relation for the generalized Apostol type polynomials \( F_n^{(\alpha)}(x; \lambda; u, v) \) holds true:

\[
\left( \frac{\alpha v}{n + 1} - 1 \right) F_{n+1}^{(\alpha)}(x; \lambda; u, v) = \frac{\alpha \lambda}{2n} \cdot \frac{n!}{(n + v)!} F_{n+1}^{(\alpha+1)}(x + 1; \lambda; u, v) - x F_n^{(\alpha)}(x; \lambda; u, v).
\]
Proof. Differentiating both sides of (1.14) with respect to \( t \), and using some elementary algebra and the identity principle of power series, recursion (2.2) easily follows. \( \square \)

By setting \( \lambda := -\lambda, u = 0 \) and \( v = 1 \) in Theorem 2.1, and then multiplying \((-1)^n\) on both sides of the result, we have:

**Corollary 2.2** For any integral \( n \geq 1, \lambda \in \mathbb{C} \) and \( \alpha \in \mathbb{N} \), the following recurrence relation for the generalized Apostol-Bernoulli polynomials \( B_n^{(\alpha)}(x; \lambda) \) holds true:

\[
\left[ \alpha - (n + 1) \right] B_{n+1}^{(\alpha)}(x; \lambda) = \alpha \lambda B_{n+1}^{(\alpha)}(x + 1; \lambda) - x B_n^{(\alpha)}(x; \lambda). \tag{2.3}
\]

By setting \( u = 1 \) and \( v = 1 \) in Theorem 2.1, we have the following corollary.

**Corollary 2.3** For any integral \( n \geq 1, \lambda \in \mathbb{C} \) and \( \alpha \in \mathbb{N} \), the following recurrence relation for the generalized Apostol-Euler polynomials \( E_n^{(\alpha)}(x; \lambda) \) holds true:

\[
E_{n+1}^{(\alpha)}(x; \lambda) = x E_n^{(\alpha)}(x; \lambda) - \frac{\alpha \lambda}{2} E_{n+1}^{(\alpha)}(x + 1; \lambda). \tag{2.4}
\]

By setting \( u = 1 \) and \( v = 1 \) in Theorem 2.1, we have the following corollary.

**Corollary 2.4** For any integral \( n \geq 1, \lambda \in \mathbb{C} \) and \( \alpha \in \mathbb{N} \), the following recurrence relation for the generalized Apostol-Genocchi polynomials \( G_n^{(\alpha)}(x; \lambda; u, v) \) holds true:

\[
\left[ \alpha - (n + 1) \right] G_{n+1}^{(\alpha)}(x; \lambda; u, v) = \alpha \lambda G_{n+1}^{(\alpha)}(x + 1; \lambda) - 2(n + 1)x G_n^{(\alpha)}(x; \lambda). \tag{2.5}
\]

From (1.14) and (1.22), we know that the generalized Appostol type polynomials \( F_n^{(\alpha)}(x; \lambda; u, v) \) is Appell polynomials with

\[
A(t) = \left( \frac{2u^v}{\lambda e^t + 1} \right)^\alpha. \tag{2.6}
\]

From the Eq. (23) of [15], we know that \( G_0(1; \lambda) = 0 \). So from (2.6) and (1.12), we can obtain that if \( v = 0 \), we have

\[
A'(t) = \frac{\lambda \alpha}{2} \sum_{n=0}^{\infty} \frac{G_{n+1}(1; \lambda)}{n+1} t^n. \tag{2.7}
\]

By using (1.24) and (1.26), we can obtain the multiplicative and derivative operators of the generalized Appostol type polynomials \( F_n^{(\alpha)}(x; \lambda; u, v) \)

\[
\hat{M} = \left( x + \frac{\lambda u}{2} G_1(1; \lambda) \right) + \frac{\lambda \alpha}{2} \sum_{k=0}^{n-1} \frac{G_{n-k+1}(1; \lambda)}{(n-k+1)!} D_x^{n-k}, \tag{2.8}
\]

\[
\hat{P} = D_x. \tag{2.9}
\]

From (2.1), we can obtain

\[
\frac{\partial^n}{\partial x^n} F_n^{(\alpha)}(x; \lambda; u, v) = \frac{n!}{(n-p)!} F_{n-p}^{(\alpha)}(x; \lambda; u, v). \tag{2.10}
\]

