Asymptotic Behaviour of a $q$-Binomial Type Distribution Based on $q$-Krawtchouk Orthogonal Polynomials

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ABSTRACT

In this manuscript, we introduce for $0 < q < 1$ a new deformed $q$-binomial distribution by virtue of the normalized $q$-Krawtchouk orthogonal polynomials. We call this $q$-discrete probability distribution $q$-binomial distribution of type I, say $B(n, p, q)$ with $p > 0$. Interpretation of this $q$-discrete probability distribution is provided by considering, for suitable values of $q$, $n$-dimensional vector spaces over the finite field of $1/q$ elements. Furthermore, we study its asymptotic behaviour for $n \to \infty$ by showing that is compatible with a standardized Stieltjes-Wigert distribution.

1. INTRODUCTION

Let $v_n$ be a discrete probability measure in $S = \{x_1, i = 1, 2, \ldots, n; n = 0, 1, \ldots\}$ with finite moments of all orders. Then it is well known that their exist a sequence of normalized discrete orthogonal polynomials $\{P_m(x)\}$ with respect to the measure $v_n$ satisfying the orthogonality relation

$$\sum_{x \in S} v_n(x) P_m(x) P_\nu(x) = \lambda_{m, \nu} \delta_{m, \nu},$$

where $\delta_{m, \nu}$ the Kronecker delta and $\lambda_{m, \nu}$ a non-negative sequence of $n$ and $\nu$ and the three term recurrence relation

$$xp_m(x) = P_{m+1}(x) + a_m P_m(x) + b_m P_{m-1}(x) \quad (m \geq 1),$$

where $a_m \in \mathbb{R}$ and $b_m > 0$ and with initial conditions $P_0(x) = 1$ and $P_1(x) = x - a_0$. Conversely, Favard's theorem ensures the existence of a discrete probability measure $v_n$ on $S$ for which the sequence of polynomials
determined by the recurrence relation (2) are orthogonal. The mean value and the variance of the discrete random variable $X$ in the discrete spectrum $S$ with probability function $v_n(x)$ are given respectively by $\mu = a_0$ and $\sigma^2 = b_1$. If $a_m = 0$ then all moments of odd order are zero. (see Saitoh and Yoshida (2000), Christiansen (2004)).

Saitoh and Yoshida (2000) introduced a $q$-deformed binomial distribution for $0 < q < 1$, by virtue of a $q$-deformed sequence of Krawtchouk orthogonal polynomials and studied its asymptotic behaviour by showing that is compatible with a $q$-deformed Gaussian distribution in a quantum probability space.

Various $q$-analogues for $0 < q < 1$, of the classical binomial distribution have also been studied by many authors. Among them we refer to Kemp (1992, 2002), Sicong (1994) and Charalambides (2005).

In this manuscript, we introduce for $0 < q < 1$ a new deformed $q$-binomial distribution by virtue of the normalized $q$-Krawtchouk orthogonal polynomials. Interpretation of this $q$-discrete probability distribution is provided by considering, for suitable values of $q$, $n$-dimensional vector spaces over the finite field of $1/q$ elements. Furthermore, we study its asymptotic behaviour for $n \to \infty$ by showing that is compatible with a standardized Stieltjes-Wigert distribution.

2. INTRODUCTORY DEFINITIONS AND NOTATIONS

For the needs of this manuscript we recall some usual definitions and notation used in $q$-analysis (see Koekoek and Swarttouw (1998)). Let $0 < q < 1$, $x$ a real number and $k$ a positive integer. The $q$-shifted factorial is defined by

$$(a; q)_n := \prod_{j=1}^{n} (1 - aq^{j-1})$$

and the general $q$-shifted factorial is given by

$$(a; q)_\infty := \prod_{j=1}^{\infty} (1 - aq^{j-1}), \quad (a; q)_0 = 1.$$ 

Also,

$$(a, b; q)_n := (a; q)_n (b; q)_n.$$ 

The $q$-number is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}$$

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and the $q$-binomial coefficient by

$$\left(\begin{array}{c} n \\ k \end{array}\right)_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

or

$$\left(\begin{array}{c} n \\ k \end{array}\right)_{1/q} = q^{-k(n-k)} \left(\begin{array}{c} n \\ k \end{array}\right)_q.$$

