Generalized Hahn’s theorem

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Abstract

Let \( \{ P_n(x) \}_{n=0}^{\infty} \) be an orthogonal polynomial system and

\[
L[\cdot] = \sum_{i=0}^{k} a_i(x) D^i \quad \left( D = \frac{d}{dx} \right)
\]

a linear differential operator of order \( k \geq 0 \) with polynomial coefficients. We find necessary and sufficient conditions for a polynomial sequence \( \{ Q_n(x) \}_{n=0}^{\infty} \) defined by \( Q_n(x) := L[P_n^r(x)], n \geq 0 \), to be also an orthogonal polynomial system. We also give a few applications of this result together with the complete analysis of the cases: (i) \( k = 0, 1, 2 \) and \( r = 0 \), and (ii) \( k = r = 1 \).

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1. Introduction

In 1935, Hahn [4] proved: if \( \{ P_n(x) \}_{n=0}^{\infty} \) and \( \{ P_n'(x) \}_{n=0}^{\infty} \) are positive-definite orthogonal polynomial systems (OPS’), then \( \{ P_n(x) \}_{n=0}^{\infty} \) must be one of classical OPS’ (Jacobi, Laguerre, or Hermite). Krall [8] and Webster [23] extended Hahn’s theorem to quasi-definite OPS’ (including Bessel polynomials [12]). Later, Hahn [5] and Krall [9] also showed: If for any fixed integer \( r \geq 1 \), \( \{ P_n(x) \}_{n=0}^{\infty} \) and \( \{ P_n^r(x) \}_{n=0}^{\infty} \) are OPS’, then \( \{ P_n(x) \}_{n=0}^{\infty} \) must be a classical OPS. Recently, it is extended further as: If \( \{ P_n(x) \}_{n=0}^{\infty} \) is an OPS and \( \{ P_n^r(x) \}_{n=0}^{\infty} \) is a WOPS, then \( \{ P_n(x) \}_{n=0}^{\infty} \) must be a classical OPS (cf. [16,19]).

Generalizing Hahn’s theorem, we now ask: Given an OPS \( \{ P_n(x) \}_{n=0}^{\infty} \) and a linear differential operator \( L[\cdot] = \sum_{i=0}^{k} a_i(x) D^i \) with polynomial coefficients, when is the polynomial sequence \( \{ Q_n(x) \}_{n=0}^{\infty} \)
defined by
\[ Q_n(x) := L[P^{(r)}_{n+r}(x)] = \sum_{i=0}^{k} a_i(x)P^{(i+r)}_{n+r} \]
also an OPS? Here, \( r \) is any nonnegative integer.

Krall and Sheffer [14] raised and solved the above problem for \( r = 0, 1 \) using the moments and the characterization of OPS’ via formal generating series \( G(x, t) = \sum_{n=0}^{\infty} P_n(x)t^n \) of a PS \( \{P_n(x)\}_{n=0}^{\infty} \) (cf. [13]). Their method is quite complicated so that it seems to be impossible to be extended to the case \( r \geq 2 \). We solve the problem completely for any \( r > 0 \) by using the formal calculus of moment functionals (see Theorems 3.1 and 3.2), by which we can refine the characterizations of classical orthogonal polynomials in [19] (see Theorem 3.4). Finally, we analyse completely the cases for \( k = 0, 1, 2 \) and \( r = 0 \) or \( k = r = 1 \) and as by products, we obtain some new relations between classical orthogonal polynomials and classical-type orthogonal polynomials.

2. Preliminaries

All polynomials in this work are assumed to be real polynomials in one variable and we let \( \mathcal{P} \) be the space of all real polynomials. We denote the degree of a polynomial \( \pi(x) \) by \( \deg(\pi) \) with the convention that \( \deg(0) = -1 \). By a polynomial system (PS), we mean a sequence of polynomials \( \{P_n(x)\}_{n=0}^{\infty} \) with \( \deg(P_n) = n, \ n \geq 0 \). Note that a PS forms a basis of \( \mathcal{P} \).

We call any linear functional \( \sigma \) on \( \mathcal{P} \) a moment functional and denote its action on a polynomial \( \pi(x) \) by \( \langle \sigma, \pi \rangle \). For a moment functional \( \sigma \), we call
\[ \sigma_n := \langle \sigma, x^n \rangle, \quad n = 0, 1, \ldots \]
the moments of \( \sigma \). We say that a moment functional \( \sigma \) is quasi-definite (respectively, positive-definite) \( \mathcal{P} \) if its moments \( \{\sigma_n\}_{n=0}^{\infty} \) satisfy the Hamburger condition
\[ \Delta_n(\sigma) := \det[\sigma_{i+j}]_{i,j=0}^{n} \neq 0 \quad (\text{respectively, } \Delta_n(\sigma) > 0), \quad n \geq 0. \]
Any PS \( \{\phi_n(x)\}_{n=0}^{\infty} \) determines a moment functional \( \sigma \) (uniquely up to a nonzero constant multiple), called a canonical moment functional of \( \{\phi_n(x)\}_{n=0}^{\infty} \), by the conditions
\[ \langle \sigma, \phi_0 \rangle \neq 0 \quad \text{and} \quad \langle \sigma, \phi_n \rangle = 0, \quad n \geq 1. \]