Then by using (1.20), (2.8) and (2.10), we obtain the following result.
Theorem 2.5 For any integral n ≥ 1, λ ∈ C and α ∈ N, the following recurrence relation for the generalized Apostol type polynomials F_n^{(α)}(x; λ; u, 0) holds true:

\[
F_{n+1}^{(α)}(x; λ; u, 0) = \left(x + \frac{λα}{2} G_1(1; λ)\right) F_n^{(α)}(x; λ; u, 0) + \frac{λα}{2} \sum_{k=0}^{n-1} \binom{n}{k} \frac{G_{n-k+1}(1; λ)}{n-k+1} F_{n-k}^{(α)}(x; λ; u, 0).
\]

(2.11)

By setting u = 1 in Theorem 2.5, we have the following corollary.

Corollary 2.6 For any integral n ≥ 1, λ ∈ C and α ∈ N, the following recurrence relation for the generalized Apostol-Euler polynomials E_n^{(α)}(x; λ) holds true:

\[
E_{n+1}^{(α)}(x; λ) = \left(x + \frac{λα}{2} G_1(1; λ)\right) E_n^{(α)}(x; λ) + \frac{λα}{2} \sum_{k=0}^{n-1} \binom{n}{k} \frac{G_{n-k+1}(1; λ)}{n-k+1} E_{n-k}^{(α)}(x; λ).
\]

(2.12)

Furthermore, applying Lemma 1.7 to F_n^{(α)}(x; λ; u, 0), we have the following theorem.

Theorem 2.7 The generalized Apostol type polynomials F_n^{(α)}(x; λ; u, 0) satisfy the differential equation:

\[
\frac{λα}{2} \frac{G_n(1; λ)}{n!} y^{(n)} + \frac{λα}{2} \frac{G_{n-1}(1; λ)}{(n-1)!} y^{(n-1)} + \ldots
\]

\[
+ \frac{λα}{2} \frac{G_1(1; λ)}{2} y'' + \left(x + \frac{λα}{2} G_1(1; λ)\right) y' - ny = 0.
\]

(2.13)

Specially, by setting u = 1 in Theorem 2.7, then we have the following corollary.

Corollary 2.8 The generalized Apostol-Euler polynomials E_n^{(α)}(x; λ) satisfy the differential equation:

\[
\frac{λα}{2} \frac{G_n(1; λ)}{n!} y^{(n)} + \frac{λα}{2} \frac{G_{n-1}(1; λ)}{(n-1)!} y^{(n-1)} + \ldots
\]

\[
+ \frac{λα}{2} \frac{G_1(1; λ)}{2} y'' + \left(x + \frac{λα}{2} G_1(1; λ)\right) y' - ny = 0.
\]

(2.14)

3 Connection problems

From (1.14) and (1.28), we know that the generalized Apostol type polynomials F_n^{(α)}(x; λ; u, v) are an D_x-Appell polynomial set, where D_x denotes the derivative operator.

From Table 1 in [19], we know that the derivative operators of monomials x^n and the Gould-Hopper polynomials g_n^m(x, h) [30] are all D_x. And their transfer power series A(t) are 1 and e^{ht}, respectively.

Applying Lemma 1.7 to P_n(x) = x^n and Q_n(x) = F_n^{(α)}(x; λ; u, v), we have the following theorem.
Theorem 3.1

\[ F_n^{(\alpha)}(x; \lambda; u, v) = \sum_{m=0}^{n} \binom{n}{m} F_{n-m}^{(\alpha)}(\lambda; u, v)x^m, \]  

(3.1)

where \( F_n^{(\alpha)}(\lambda; u, v) \) is the so-called Apostol type numbers of order \( \alpha \) defined by (1.15).

By setting \( \lambda := -\lambda, u = 0 \) and \( v = 1 \) in Theorem 3.1, and then multiplying \((-1)^m\) on both sides of the result, we have the following corollary.

Corollary 3.2

\[ B_n^{(\alpha)}(x; \lambda) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m}^{(\alpha)}(\lambda)x^m, \]  

(3.2)

which is just Eq. (3.1) of [23].

