Congruences for higher-order Euler numbers

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Abstract: In this paper, we prove some congruences for higher-order Euler numbers.

Key words: Higher-order Euler numbers; Euler numbers, congruences.

1. Introduction and results. For an integer \( k \), the Euler number \( E_{2n}^{(k)} \) of order \( k \) \((k = \text{index} \text{ may be} \text{ negative})\) is defined by (see [2, 5])

\[
(\sec x)^k = \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(k)} \frac{x^{2n}}{(2n)!},
\]

or equivalently

\[
\left( \frac{2}{e^x + e^{-x}} \right)^k = \sum_{n=0}^{\infty} E_{2n}^{(k)} \frac{x^{2n}}{(2n)!}.
\]

The numbers \( E_{2n}^{(1)} = E_{2n} \) are the ordinary Euler numbers. By (1.1) or (1.2), we can get

\[
E_{2n}^{(k)} = (2n)! \sum_{v_1 + v_2 + \ldots + v_k = n, v_1 \geq 0, v_2 \geq 0, \ldots, v_k \geq 0} \frac{E_{2v_1} E_{2v_2} \cdots E_{2v_k}}{(2v_1)!(2v_2)!(2v_k)!}
\]

when \( k \) is positive.

The Euler numbers \( E_{2n} \) satisfy the recurrence relation

\[
E_0 = 1, \quad E_{2n} = -\sum_{k=0}^{n-1} \binom{2n}{2k} E_{2k}.
\]

By induction, all the Euler numbers \( E_0, E_2, E_4, \ldots \) are integers. By (1.3), we know \( E_{2n}^{(k)} \) is an integer.

Recently, several researchers have studied the congruences for Euler numbers. For example:

In [7], Wenpeng Zhang obtained an interesting congruence for Euler numbers,

\[
E_{p-1} \equiv 1 + (-1)^{(p+1)/2} \pmod{p},
\]

where \( p \) is any odd prime.

In [4], Guodong Liu obtained an congruence for Euler numbers,

\[
\sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(r)} \equiv (1)^{(p+1)/2} \left( \frac{2k}{k} \right) \pmod{p}.
\]

Taking \( r = 2 \) in Theorem 2, we may immediately deduce the following

**Corollary 1.** Let \( n \geq 0 \) be any integers, \( p \) be any odd prime. Then we have

\[
\sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(2)} \equiv (-1)^{(p+1)/2} \pmod{p}.
\]

Setting \( p = 3, 5, 7, 11 \) in Corollary 1, we can get

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(1.10) $E_{2n+2}^{(2)} \equiv 1 \pmod{3}$.

(1.11) $E_{2n+4}^{(2)} + E_{2n+4}^{(2)} \equiv -1 \pmod{5}$.

(1.12) $E_{2n+6}^{(2)} + E_{2n+4}^{(2)} + E_{2n+6}^{(2)} \equiv 1 \pmod{7}$.

(1.13) $E_{2n+2}^{(2)} + E_{2n+4}^{(2)} + E_{2n+6}^{(2)} \equiv 1 \pmod{11}$.

2. Some lemmas.

Lemma 1. Let $n \geq 1$, $k \geq 1$ be integers. Then we have

(2.1) $E_{2n}^{(k)} \equiv 0 \pmod{k}$.

Proof. By (1.1), we have

$$
\sum_{n=1}^{\infty} (-1)^n E_{2n}^{(k)} \frac{x^{2n-1}}{(2n-1)!} = k (\sec x)^k \tan x.
$$

By $(\sec x)^2 = \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(2)}(x^{2n}/((2n)!)) = \sum_{n=1}^{\infty} (-1)^{n-1} E_{2n-2}^{(2)}(x^{2n-2}/((2n-2)!))$, we get

(2.2) $\tan x = \sum_{n=1}^{\infty} (-1)^{n-1} E_{2n-2}^{(2)} \frac{x^{2n-1}}{(2n-1)!}$.

By (2.2) and (2.3), we have

$$
\sum_{n=1}^{\infty} (-1)^n E_{2n}^{(k)} \frac{x^{2n-1}}{(2n-1)!} = k \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(k)} \frac{x^{2n}}{(2n)!} \times \sum_{n=1}^{\infty} (-1)^{n-1} E_{2n-2}^{(2)} \frac{x^{2n-1}}{(2n-1)!}.
$$

(2.4) $E_{2n}^{(k)} = k \sum_{n=1}^{\infty} (-1)^{n-1} \sum_{i=0}^{2n-1} \left( \frac{2n-1}{2i} \right) E_{2i}^{(k)} E_{2n-2i-2}^{(2)} \frac{x^{2n-1}}{(2n-1)!}.

Comparing the coefficients of $x^{2n-1}$ on both sides of (2.4), we get

(2.5) $E_{2n}^{(k)} = -k \sum_{i=0}^{2n-1} \left( \frac{2n-1}{2i} \right) E_{2i}^{(k)} E_{2n-2i-2}^{(2)} \equiv 0 \pmod{k}$.