Also

$$(-t; q)_n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right)_q q^{\binom{k}{2}} t^k.$$

3. $q$-BINOMIAL DISTRIBUTION OF TYPE I–ASYMPTOTIC BEHAVIOUR

Let $0 < q < 1$ be a rational number such that $1/q$ be an integer larger than 1. Also, let $V_n$ be an $n$-dimensional vector space over the finite field $GF(1/q)$ of $1/q$ elements. Then $V_n$ contains a total of $1/q^n$ vectors and the number of $x$-dimensional subspaces of $V_n$, $x = 0, 1, \ldots, n$ is given by $\binom{n}{x}_{1/q}$ (see Exton(1983)).

Supposing that each $x$-dimensional subspace assigns a weight, say $g_q(x; V_n)$, with

$$g_q(x; V_n) = q^{\binom{x+1}{2}} p^{-x}, \quad p > 0, \quad x = 0, 1, \ldots, n$$

we have that the probability of appearance of a $x$-dimensional subspace of $V_n$ is given by

$$f_X(x) = \frac{\binom{n}{x}_{1/q} q^{\binom{x+1}{2}} p^{-x}}{\sum_{x=0}^{n} \binom{n}{x}_{1/q} q^{\binom{x+1}{2}} p^{-x}} \quad (3)$$

or

$$f_X(x) = \left(\begin{array}{c} n \\ x \end{array}\right)_q q^{\binom{x}{2}} (p^{-1} q^{-n})^x \prod_{i=1}^{n} (1 + p^{-1} q^{i-n-1} - 1), \quad x = 0, 1, \ldots, n, \quad (4)$$

with $0 < q < 1$, $p > 0$.

Next, we define the following deformation of the $q$-analogue binomial distribution (4) considering the random variable $Y = q^{-X}$ with probability function

$$u_Y(y) = \frac{\binom{n}{\ln q}_{\ln q} q^{\frac{\ln q}{2}} (\ln q)}{\prod_{i=1}^{n} (1 + \alpha_n q^{i-1})}, \quad y = q^0, q^{-1}, \ldots, q^{-n}, \quad (5)$$
where
\[ \alpha_n = p^{-1}q^{-n}, \quad p > 0 \] (6)
such that
\[ \alpha_n > 1, \quad \alpha_n q^n \to 0 \] (7)
and
\[ \alpha_n \to \infty \text{ as } n \to \infty. \] (8)

**Proposition 3.1.** The probability function \( u_Y(y) \) is induced by the normalized \( q \)-Krawtchouk orthogonal polynomials, say \( p_m(y; p, n, q) \), \( 0 < q < 1 \) given by
\[ p_m(y; p, n, q) = \frac{(q^{-n}; q)_m}{(-pq^n; q)_m} K_m(y; p, n, q), \] (9)
where \( y = q^{-x}, \quad x = 0, 1, \ldots, n \) and \( K_m(y; p, n, q) \) the \( q \)-Krawtchouk orthogonal polynomials with parameter \( p = \alpha_n^{-1}q^{-n} \).
Moreover the random variable \( Y \) has the following \( r \)th order moments
\[ \mu_r = \sum_{y \in S_1} y^r u_Y(y) = q^{-rn} \frac{\prod_{i=1}^r (1 + \frac{1}{\alpha_n^{-1}q^{i}})}{\prod_{i=1}^r (1 + \frac{1}{\alpha_n^{-1}q^{-n+i}})}. \] (10)