**Definition 2.1.** A PS \( \{P_n(x)\}_{n=0}^{\infty} \) is a weak orthogonal polynomial system (WOPS) if there is a nontrivial moment functional \( \sigma \) such that
\[ \langle \sigma, P_mP_n \rangle = 0 \quad \text{if } 0 \leq m < n. \quad (2.1) \]
If we further have
\[ \langle \sigma, P_n^2 \rangle = K_n, \quad n \geq 0, \]
where \( K_n \) are nonzero real constants, then we call \( \{P_n(x)\}_{n=0}^{\infty} \) an orthogonal polynomial system (OPS). In either case, we say that \( \{P_n(x)\}_{n=0}^{\infty} \) is a WOPS or an OPS relative to \( \sigma \) and call \( \sigma \) an orthogonalizing moment functional of \( \{P_n(x)\}_{n=0}^{\infty} \).
It is immediate from (2.1) that for any WOPS \( \{P_n(x)\}_{n=0}^{\infty} \), its orthogonalizing moment functional \( \sigma \) must be a canonical moment functional of \( \{P_n(x)\}_{n=0}^{\infty} \). It is well known (see [Chapters 1 and 2]) that a moment functional \( \sigma \) is quasi-definite if and only if there is an OPS \( \{P_n(x)\}_{n=0}^{\infty} \) relative to \( \sigma \) and then each \( P_n(x) \) is uniquely determined up to a nonzero multiplicative constant. For a moment functional \( \sigma \) and a polynomial \( \pi(x) \), we let \( \sigma' \) (the derivative of \( \sigma \)) and \( \pi \sigma \) (the left multiplication of \( \sigma \) by \( \pi(x) \)) be the moment functionals defined by

\[
\langle \sigma', \phi \rangle = -\langle \sigma, \phi' \rangle
\]

and

\[
\langle \pi \sigma, \phi \rangle = \langle \sigma, \pi \phi \rangle, \quad \phi \in \mathcal{P}.
\]

Then it is easy to obtain the following (see [16,18]).

**Lemma 2.1.** For a moment functional \( \sigma \) and a polynomial \( \pi(x) \), we have

(i) Leibniz’ rule: \( (\pi \sigma)' = \pi' \sigma + \pi \sigma' \);

(ii) \( \sigma' = 0 \) if and only if \( \sigma = 0 \).

Assume that \( \sigma \) is quasi-definite and \( \{P_n(x)\}_{n=0}^{\infty} \) is an OPS relative to \( \sigma \). Then

(iii) \( \pi \sigma = 0 \) if and only if \( \pi(x) = 0 \);

(iv) for any other moment functional \( \tau \), \( \langle \tau, P_n \rangle = 0 \), \( n \geq k + 1 \) for some integer \( k \geq 0 \) if and only if \( \tau = \phi \sigma \) for some polynomial \( \phi(x) \) of degree \( \leq k \).

It is well known [1,17] that there are essentially four distinct classical OPS’ satisfying second-order differential equations with polynomial coefficients

\[
\mathcal{L}[y](x) = \ell_2(x)y''(x) + \ell_1(x)y'(x) + \ell_0(x) \quad \text{or} \quad \mathcal{L}[y](x) = \ell_1(x)y'(x) + \ell_0(x) y(x)
\]

(2.2)

They are:

(i) Hermite polynomials \( \{H_n(x)\}_{n=0}^{\infty} \) (orthogonal relative to \( e^{-x^2} \) \( dx \)) satisfying

\[
y''(x) - 2xy'(x) = -2ny(x).
\]

(ii) Laguerre polynomials \( \{L_n^{(a)}(x)\}_{n=0}^{\infty} \) (orthogonal relative to \( x^a e^{-x} dx \)) satisfying

\[
xy''(x) + (a + 1 - x)y'(x) = -ny(x) \quad (a \notin \{-1,-2,\ldots\}).
\]

(iii) Jacobi polynomials \( \{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty} \) (orthogonal relative to \( (1-x)^{\alpha} (1+x)^{\beta} dx \)) satisfying

\[
(1-x^2)y''(x) + (\beta - \alpha - (\alpha + \beta + 2)x)y'(x) = -n(n + \alpha + \beta + 1)y(x)
\]

\[
(\alpha, \beta, \alpha + \beta + 1 \notin \{-1,-2,\ldots\}).
\]

(iv) Bessel polynomials \( \{B_n^{(\alpha)}(x)\}_{n=0}^{\infty} \) (see [12,15]) satisfying

\[
x^2y''(x) + (\alpha x^2 + 2)y'(x) = n(n + \alpha - 1)y(x) \quad (\alpha \notin \{0,-1,-2,\ldots\}).
\]

Here, \( x^2 \) is the distribution with support in \([0,\infty)\), which is obtained by the regularization of the function

\[
f_{x^2}(x) = \begin{cases}
  x^2 & \text{if } x > 0, \\
  0 & \text{if } x \leq 0
\end{cases}
\]

(see in [6, Chapter 3.3.2]).
More generally, Krall [10] (see also [18,21]) found necessary and sufficient conditions for an OPS to be eigenfunctions of differential equations with polynomial coefficients:

**Proposition 2.2.** Let \( \{P_n(x)\}_{n=0}^\infty \) be an OPS relative to \( \sigma \) and \( L_N[\cdot ] = \sum_{i=1}^N \ell_i(x)D^i \) (\( D = d/dx \)) be a linear differential operator of order \( N \geq 1 \) with polynomial coefficients \( \ell_i(x) \) of order \( \leq i \). Then

\[
L_N[P_n](x) = \sum_{i=1}^N \ell_i(x)P_n^{(i)}(x) = \lambda_n P_n(x), \quad n \geq 0,
\]

where

\[
\lambda_n = \sum_{i=1}^N \frac{1}{i!} \ell_i^{(i)}(x) n(n-1) \cdots (n-i+1)
\]

if and only if \( \sigma \) satisfies

\[
R_k(\sigma) := \sum_{i=2k+1}^N (-1)^i \binom{i-1}{k} \left( \ell_i(\sigma)^{(i-2k-1)} \right) = 0, \quad k = 0, 1, \ldots, r-1.
\]

Moreover, in this case, \( N = 2r \) must be even.