This completes the proof of Lemma 1. □

Lemma 2. Let $n \geq 0$, $k \geq 1$, $m \geq 0$ be integers. Then we have

(2.6) $E_{2n}^{(k+m)} \equiv E_{2n}^{(k)} \pmod{m}$.

Proof. By (1.1), we have

(2.7) $\sum_{n=0}^{\infty} (-1)^n E_{2n}^{(k+m)} \frac{x^{2n}}{(2n)!} = (\sec x)^{k+m}$

$= (\sec x)^k (\sec x)^m$

$= \left( \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(k)} \frac{x^{2n}}{(2n)!} \right) \left( \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(m)} \frac{x^{2n}}{(2n)!} \right)$

$= \sum_{n=0}^{\infty} (-1)^n \sum_{j=0}^{n} \frac{2n}{2j} E_{2j}^{(k)} E_{2n-2j}^{(m)} \frac{x^{2n}}{(2n)!}$.

Comparing the coefficients of $x^{2n}$ on both sides of (2.7), we get

(2.8) $E_{2n}^{(k+m)} = \sum_{j=0}^{n} \frac{2n}{2j} E_{2j}^{(k)} E_{2n-2j}^{(m)}$.

By (2.8) and Lemma 1, we have

(2.9) $E_{2n}^{(k+m)} \equiv E_{2n}^{(k)} \pmod{m}$.

This completes the proof of Lemma 2. □

Lemma 3. Let $n \geq 1$, $k \geq 1$, $m \geq 0$ be integers. Then we have

(2.10) $E_{2n}^{(k)} \equiv \frac{1}{2^n} \sum_{i=0}^{m} \left( \frac{m}{i} \right) (m-2i)^{2n} \pmod{(m+k)}$.

Proof. By (1.1), we have

$$
\sum_{n=0}^{\infty} (-1)^n E_{2n}^{(k+m)} \frac{x^{2n}}{(2n)!} = (\sec x)^k
$$

$= (\sec x)^k (\sec x)^m$

(2.11) $= \left( \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(m+k)} \frac{x^{2n}}{(2n)!} \right)$

$= \sum_{n=0}^{\infty} (-1)^n \sum_{j=0}^{n} \frac{2n}{2j} E_{2j}^{(m+k)} E_{2n-2j}^{(m)} \frac{x^{2n}}{(2n)!}$.

Comparing the coefficients of $x^{2n}$ on both sides of (2.11), we get
By (2.12) and Lemma 1, we have

$$E_{2m}^{(k)} \equiv E_{2m}^{(-m)} \pmod{(m + k)}.$$  

(2.13)

On the other hand, by (1.2), we have

$$\sum_{n=0}^{\infty} E_{2n}^{(-m)} x^{2n} (2n)! = \left( e^x + e^{-x} \right)^m 2^{-m} \sum_{i=0}^{m} \binom{m}{i} (m-2i)^2 x^i$$

Comparing the coefficients of $x^{2n}$ on both sides of (2.14), we get

$$E_{2n}^{(k)} = 2^{-m} \sum_{i=0}^{m} \binom{m}{i} (m-2i)^2 n!.$$  

(3.2)

By (2.13) and (3.2), we immediately obtain (2.10).

This completes the proof of Lemma 3. □

3. Proof of the theorems.

Proof of Theorem 1. By Lemma 2 and Lemma 3, we have

$$\sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(r)} \equiv \sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(2k+2)} \equiv \frac{1}{2p-2k-2} \sum_{j=0}^{p-2k-1} \left( p - 2k - 1 \right) j$$

(3.1)

$$\times \sum_{i=1}^{(p-1)/2} (p - 2k - 1 - 2j)^{2n+2i} \equiv \frac{1}{2p-2k-2} \sum_{j=0}^{p-2k-1} \left( p - 2k - 1 \right) j$$

$$\times (p - 2k - 1 - 2j) 2^{2n+2} \times (p - 2k - 1 - 2j)^{p-1} - 1 \equiv 0 \pmod{p}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. By Lemma 2 and Lemma 3, we have

$$\sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(r)} \equiv \sum_{i=1}^{(p-1)/2} E_{2n+2i}^{(2k+2)}$$

$$\equiv \frac{1}{2p-2k-2} \sum_{j=0}^{p-2k-2} \left( p - 2k - 2 \right) j$$

$$\times \sum_{i=1}^{(p-1)/2} (p - 2k - 2 - 2j)^{2n+2i} \equiv \frac{1}{2p-2k-2} \sum_{j=0}^{p-2k-2} \left( p - 2k - 2 \right) j$$

$$\times (p - 2k - 2 - 2j)^{2n+2} \times (p - 2k - 2 - 2j)^{p-1} - 1 \equiv -2^{2k} \left( p - 2k - 2 \right) \left( p - 2k - 1 \right)^2 \left( p - 2k - 3 / 2 \right)$$

$$\equiv -2^{2k} \left( p - 2k - 1 \right) \left( p - 2k - 1 / 2 \right) \left( p - 2k - 3 / 2 \right)$$

$$\equiv (2k)^{(p+1)/2} \left( 2k / k \right) \pmod{p}.$$
References