By setting \( u = 0 \) and \( v = 0 \) in Theorem 3.1, we have the following corollary.

Corollary 3.3

\[ E_n^{(\alpha)}(x; \lambda) = \sum_{m=0}^{n} \binom{n}{m} E_{n-m}^{(\alpha)}(\lambda)x^m. \]  

(3.3)

By setting \( u = 1 \) and \( v = 1 \) in Theorem 3.1, we have the following corollary.

Corollary 3.4

\[ G_n^{(\alpha)}(x; \lambda) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}^{(\alpha)}(\lambda)x^m, \]  

(3.4)

which is just Eq. (24) of [15].

Applying Lemma 1.7 to \( P_n(x) = F_n(x; \lambda; u, v) \) and \( Q_n(x) = F_n^{(\alpha)}(x; \lambda; u, v) \), we have the following theorem.

Theorem 3.5

\[ F_n^{(\alpha)}(x; \lambda; u, v) = \sum_{m=0}^{n} \binom{n}{m} F_{n-m}^{(\alpha-1)}(\lambda; u, v)F_m(x; \lambda; u, v), \]  

(3.5)

where \( F_n^{(\alpha)}(\lambda; u, v) \) is the so-called Apostol type numbers of order \( \alpha \) defined by (1.15).

By setting \( \lambda := -\lambda, u = 0 \) and \( v = 1 \) in Theorem 3.5, and then multiplying \((-1)^m\) on both sides of the result, we have the following corollary.

Corollary 3.6

\[ B_n^{(\alpha)}(x; \lambda) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m}^{(\alpha-1)}(\lambda)B_m(x; \lambda), \]  

(3.6)

which is just Eq. (3.2) of [23].
By setting \( u = 1 \) and \( v = 0 \) in Theorem 3.5, we have the following corollary.

**Corollary 3.7**

\[
E_n^{(a)}(x; \lambda) = \sum_{m=0}^{n} \binom{n}{m} E_{n-m}^{(a-1)}(\lambda) E_m(x; \lambda). \tag{3.7}
\]

By setting \( u = 1 \) and \( v = 1 \) in Theorem 3.5, we have the following corollary.

**Corollary 3.8**

\[
G_n^{(a)}(x; \lambda) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}^{(a-1)}(\lambda) G_m(x; \lambda). \tag{3.8}
\]

Applying Lemma 1.7 to \( P_n(x) = g_m^{(a)}(x, h) \) and \( Q_n(x) = F_n^{(a)}(x; \lambda; u, v) \), we have the following theorem.

**Theorem 3.9**

\[
F_n^{(a)}(x; \lambda; u, v) = \sum_{r=0}^{n} \frac{n!}{r!} \left[ \sum_{k=0}^{\left\lceil \frac{(n-r)/m} \right\rceil} (-1)^k \frac{h^k}{k!(n-r-mk)!} F_n^{(a)}(\lambda; u, v) \right] g_{r}^{(a)}(x, h). \tag{3.9}
\]

By setting \( \lambda := -\lambda \), \( u = 0 \) and \( v = 1 \) in Theorem 3.9, and then multiplying \((-1)^a\) on both sides of the result, we have the following corollary.

**Corollary 3.10**

\[
B_n^{(a)}(x; \lambda) = \sum_{r=0}^{n} \frac{n!}{r!} \left[ \sum_{k=0}^{\left\lceil \frac{(n-r)/m} \right\rceil} (-1)^k \frac{h^k}{k!(n-r-mk)!} B_n^{(a)}(\lambda) \right] g_{r}^{(a)}(x, h), \tag{3.10}
\]

which is just Eq. (3.3) of [23].

By setting \( u = 1 \) and \( v = 0 \) in Theorem 3.9, we have the following corollary.

**Corollary 3.11**

\[
E_n^{(a)}(x; \lambda) = \sum_{r=0}^{n} \frac{n!}{r!} \left[ \sum_{k=0}^{\left\lceil \frac{(n-r)/m} \right\rceil} (-1)^k \frac{h^k}{k!(n-r-mk)!} E_n^{(a)}(\lambda) \right] g_{r}^{(a)}(x, h). \tag{3.11}
\]

By setting \( u = 1 \) and \( v = 1 \) in Theorem 3.9, we have the following corollary.