Using this characterization, Krall [11] classified all OPS’ that are eigenfunctions of fourth-order differential equations. They are the four classical OPS’ above and the three new OPS’, now known as classical-type OPS’ [7]:

(v) Legendre-type polynomials \( \{P_n^{(\sigma)}(x)\}_{n=0}^\infty \) (orthogonal relative to \( (H(1-x^2) + (1/x)(\delta(x-1) + \delta(x+1)))dx \)) satisfying

\[
(x^2 - 1)^2 y^{(4)} + 8x(x^2 - 1) y^{(3)} + 4(3x + 3)(x^2 - 1) y'' + 8x y' = \lambda_n y \quad (\sigma \neq -n(n-1)/2, n \geq 0).
\]

(vi) Laguerre-type polynomials \( \{R_n(x)\}_{n=0}^\infty \) (orthogonal relative to \( (e^{-x}H(x) + (1/R)\delta(x)) dx \)) satisfying

\[
x^2 y^{(4)} - (2x^2 - 4x)y^{(3)} + [x^2 - (2R + 6)x] y'' + [(2R + 2)x - 2R] y' = \lambda_n y \quad (R \neq 0, -1, -2, \ldots). \tag{2.3}
\]

(vii) Jacobi-type polynomials \( \{S_n^{(\sigma)}(x)\}_{n=0}^\infty \) (orthogonal relative to \( ((1-x)H(x) + (1/M)\delta(x)) dx \)) satisfying

\[
(x^2 - x)^2 y^{(4)} + 2(x-1)[(x+4)x-2] y^{(3)} + x[(x^2 + 9x + 14 + 2M)x - (6x + 12 + 2M)] y''
+ [(x + 2)(2x + 2 + 2M)x - 2M] y' = \lambda_n y \quad (\sigma \neq -1, -2, \ldots, \text{ and } n^2 + xn + M \neq 0, n \geq 0).
\]

Here, \( H(x) \) is the Heaviside step function.

In [20], we showed that if a fourth- (or higher) order differential equation has a classical OPS \( \{P_n(x)\}_{n=0}^\infty \) as solutions, then the differential equation must be a linear combination of iterations of a second-order differential equation (2.2) having \( \{P_n(x)\}_{n=0}^\infty \) as solutions.
3. Main results

In the following, we always let \( \{P_n(x)\}_{n=0}^{\infty} \) be a monic OPS relative to \( \sigma \) and \( L[\cdot] = \sum_{i=0}^{k} a_i(x)D^i \) (\( D = d/dx \)) a linear differential operator of order \( k \) with polynomial coefficients \( a_i(x) = \sum_{j=0}^{i} a_{ij}x^j \), \( 0 \leq i \leq k(a_k(x) \neq 0) \). For an integer \( r \geq 0 \), we also let
\[
Q_n(x) = L[P_{n+r}(x)] = a_n x^n + \text{lower degree terms}, \quad n \geq 0
\] (3.1)
and assume that
\[
a_n := \sum_{i=0}^{k} a_n(n+r)(i+r) \neq 0, \quad n \geq 0
\] (3.2)
so that \( \{Q_n(x)\}_{n=0}^{\infty} \) is also a PS, where

\[
n_{(i)} = \begin{cases} 
1 & \text{if } i = 0, \\
n(n-1) \cdots (n-i+1) & \text{if } i \geq 1.
\end{cases}
\]

We now ask: When is the PS \( \{Q_n(x)\}_{n=0}^{\infty} \) also an OPS?

Then our main result is:

**Theorem 3.1.** The PS \( \{Q_n(x)\}_{n=0}^{\infty} \) defined by (3.1) is a WOPS if and only if there is a moment functional \( \tau \neq 0 \) and \( k+r+1 \) polynomials \( \{b_r(x)\}_{r=0}^{k+2r} \) with \( \deg(b_i) \leq i \) satisfying
\[
\sum_{i=j}^{k+r} (-1)^j \binom{i}{j} (a_{i-j}(x)\tau)^{(i-j)} = b_{j+r}(x)\sigma, \quad 0 \leq j \leq k + r
\] (3.3)
or equivalently
\[
\sum_{i=j}^{k+r} (-1)^j \binom{i}{j} (b_{i+r}(x)\sigma)^{(i-j)} = a_{j-r}(x)\tau, \quad 0 \leq j \leq k + r,
\] (3.4)
where \( a_i(x) = 0 \) for \( i < 0 \). In this case, \( \deg(b_r) = r \) and
\[
\langle \tau, a_i \rangle = (-1)^{i+r} \langle \sigma, b_{i+2r} \rangle, \quad 0 \leq i \leq k
\] (3.5)
so that \( \langle \sigma, b_{2r} \rangle \neq 0 \) and \( b_{2r}(x) \neq 0 \). Furthermore, \( \{Q_n(x)\}_{n=0}^{\infty} \) is an OPS if and only if the polynomials \( \{b_r(x)\}_{r=0}^{k+2r} \) satisfy, in addition to (3.3),
\[
\sum_{i=0}^{k+r} b_{i+r, i+r} n_{(i)} \neq 0, \quad n \geq 0,
\] (3.6)
where \( b_r(x) = \sum_{j=0}^{r} b_{ij}x^j \). In this case, \( \deg(b_r) = r \) and \( b_{k+2r}(x) \neq 0 \).

**Proof.** Assume that \( \{Q_n(x)\}_{n=0}^{\infty} \) is a WOPS and let \( \tau \) be a canonical moment functional of \( \{Q_n(x)\}_{n=0}^{\infty} \).