**Proof.** The normalized \( q \)-Krawtchouk orthogonal polynomials \( p_m(q^{-x}; p, n, q) \), \( 0 < q < 1 \) with parameter \( p = \alpha_n^{-1}q^{-n} \) satisfies the orthogonality relation
\[ \sum_{x=0}^n \frac{(q^{-n}; q)_x}{(q; q)_x} (-1)^x \alpha_n^x q^{nx} p_m(q^{-x}; p, n, q) p_x(q^{-x}; p, n, q) = \gamma_{n, \nu}(1 + \alpha_n^{-1}q^{-n})(-\alpha_n^{-1}q^{-n+1}; q)_n \alpha_n^\nu q^{n^2} q^{-\binom{n+1}{2}} \delta_{m, \nu}, \]
where \( \delta_{m, \nu} \) the Kronecker delta and \( \gamma_{n, \nu} \) a non-zero sequence of \( n \) and \( \nu \) (see Koekoek and Swarttouw(1998)). Thus their weight function, say \( u(x; n, q) \), can be written as
\[ u(x; n, q) = \frac{(q^{-n}; q)_x}{(q; q)_x} (-1)^x \alpha_n^x q^{nx} \]
\[ \cdot \frac{1 + \alpha_n^{-1}q^{-n})(-\alpha_n^{-1}q^{-n+1}; q)_n \alpha_n^\nu q^{n^2} q^{-\binom{n+1}{2}}} {1 + \alpha_n^{-1}q^{-n+1}; q)_n \alpha_n^\nu q^{n^2} q^{-\binom{n+1}{2}}}. \] (11)

Using the identities
\[ \frac{(q^{-n}; q)_x}{(q; q)_x} = (-1)^x q^{-nx} \alpha_n^{-x} \begin{pmatrix} n \\ x \end{pmatrix} q^{\binom{x}{2}}. \]
and

\[(1 + \alpha_n^{-1} q^{-n})(-\alpha_n^{-1} q^{-n+1}; q)_n = \alpha_n^{-n} q^{-\frac{n^2}{2}} \prod_{i=1}^n (1 + \alpha_n q^i)\]

we have from (11)

\[u(x; n, q) = u_Y(y), \quad y = q^{-x}.\]

Moreover, by the probability function (5) the rth moment of the random variable Y is

\[\mu_r = \sum_{y \in S_1} y^r u_Y(y)\]

\[= \sum_{y \in S_1} y^r q^{\left(\frac{\ln y}{\ln q}\right)} q^{\left(\frac{\ln q}{2}\right)} \alpha_n^{-\frac{\ln n}{\ln q}} \prod_{i=1}^n (1 + \alpha_n q^{i-1})\]

\[= \frac{\sum_{x=0}^n q^{-x} q^{\left(\frac{n-x}{2}\right)} q^{\left(\frac{x}{2}\right)} \alpha_n^{-x}}{\prod_{i=1}^n (1 + \alpha_n q^{i-1})} = \frac{\prod_{i=1}^n (1 + \alpha_n q^{i-r-1})}{\prod_{i=1}^n (1 + \alpha_n q^{i-1})}\]

\[= \frac{(1 + \alpha_n q^{-r})(1 + \alpha_n q^{-r+1}) \cdots (1 + \alpha_n^{-1} q^{-1}) \cdots (1 + \alpha_n q^{n-r-1})}{(1 + \alpha_n q^{-1}) \cdots (1 + \alpha_n q^{n-r+1})(1 + \alpha_n q^{n-r}) \cdots (1 + \alpha_n q^{n-1})} \quad (12)\]

From the last expression the equation (10) is obtained.

**Definition 3.1.** For \(0 < q < 1\), the q-discrete probability distribution with probability function \(u_Y(y)\) defined in the spectrum \(S_1 = \{ q^{-k}, k = 0, 1, \ldots, n \} \) and based on normalized q-Krawtchouk orthogonal polynomials is called q-Binomial of type I distribution and is denoted by \(B(n, p, q)\).