**Corollary 3.12**

\[
G_n^{(a)}(x; \lambda) = \sum_{r=0}^{n} \frac{n!}{r!} \left[ \sum_{k=0}^{\left\lceil \frac{(n-r)/m} \right\rceil} (-1)^k \frac{h^k}{k!(n-r-mk)!} G_n^{(a)}(\lambda) \right] g_{r}^{(a)}(x, h). \tag{3.12}
\]

When \( \omega = 1 \), applying Lemma 1.7 to \( P_n(x) = E_n^{(a-1)}(x; \lambda) \) and \( Q_n(x) = F_n^{(a)}(x; \lambda; u, v) \), we have the following theorem.
Theorem 3.13 If \( \nu \alpha = 1 \), then we have

\[
F_n^{(\alpha)}(x; \lambda; u, v) = \sum_{m=0}^{n} \binom{n}{m} 2^{(u-1)\alpha} G_{n-m}^{(\alpha-1)}(\lambda) E_{m}^{(\alpha-1)}(x; \lambda). 
\] (3.13)

By setting \( \lambda := -\lambda \), \( u = 0 \) and \( v = 1 \) in Theorem 3.13, and then multiplying \((-1)^{\nu}\) on both sides of the result, we have the following corollary.

Corollary 3.14

\[
B_n^{(\alpha)}(x; \lambda) = -\frac{1}{2} \sum_{m=0}^{n} \binom{n}{m} G_{n-m}^{(\alpha-1)}(\lambda) x^m. 
\] (3.14)

By setting \( u = 1 \) and \( v = 1 \) in Theorem 3.13, we have the following corollary.

Corollary 3.15

\[
G_n^{(\alpha)}(x; \lambda) = -\frac{1}{2} \sum_{m=0}^{n} \binom{n}{m} G_{n-m}^{(\alpha-1)}(\lambda) x^m, 
\] (3.15)

which is just the case of \( \alpha = 1 \) in (3.4).

When \( \nu = 1 \) or \( \alpha = 0 \), applying Lemma 1.7 to \( P_n(x) = G_n^{(\alpha-1)}(x; \lambda) \) and \( Q_n(x) = F_n^{(\nu)}(x; \lambda; u, v) \), we can obtain the following theorem.

Theorem 3.16 If \( \nu = 1 \) or \( \alpha = 0 \), we have

\[
F_n^{(\alpha)}(x; \lambda; u, v) = \sum_{m=0}^{n} \binom{n}{m} 2^{(u-1)\alpha} G_{n-m}^{(\alpha-1)}(\lambda) E_{m}^{(\alpha-1)}(x; \lambda). 
\] (3.16)

By setting \( \lambda := -\lambda \), \( u = 0 \) and \( v = 1 \) in Theorem 3.13, and then multiplying \((-1)^{\nu}\) on both sides of the result, we have the following corollary.

Corollary 3.17

\[
\mathcal{B}_n^{(\alpha)}(x; \lambda) = \sum_{m=0}^{n} \binom{n}{m} \left( -\frac{1}{2} \right)^{\alpha} G_{n-m}^{(\alpha-1)}(\lambda) E_{m}^{(\alpha-1)}(x; \lambda). 
\] (3.17)

When \( \alpha = 1 \) in (3.17), it is just (3.15).

By setting \( u = 1 \) and \( v = 1 \) in Theorem 3.16, we have the following corollary.

Corollary 3.18

\[
\mathcal{G}_n^{(\alpha)}(x; \lambda) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}^{(\alpha-1)}(\lambda) E_{m}^{(\alpha-1)}(x; \lambda), 
\] (3.18)

which is equal to (3.8).
If $\alpha = 0$ in Theorem 3.16, we have:

**Corollary 3.19**

\[ x^n = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}(\lambda) G_m^{(-1)}(x; \lambda). \]  

(3.19)

### 4 Hermite-based generalized Apostol type polynomials

Finally, we give a generation of the generalized Apostol type polynomials.