Then \( \langle \tau, Q_nQ_m \rangle = 0, \quad 0 \leq m < n \). We shall prove that there are polynomials \( \{b_r(x)\}_{r=0}^{k+2r} \) with \( \deg(b_i) \leq i \) satisfying (3.3) by induction on \( i = 0, 1, \ldots, k + r \). For \( n \geq 1 \),
\[
0 = \langle \tau, Q_n(x) \rangle = \left\langle \tau, \sum_{i=0}^{k} a_i(x)P_{n+r}^{(i+r)}(x) \right\rangle = \left\langle \sum_{i=0}^{k+r} (-1)^i (a_{i-r}(\tau)^{(i)}P_{n+r} \right\rangle.
\]
By Lemma 2.1(iv), there is a polynomial \( b_r(x) \) of degree \( \leq r \) such that

\[
b_r(x) \sigma = \sum_{i=0}^{k+r} (-1)^i (a_i \tau)^{i(j)}
\]

so that (3.3) holds for \( j = 0 \). Assume that for some \( \ell \) with \( 0 \leq \ell < k + r \), there exist polynomials \( \{b_i(x)\}_{i=\ell}^{k+r} \) of degree \( \leq i \) such that (3.3) holds for \( j = 0, 1, \ldots, \ell \). Then for \( n \geq \ell + 2 \),

\[
0 = \langle \tau, Q_{\ell+1} Q_n \rangle = \left\langle \tau, Q_{\ell+1} \sum_{i=0}^{k} a_i \tau^{i(\ell+r)} P_{n+r} \right\rangle = \sum_{i=0}^{k} (-1)^i (Q_{\ell+1} a_i \tau^{(i+r)}, P_{n+r})
\]

\[
= \sum_{i=0}^{k} (-1)^i \left( \sum_{j=0}^{\ell} \binom{i}{j} Q_{\ell+1}^{(j)} (a_i \tau)^{i(\ell-j)}, P_{n+r} \right)
\]

\[
= \sum_{i=0}^{k} Q_{\ell+1}^{(i)} \sum_{j=0}^{\ell} (-1)^i \binom{i}{j} (a_i \tau)^{i(\ell-j)}, P_{n+r} \right)
\]

\[
= \sum_{i=0}^{k} \sum_{j=0}^{\ell} Q_{\ell+1}^{(j)} (-1)^i \binom{i}{j} (a_i \tau)^{i(\ell-j)}, P_{n+r} \right)
\]

\[
= Q_{\ell+1}^{(\ell+1)} \left( \sum_{i=0}^{k} (-1)^i \binom{i}{\ell+1} (a_i \tau)^{i(\ell-1)}, P_{n+r} \right) + \sum_{j=0}^{\ell} \sum_{i=0}^{k} Q_{\ell+1}^{(j)} b_{i+j, r} P_{n+r} \right)
\]

\[
= \alpha_{\ell+1} (\ell+1)! \left( \sum_{i=0}^{k} (-1)^i \binom{i}{\ell+1} (a_i \tau)^{i(\ell-1)}, P_{n+r} \right) + \left( \sigma, \sum_{j=0}^{\ell} Q_{\ell+1}^{(j)} b_{j+r} P_{n+r} \right).
\]

Since \( \text{deg} \left( \sum_{j=0}^{\ell} Q_{\ell+1}^{(j)} b_{j+r} \right) \leq r + \ell + 1 < n + r \), \( \langle \tau, \sum_{j=0}^{\ell} Q_{\ell+1}^{(j)} b_{j+r} P_{n+r} \rangle = 0 \), so that

\[
\sum_{i=0}^{k+r} (-1)^i \binom{i}{\ell+1} (a_i \tau)^{i(\ell-1)}, P_{n+r} \rangle = 0, \quad n \geq \ell + 2.
\]

Therefore, by Lemma 2.1(iv), there is a polynomial \( b_{r+\ell+1}(x) \) with \( \text{deg} (b_{r+\ell+1}) \leq r + \ell + 1 \) such that

\[
\sum_{i=0}^{k+r} (-1)^i \binom{i}{\ell+1} (a_i \tau)^{i(\ell-1)} = b_{r+\ell+1} \sigma, \quad \text{that is}, (3.3) \text{ also holds for } j = \ell + 1.
\]

Conversely, assume that there are moment functionals \( \tau \neq 0 \) and polynomials \( \{b_i(x)\}_{i=\ell}^{k+r} \) with \( \text{deg} (b_i) \leq i \) satisfying (3.3). Then

\[
\langle \tau, Q_m Q_n \rangle = \left\langle \tau, Q_m \sum_{i=0}^{k} a_i \tau^{i(\ell+r)} P_{n+r} \right\rangle = \sum_{i=0}^{k} (-1)^i (Q_m a_i \tau^{i(\ell+r)}, P_{n+r})
\]

\[
= \sum_{i=0}^{k} (-1)^i \sum_{j=0}^{i+r} \binom{i+r}{j} Q_m^{(j)} (a_i \tau^{i(\ell+j)}), P_{n+r} \right)
\]
In particular, since deg($P_{n+r}$) $\leq r + m < n + r$. Thus \{Q_n(x)\}_{n=0}^\infty is a WOPS relative to $\tau$.

(3.3) $\Rightarrow$ (3.4): For $j = 0, 1, \ldots, k + r$

$$
\sum_{i=0}^{k+r} (-1)^i \binom{i}{j} (b_{i+r}(x) \sigma)^{(i-j)} = \sum_{i=0}^{k+r} (-1)^i \binom{i}{j} \left[ \sum_{\ell=j}^{k+r} (-1)^{\ell-j} \binom{\ell}{j} (a_{\ell-r} \tau)^{(\ell-j)} \right]
$$

$$
= \sum_{\ell=j}^{k+r} (-1)^{\ell-j} \binom{\ell}{j} \sum_{i=0}^{\ell-j} (-1)^i \binom{i}{j} (a_{\ell-r} \tau)^{(\ell-j)}
$$

$$
= \sum_{\ell=j}^{k+r} (-1)^{\ell-j} \binom{\ell}{j} \delta_{\ell,j}(a_{\ell-r}(x) \tau)^{(\ell-j)}
$$

$$
= a_{j-r} \tau \quad (a_j(x) \equiv 0 \text{ if } j < 0)
$$

since $\sum_{i=0}^{\ell-j} (-1)^i \binom{i}{j} = \delta_{\ell,j}$.