**Remark 3.1.** Using the moments of rth order (10) the random variable Y of the q-binomial of type I distribution has respectively the following mean value and variance

\[\mu_Y = q^{-n} \frac{1 + \alpha_n^{-1} q}{1 + \alpha_n^{-1} q^{-n+1}} \quad (13)\]

and

\[\sigma_Y^2 = q^{-2n+1} \alpha_n^{-1} \frac{(1 + \alpha_n^{-1} q)(q^{-n} - 1)(1 - q)}{(1 + \alpha_n^{-1} q^{-n+1})^2 (1 + \alpha_n^{-1} q^{-n+2})} \quad (14)\]
Equivalently, since the \( q \)-binomial of type I distribution is induced from the normalized \( q \)-Krawtchouk orthogonal polynomials \( p_m(y; p, n, q) \), \( y = q^{-x} \), \( p = \alpha_n^{-1} q^{-n} \), the mean value and variance of the random variable \( Y \) can be derived from the recurrence relation of these polynomials. Analytically, the recurrence relation of the orthogonal polynomials \( p_m(y; p, n, q) \) with \( p = \alpha_n^{-1} q^{-n} \) is

\[
y p_m(y; p, n, q) = p_{m+1}(y; p, n, q) + a_m^{(p, n, q)} p_m(y; p, n, q) + b_m^{(p, n, q)} p_{m-1}(y; p, n, q) \quad (m \geq 1)
\]

(15)

with initial conditions

\[
p_0(y; p, n, q) = 1, \quad p_1(y; p, n, q) = y - a_0,
\]

where

\[
a_m^{(p, n, q)} = 1 - (A_m + C_m), \quad b_m^{(p, n, q)} = A_{m-1} C_m
\]

(16)

and

\[
A_m = \frac{(1 - q^{m-n})(1 + \alpha_n^{-1} q^{m-n})}{(1 + \alpha_n^{-1} q^{2m-n})(1 + \alpha_n^{-1} q^{2m-n+1})}
\]

\[
C_m = -\alpha_n^{-1} q^{2m-2n-1} \frac{(1 + \alpha_n^{-1} q^{m-n})(1 - q^m)}{(1 + \alpha_n^{-1} q^{2m-n-1})(1 + \alpha_n^{-1} q^{2m-n})}
\]

(17)

(see Koekoek and Swarttouw (1998)).

So,

\[
\mu_Y = 1 - A_0 - C_0, \quad \sigma_Y^2 = A_0 C_1
\]

and on using (17) we recapture (13) and (14).

**Theorem 3.1.** The limit distribution for \( n \to \infty \) of the standardized \( q \)-binomial of type I distribution \( B(n, p, q) \) is the standardized Stieltjes-Wigert distribution with probability density function

\[
u^SW_q(y) = \frac{q^{-11/8}(q^{-3/2}(1-q)^{1/2}y + q^{-1})^{-1/2}}{\sqrt{2\pi \log q^{-1}}} e^{\frac{(\log(q^{-3/2}(1-q)^{1/2}y+q^{-1}))^2}{2\log q^{-1}}}, \quad y \geq -q^{1/2}(1-q)^{-1/2}.
\]

(18)

**Proof.** Shifting so as to have zero mean in the \( q \)-Binomial of type I distribution with mean value \( \mu_Y \), its orthogonal polynomials are \( p_m(y + \mu_Y; p, n, q) \),
$p = \alpha_n^{-1}q^{-n}$. The coefficients of the recurrence relation of $p_m(y + \mu Y; p, n, q)$ are represented as $a_{m}^{(n,p,q)} - \mu Y = A_0 + C_0 - (A_m + C_m)$ and $b_{m}^{(n,p,q)} = A_{m-1}C_m$. In addition, to standardized so as to be of variance 1, it can be realized by replacing the recurrence relation coefficients respectively by

$$\frac{a_{m}^{(n,p,q)} - \mu Y}{\sigma_Y} = \frac{A_0 + C_0 - (A_m + C_m)}{\sqrt{A_0C_1}}$$  \hspace{1cm} (19)$$

and

$$\frac{b_{m}^{(n,p,q)}}{\sigma_Y^2} = \frac{A_{m-1}C_m}{A_0C_1},$$ \hspace{1cm} (20)

where $A_m$ and $C_m$ are defined in (17). Thus the orthogonal polynomials, say $Q_m(y; p, q, n), p = \alpha_n^{-1}q^{-n}$, of the standardized $q$-binomial of type I distribution are determined by the recurrence relation