The two-variable Hermite-Kampé de Fériet polynomials (2VHKdFP) $H_n(x, y)$ are defined by the series [31]

\[ H_n(x, y) = n! \sum_{r=0}^{[n/2]} \frac{x^{n-2r} y^r}{r!(n-2r)!} \]  

(4.1)

with the following generating function:

\[ \exp \left( xt + yt^2 \right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y). \]  

(4.2)

And the 2VHKdFP $H_n(x, y)$ are also defined through the operational identity

\[ \exp \left( y \frac{\partial^2}{\partial x^2} \right) [x^n] = H_n(x, y). \]  

(4.3)

Acting the operator $\exp (y \frac{\partial^2}{\partial x^2})$ on (1.14), and by the identity [32]

\[ \exp \left( y \frac{\partial^2}{\partial x^2} \right) \left[ \exp (-ax^2 + bx) \right] = \frac{1}{\sqrt{1 + 4ay}} \exp \left( \frac{-ax^2 - bx - b^2 y}{1 + 4ay} \right), \]  

(4.4)

we define the Hermite-based generalized Apostol type polynomials $H_n^{(\alpha)}(x, y; \lambda; u, v)$ by the generating function

\[ \left( \frac{2ue^t}{\lambda e^t + 1} \right)^\alpha \cdot e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x, y; \lambda; u, v) \frac{t^n}{n!} \quad (|t| < |\log (-\lambda)|). \]  

(4.5)

Clearly, we have

\[ H_n^{(\alpha)}(x, y; \lambda; u, v) = H_n^{(I)}(x, y; \lambda; u, v). \]

From the generating function (4.5), we easily obtain

\[ \frac{\partial}{\partial x} H_n^{(\alpha)}(x, y; \lambda; u, v) = n H_n^{(\alpha)}(x, y; \lambda; u, v) \]  

(4.6)

and

\[ \frac{\partial}{\partial y} H_n^{(\alpha)}(x, y; \lambda; u, v) = n(n-1) H_{n-2}^{(\alpha)}(x, y; \lambda; u, v), \]  

(4.7)
which can be combined to get the identity
\[
\frac{\partial^2}{\partial x^2} H_n F_n^{(a)} (x, y; \lambda; u, v) = \frac{\partial}{\partial y} H_n F_n^{(a)} (x, y; \lambda; u, v). \tag{4.8}
\]

Acting with the operator \( \exp y \frac{\partial^2}{\partial x^2} \) on both sides of (3.1), (3.5), (3.13), (3.18), and by using (4.3), we obtain
\[
H_n F_n^{(a)} (x; \lambda; u, v) = \sum_{m=0}^{n} \binom{n}{m} F_n^{(a-1)} (\lambda; u, v) H_m (x), \tag{4.9}
\]
\[
H_n G_n^{(a)} (x; \lambda) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m} (\lambda) H_m^{(a-1)} (x), \quad \text{where } \nu \neq 1, \tag{4.10}
\]
\[
H_n F_n^{(a)} (x; \lambda; u, v) = \sum_{m=0}^{n} \binom{n}{m} \frac{2^{(a-1)} \alpha}{m!} G_{n-m} (\lambda) H_m (x), \tag{4.11}
\]
\[
H_n G_n^{(a)} (x; \lambda) = \sum_{m=0}^{n} \binom{n}{m} \frac{2^{(a-1)} \alpha}{m!} H_m (x), \quad \text{where } \nu = 1 \text{ or } \alpha = 0, \tag{4.12}
\]

where \( H_n F_n^{(a)} (x; \lambda) \) and \( H_n G_n^{(a)} (x; \lambda) \) are the Hermite-based generalized Apostol-Euler polynomials and the Hermite-based generalized Apostol-Genocchi polynomials respectively, defined by the following generating functions:
\[
\left( \frac{2 \alpha}{\lambda e^t + 1} \right)^\alpha e^{xt+t^2} = \sum_{n=0}^{\infty} H_n F_n^{(a)} (x; \lambda) \frac{t^n}{n!}, \quad (|t| < |\log (-\lambda)|), \]
\[
\left( \frac{2t}{\lambda e^t + 1} \right)^\alpha e^{xt+t^2} = \sum_{n=0}^{\infty} H_n G_n^{(a)} (x; \lambda) \frac{t^n}{m!}, \quad (|t| < |\log (-\lambda)|). \]