(3.4) $\Rightarrow$ (3.3): The proof is similar as above.

Now we shall show (3.5). Since deg($b_{j+r}$) $\leq j + r$, there are constants $\{c_j^{j+r}\}_{k=0}^j$ such that $b_{j+r}(x) = \sum_{k=0}^{j+r} c_j^{j+r} P_k(x)$ so that $b_{j+r,j+r} = c_j^{j+r}$. Then by applying (3.3) to $b_{j+r}(x)$, we have

$$
b_{j+r,j+r} = \frac{\langle \sum_{i=j}^{k+r} (-1)^i \binom{i}{j} (a_{i-r} \tau)^{(i-j)}, P_{j+r} \rangle}{\langle \sigma, P_{j+r}^2 \rangle}, \quad 0 \leq j \leq k + r.
$$

In particular,

$$
b_{r} = \frac{\langle \tau, a_0 P_r^{(r)} \rangle}{\langle \sigma, P_r^2 \rangle} = \frac{r! a_0 \langle \tau, 1 \rangle}{\langle \sigma, P_r^2 \rangle} \neq 0
$$

so that deg($b_r$) = $r$. Applying (3.4) to $P_0(x) = 1$, we can obtain (3.5).

Now assume that \{Q_n(x)\}_{n=0}^\infty is a WOPS relative to $\tau$. Then by (3.7), \{Q_n(x)\}_{n=0}^\infty is an OPS relative to $\tau$ if and only if $\langle \tau, Q_n^2 \rangle = \langle \sigma, (\sum_{j=0}^{k+r} Q_n^{(j)} b_{j+r}) P_{n+r} \rangle \neq 0$, $n \geq 0$, which is equivalent to the condition (3.6).

In this case, (3.3) for $j = k + r$ implies that $b_{k+r}(x) \sigma = (-1)^{k+r} a_k(x) \tau \neq 0$. Thus $b_{k+r}(x) \neq 0$ since $a_k(x) \neq 0$ and $\tau$ is quasi-definite. □
Set $j = r$ in (3.4). Then

\[ a_0 \tau = \sum_{i=r}^{k+r} (-1)^i \binom{i}{r} (b_{i+r}, \sigma)^{(i-r)}. \]  

(3.8)

Hence, we may restate Theorem 3.1 as:

**Theorem 3.2.** The PS \( \{Q_n(x)\}_{n=0}^{\infty} \) defined by (3.1) is a WOPS if and only if there are \( k + r + 1 \) polynomials \( \{b_i(x)\}_{i=r}^{k+2r} \) with deg(\( b_i \)) \( \leq i \), which are not all zero, satisfying

\[ a_0(x) \sum_{i=j}^{k+r} (-1)^i \binom{i}{j} (b_{i+r}, \sigma)^{(i-j)} = \begin{cases} 0 & \text{if } 0 \leq j \leq r - 1, \\ a_{j-r}(x) \sum_{i=j}^{k+r} (-1)^i \binom{i}{j} (b_{i+r}, \sigma)^{(i-j)} & \text{if } r + 1 \leq j \leq k + r. \end{cases} \]  

(3.9)

In this case, deg(\( b_r \)) = \( r \), \( b_{2r}(x) \neq 0 \), and \( \{Q_n(x)\}_{n=0}^{\infty} \) is a WOPS relative to

\[ \tau := \frac{1}{a_0} \sum_{i=r}^{k+r} (-1)^i \binom{i}{r} (b_{i+r}, \sigma)^{(i-r)}. \]

Moreover, \( \{Q_n(x)\}_{n=0}^{\infty} \) is an OPS if and only if \( \{b_i(x)\}_{i=r}^{k+2r} \) also satisfy the condition (3.6). In this case, we also have \( b_{k+2r}(x) \neq 0 \).

**Proof.** Assume that there are \( k + r + 1 \) polynomials \( \{b_i(x)\}_{i=r}^{k+2r} \) with deg(\( b_i \)) \( \leq i \), which are not all zero, and (3.9) holds. Define \( \tau \) by (3.8). Then (3.4) holds so that we only need to show \( \tau \neq 0 \). If \( \tau = 0 \), then \( \sum_{i=r}^{k+r} (-1)^i \binom{i}{j} (b_{i+r}, \sigma)^{(i-j)} = 0 \), \( 0 \leq j \leq k + r \). Then for \( j = k + r \), \( (-1)^{k+r} (b_{k+2r}, \sigma) = 0 \) so that \( b_{k+2r}(x) = 0 \). By induction on \( j = k + r, k + r - 1, \ldots, 0 \), we can see \( b_i(x) = 0 \), for \( r \leq i \leq k + 2r \), which is a contradiction. The converse is trivial by Theorem 3.1. □

**Theorem 3.3.** If the PS \( \{Q_n(x)\}_{n=0}^{\infty} \) defined by (3.1) is also an OPS, then there are nonzero constants \( \lambda_n \), \( n \geq r \), such that

\[ M[Q_n(x)] = \lambda_n P_n(x), \quad n \geq r, \]  

(3.10)

where \( M[ \cdot ] = \sum_{i=0}^{k+r} b_{i+r}(x) D^i \) and both \( \{P_n(x)\}_{n=0}^{\infty} \) and \( \{Q_n(x)\}_{n=0}^{\infty} \) must be eigenfunctions of linear differential operators of order \( 2(k + r) \):

\[ MLD \gamma P_n(x) = \lambda_n P_n(x), \quad n \geq 0, \]  