$$yQ_m(y; p, q, n) = Q_{m+1}(y; p, q, n) + \frac{A_0 + C_0 - (A_m + C_m)}{\sqrt{A_0C_1}}Q_m(y; p, q, n) + \frac{A_{m-1}C_m}{A_0C_1}Q_{m-1}(y; p, q, n), \quad (m \geq 1)$$ \hspace{1cm} (21)

with initial conditions

$$Q_0(y; p, q, n)(y) = 1 \quad Q_1(y; p, q, n)(y) = y.$$ 

Using (13), (14) and (17) the coefficients (19) and (20) respectively become

$$\frac{a_{m}^{(n,p,q)} - \mu Y}{\sigma_Y} = \frac{1 - (1-q^{m-n})(1+\alpha_n^{-1}q^{m-n})(1+\alpha_n^{-1}q^{m-n+1})(1+\alpha_n^{-1}q^{2m-n})(1+\alpha_n^{-1}q^{2m-n+1})(1-q^m)}{(1+\alpha_n^{-1}q^{m-n})(1+\alpha_n^{-1}q^{2m-n})(1+\alpha_n^{-1}q^{2m-n+1})(1-q^m)} \quad (22)$$

and

$$\frac{b_{m}^{(n,p,q)}}{\sigma_Y^2} = q^{2n-2}(1-q^{m-n})(1+\alpha_n^{-1}q^{m-n})(1+\alpha_n^{-1}q^{m-n+1})(1+\alpha_n^{-1}q^{2m-n})(1+\alpha_n^{-1}q^{2m-n+1})(1-q^m).$$ \hspace{1cm} (23)
Taking the limit $n \to \infty$ of (22) and (23) and using (7) and (8), we obtain the recurrence relation of the orthogonal polynomials, say $Q_m(y)$, for the limit distribution, as

$$yQ_m(y) = Q_{m+1}(y) + q^{3/2}(1-q)^{-1/2}(q^{-2m+1}(1 + q - q^{m+1}) - q^{-1})Q_m(y) + q^{-4m+1}(1-q)(1-q^m)Q_{m-1}(y), \quad (m \geq 1) \quad (24)$$

with initial conditions

$$Q_0(y) = 1 \quad Q_1(y) = y.$$ 

The continuous Stieltjes-Wigert distribution with probability density function

$$p_q^{SW}(x) = \frac{q^{1/8}}{\sqrt{2\pi \log q^{-1}}} \frac{1}{e^{(\log x)^2/2}}, \quad x > 0 \quad (25)$$

with

$$\mu = q^{-1}, \quad \sigma^2 = q^{-3}(1-q), \quad (26)$$

is based on the normalized Stieltjes-Wigert orthogonal polynomials, say $p_m^{SW}(x)$, satisfying the recurrence relation

$$xp_m^{SW}(x) = p_{m+1}^{SW}(x) + q^{-2m+1}(1 + q - q^{m+1})p_m^{SW}(x) + q^{-4m+1}(1-q^m)p_{m-1}^{SW}(x), \quad (m \geq 1) \quad (27)$$

with initial conditions

$$p_0^{SW}(x) = 1, \quad p_1^{SW}(x) = x - q^{-1}$$

(see Christiansen (2003)).

By standardizing (27) and (25) we obtain respectively (24) and (18) which complete our proof.

ΠΕΡΙΛΗΨΗ

Στην εργασία αυτή, εισάγεται για $0 < q < 1$ μία μετασχηματισμένη $q$-διανυσματική κατανομή μέσω των $q$-κανονικοποιημένων $q$-Krawtchouk ορθογώνιων πολυώνυμων. Η $q$-διακριτή αυτή κατανομή ονομάζεται $q$-διανυσματική κατανομή τύπου I και συμβολίζεται με $B(n, p, q)$ με $p > 0$. Επίσης, δίδεται μία ερμηνεία αυτής της $q$-διακριτής κατανομής, για κατάλληλες τιμές του $q$, $n$—διάστατους διανυσματικούς
REFERENCES