(3.11)

where \( \lambda_n = 0 \), \( 0 \leq n \leq r - 1 \) and

\[ LD \gamma M[Q_n(x)] = \lambda_{n+r} Q_n(x), \quad n \geq 0. \]  

(3.12)
Proof. Define a sequence of polynomials $\{\tilde{P}_n(x)\}_{n=0}^{\infty}$ by

$$
\tilde{P}_n(x) = \begin{cases} 
  P_n(x), & 0 \leq n \leq r - 1, \\
  M[Q_{n-r}(x)] = \sum_{i=0}^{k+r} b_{i+r}(x) Q_{n-r}^{(i)}(x), & n \geq r.
\end{cases}
$$

Then $\text{deg}(\tilde{P}_n) = n$, $n \geq 0$, by (3.6) so that $\{\tilde{P}_n(x)\}_{n=0}^{\infty}$ is a PS.

Now we shall show that $\{\tilde{P}_n(x)\}_{n=0}^{\infty}$ is an OPS relative to $\sigma$. For $0 \leq m \leq n \leq r - 1$, $\langle \sigma, \tilde{P}_m \tilde{P}_n \rangle = \langle \sigma, P_m P_n \rangle = \langle \sigma, P_m^2 \rangle \delta_{mn}$. For $0 \leq m \leq n$ and $n \geq r$,

$$
\langle \sigma, \tilde{P}_m \tilde{P}_n \rangle = \left\langle \sigma, \tilde{P}_m \sum_{i=0}^{k+r} b_{i+r} Q_{n-r}^{(i)} \right\rangle = \left\langle \sum_{i=0}^{k+r} (\tilde{P}_m b_{i+r} \sigma)^{(i)}(x), Q_{n-r} \right\rangle
$$

$$
= \sum_{i=0}^{k+r} (\tilde{P}_m b_{i+r} \sigma)^{(i)}(x) Q_{n-r}
$$

$$
= \left\langle \tau, \left( \sum_{j=0}^{k+r} \tilde{P}_m^{(j)} a_{j-r} \right) Q_{n-r} \right\rangle
$$

$$
= \left\langle \tau, \left( \sum_{j=r}^{k+r} \tilde{P}_m^{(j)} a_{j-r} \right) Q_{n-r} \right\rangle = \left\langle \tau, \left( \sum_{j=0}^{k} \tilde{P}_m^{(j+r)} a_j \right) Q_{n-r} \right\rangle = \left\{ \begin{array}{ll}
0 & \text{if } m < n, \\
nonzero & \text{if } m = n
\end{array} \right.
$$

since $\text{deg}(\sum_{j=0}^{k} \tilde{P}_m^{(j+r)} a_j) = m - r$ by (3.2) and $\{Q_n(x)\}_{n=0}^{\infty}$ is an OPS relative to $\tau$.

Hence $\{\tilde{P}_n(x)\}_{n=0}^{\infty}$ is an OPS relative to $\sigma$ so that $\tilde{P}_n(x) = M[Q_{n-r}(x)] = \lambda_n P_n(x)$, for some $\lambda_n \neq 0$ for $n \geq r$. Now

$$
\text{MLD}^r[P_n] = \text{ML}[P_n^{(r)}] = M[Q_{n-r}] = \tilde{P}_n = \lambda_n P_n, \quad n \geq r.
$$

For $0 \leq n \leq r - 1$, $D'[P_n] = 0$ so that $\text{MLD}^r[P_n] = 0$. We also have

$$
\text{LD}^r M[Q_n] = \text{LD}[\tilde{P}_{n+r}] = \lambda_{n+r} P_{n+r} = \lambda_{n+r} L[P_{n+r}] = \lambda_{n+r} Q_n, \quad n \geq r.
$$

Finally since $b_{k+2}(x) \neq 0$, $M[\cdot]$ is of order $k + r$ and so $\text{MLD}^r[\cdot]$ and $\text{LD}^r M[\cdot]$ are of order $2(k + r)$. \hfill \Box

Krall and Sheffer proved Theorem 3.1 only for $r = 0$ (see [14, Theorem 2.1]) and $r = 1$ (see [14, Theorem 3.1]) and Theorem 3.4 only for $r = 0$ (see [14, Theorem 2.3]), using the moments $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\tau_n\}_{n=0}^{\infty}$ of $\sigma$ and $\tau$, respectively. They used the characterization of OPS’ via their formal (cf. [13]) generating series

$$
G(x, t) = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} \phi_n(t) x^n,
$$
where \( \phi_n(t) \) is a power series in \( t \) starting from \( t^n \). Their method seems to be too much complicated to be extended to the case \( r \geq 2 \).

It is well-known (cf. [4,5,9,23]) that an OPS \( \{P_n(x)\}_{n=0}^{\infty} \) is a classical OPS if and only if \( \{P^{(r)}_{n+r}(x)\}_{n=0}^{\infty} \) is also an OPS for some integer \( r \geq 1 \).

As a special case of Theorems 3.1, 3.2 and 3.4, we obtain:

**Theorem 3.4.** Let \( \{P_n(x)\}_{n=0}^{\infty} \) be an OPS relative to \( \sigma \) and \( r \geq 1 \) an integer. Then, the following are all equivalent.

(i) \( \{P_n(x)\}_{n=0}^{\infty} \) is a classical OPS.

(ii) \( \{P^{(r)}_{n+r}(x)\}_{n=0}^{\infty} \) is a WOPS.

(iii) There are nonzero moment functional \( \tau \) and \( r+1 \) polynomials \( \{b_k(x)\}_{k=r}^{2r} \) with \( \deg(b_k) \leq k \) such that

\[
(-1)^{r-j} \binom{r}{j} \tau^{(r-j)} = b_{j+r}, \quad 0 \leq j \leq r.
\] (3.13)

(iv) There are \( r+1 \) polynomials \( \{b_k(x)\}_{k=r}^{2r} \) with \( \deg(b_k) \leq k \) such that \( \{b_k(x)\}_{k=r}^{2r} \) are not all zero and

\[
\sum_{i=j}^{r} (-1)^{i-j} \binom{i}{j} (b_{i+r})^{(i-j)} = 0, \quad 0 \leq j \leq r - 1.
\]

Moreover, in this case, \( \deg(b_r) = r \), \( b_{2r}(x) \neq 0 \), and

\[
\sum_{k=r}^{2r} b_k(x) P^{(k)}_{n}(x) = \lambda_n P_n(x), \quad n \geq 0
\] (3.14)

for some constants \( \lambda_n \) with \( \lambda_0 = \lambda_1 = \cdots = \lambda_{r-1} = 0 \) and

\[
\sum_{i=0}^{r} (-1)^{i+r} \binom{r}{i} \langle \sigma, b_{2r} P^{(r-i)}_{i+r} \rangle \frac{n(i)}{\langle \sigma, P^2_{i+r} \rangle} \neq 0, \quad n \geq 0.
\] (3.15)

**Proof.** (i) \( \Rightarrow \) (ii): It is well known that for a classical OPS \( \{P_n(x)\}_{n=0}^{\infty} \), \( \{P^{(r)}_{n+r}(x)\}_{n=0}^{\infty} \) is also a classical OPS.

(ii) \( \Rightarrow \) (i): See Theorems 3.2 and 3.3 in [19].

(ii) \( \iff \) (iii) \( \iff \) (iv): It is a special case of Theorems 3.1 and 3.2 when \( k = 0 \) so that \( L[.]=a_0 I d (I d = \text{the identity operator}) \) and \( Q_n(x) = a_0 P^{(r)}_{n+r}(x), n \geq 0 \). In (iii), \( \deg(b_r) = r \) and \( b_{2r}(x) \neq 0 \) by Theorem 3.1. Eq. (3.14) comes from Theorem 3.4 and (3.15) comes from (3.6), (3.7), and (3.13). \( \square \)

Equivalences of (i)–(iii) in Theorem 3.4 are first proved in [19]. Moreover, the condition (3.14) also implies that \( \{P_n(x)\}_{n=0}^{\infty} \) is a classical OPS (see [19]).
4. Examples

As in Section 3, we always let \( \{P_n(x)\}_{n=0}^{\infty} \) be the monic OPS relative to \( \sigma \) and write \( a_0(x) = a_{00} = a_0 \). If \( k = r = 0 \), then \( \{Q_n(x)\}_{n=0}^{\infty} \), where \( Q_n(x) = a_0P_n(x) \), \( n \geq 0 \), and \( a_0 \neq 0 \), is also an OPS if and only if \( \{P_n(x)\}_{n=0}^{\infty} \) is an OPS.

4.1. \( k = 1 \) and \( r = 0 \)

Let \( L[\cdot] = a_1(x)D + a_0 \), where \( a_1(x) = a_{11}x + a_{10} \neq 0 \), \( a_0 \neq 0 \), and \( a_{11}n + a_{10} \neq 0 \), \( n \geq 0 \). Define a monic PS \( \{Q_n(x)\}_{n=0}^{\infty} \) by:

\[
(a_{11}n + a_{10})Q_n(x) = L[P_n(x)] = a_1(x)P'_n(x) + a_0(x)P_n(x), \quad n \geq 0.
\]

Then, \( \{Q_n(x)\}_{n=0}^{\infty} \) is also a monic OPS (relative to \( \tau := a_0^{-1}((b_1(x)\sigma' - b_0(x)\sigma)) \) if and only if there are polynomials \( b_1(x) = b_{11}x + b_{10} \) and \( b_0(x) = b_0 \) satisfying

\[
a_{10}b_1\sigma = a_1\{(b_1\sigma') - b_0\sigma\} \quad \text{and} \quad b_{11}n + b_{10} \neq 0, \quad n \geq 0.
\]

Hence, if \( \{Q_n(x)\}_{n=0}^{\infty} \) is also a monic OPS, then (3.11) and (3.14) become

\[
ML[P_n] = (a_1b_1)P'_n + (a_1b_1 + a_0b_0 + a_1b_0)P'_n + a_0b_0P_n = \lambda_nP_n,
\]

\[
LM[Q_n] = (a_1b_1)Q'_n + (a_1b_1 + a_0b_0 + a_1b_0)Q'_n + a_0b_0Q_n = \lambda_nQ_n,
\]

where \( \lambda_n = (a_{11}n + a_{10})(b_{11}n + b_{10}), n \geq 0 \). Hence both \( \{P_n(x)\}_{n=0}^{\infty} \) and \( \{Q_n(x)\}_{n=0}^{\infty} \) are classical OPS' of the same type. Note

\[
a_{1}(x)b_{1}(x) = a_{11}b_{11}x^2 + (a_{11}b_{10} + a_{10}b_{11})x + a_{10}b_{10}.
\]

Case 1: \( \deg(a_1b_1) = 0 \). Then \( a_{11}b_{11} + a_{10}b_{11} = 0 \) so that \( a_{11} = b_{11} = 0 \). Hence \( ML[\cdot] = LM[\cdot] \) and so \( P_n(x) = Q_n(x), n \geq 0 \), and

\[
a_{1}(x)P'_n(x) = a_{11}nP_n(x) = 0, \quad n \geq 0.
\]

Therefore, \( a_{1}(x) \equiv 0 \), which is a contradiction.

case 2: \( \deg(a_1b_1) = 1 \). Then we may assume \( a_1(x) = 1 \) and \( b_1(x) = x \) or \( a_1(x) = x \) and \( b_1(x) = 1 \).

Case 2.1: \( a_1(x) = 1 \) and \( b_1(x) = x \). Then for \( n \geq 0 \)

\[
ML[P_n(x)] = xP'_n(x) + (a_0x + b_0)P'_n(x) + a_0b_0P_n(x) = \lambda_nP_n(x),
\]

\[
LM[Q_n(x)] = xQ'_n(x) + (a_0x + b_0 + 1)Q'_n(x) + a_0b_0Q_n(x) = \lambda_nQ_n(x).
\]

We may also assume \( a_0 = -1 \) and \( b_0 = x + 1 \) (\( x \neq -1, -2, \ldots \)) by a real linear change of variable. Then (4.3) becomes

\[
ML[P_n(x)] = xP'_n(x) + (x + 1 - x)P'_n(x) - (x + 1)P_n(x) = \lambda_nP_n(x),
\]

\[
LM[Q_n(x)] = xQ'_n(x) + (x + 2 - x)Q'_n(x) - (x + 1)Q_n(x) = \lambda_nQ_n(x).
\]

Thus, \( \{P_n(x)\}_{n=0}^{\infty} = \{L_n^{(x)}(x)\}_{n=0}^{\infty} \) and \( \{Q_n(x)\}_{n=0}^{\infty} = \{L_n^{(x+1)}(x)\}_{n=0}^{\infty} \), where \( \{L_n^{(x)}(x)\}_{n=0}^{\infty} \) is the monic Laguerre polynomials. Hence, we have (see [22, (5.1.13)]):

\[
L_n^{(x+1)}(x) = L_n^{(x)}(x) - nL_{n-1}^{(x+1)}(x), \quad n \geq 0
\]

since \( (L_n^{(x)}(x))' = nL_{n-1}^{(x+1)}(x), \quad n \geq 0.\)
Case 2.2: \(a_1(x) = x \) and \(b_1(x) = 1\). Then we may assume \(a_0 = \alpha \) (\(\alpha \neq 0, -1, -2, \ldots\)) and \(b_0 = -1\) so that (4.1) becomes

\[
ML[P_n(x)] = xP'_n(x) + (x + 1 - x)P'_n(x) - xP_n(x) = \lambda_n P_n(x),
\]

\[
LM[Q_n(x)] = xQ'_n(x) + (x - x)Q'_n(x) - xQ_n(x) = \lambda_n Q_n(x).
\]

Thus, \(\{P_n(x)\}_{n=0}^{\infty} = \{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}\) and \(\{Q_n(x)\}_{n=0}^{\infty} = \{L_n^{(\alpha+1)}(x)\}_{n=0}^{\infty}\) so that we have (see [22, (5.1.14)]):

\[
(n + x)L_n^{(\alpha+1)}(x) = xL_n^{(\alpha)}(x)' + \alpha L_n^{(\alpha)}(x), \quad n \geq 0.
\]  

(4.5)

Case 3: \(\deg(a_1b_1) = 2\) and \((a_1b_1)(x)\) has a double root. Then from (4.2), \(a_{11}b_{10} = a_{10}b_{11}\) so that \(ML[.] = LM[.]\). Thus \(\{P_n(x)\}_{n=0}^{\infty} = \{Q_n(x)\}_{n=0}^{\infty}\) and

\[
(a_{1n}x + a_{10})P_n(x) = a_{11}nP_n(x), \quad n \geq 0,
\]

which is impossible (cf. Proposition 2.2).

Case 4: \(\deg(a_1b_1) = 2\) and \((a_1b_1)(x)\) has 2 distinct real roots. Then we may assume \((a_1b_1)(x) = 1 - x^2\) and \(a_1(x) = 1 - x\) or \(1 + x\).

Case 4.1: \(a_1(x) = 1 - x\) and \(b_1(x) = 1 + x\). Then (4.1) becomes

\[
ML[P_n(x)] = (1 - x^2)P'_n(x) + ((a_0 + b_0) - 1 - (b_0 - a_0 + 1)x)P'_n(x) + a_0b_0P_n(x) = \lambda_n P_n(x),
\]

\[
LM[Q_n(x)] = (1 - x^2)Q'_n(x) + ((a_0 + b_0) - 1 - (b_0 - a_0 + 1)x)Q'_n(x) + a_0b_0Q_n(x) = \lambda_n Q_n(x).
\]

We may also assume \(a_0 + b_0 - 1 = \beta - \alpha\) and \(b_0 - a_0 + 1 = \beta + \alpha + 2\) (\(\alpha, \beta, \alpha + \beta + 1 \neq -1, -2, \ldots\), and \(\alpha \neq 0\)). Then

\[
ML[P_n(x)] = (1 - x^2)P'_n(x) + (\beta - \alpha - (\alpha + \beta + 2)x)P'_n(x) - \alpha(\beta + 1)P_n(x) = \lambda_n P_n(x),
\]

\[
LM[Q_n(x)] = (1 - x^2)Q'_n(x) + (\beta - \alpha + 2 - (\alpha + \beta + 2)x)Q'_n(x) - \alpha(\beta + 1)Q_n(x) = \lambda_n Q_n(x).
\]

Therefore, we have

\[
\{P_n(x)\}_{n=0}^{\infty} = \{P^{(\alpha, \beta)}_n(x)\}_{n=0}^{\infty} \quad \text{and} \quad \{Q_n(x)\}_{n=0}^{\infty} = \{P^{(\alpha+1, \beta+1)}_n(x)\}_{n=0}^{\infty},
\]

where \(\{P^{(\alpha, \beta)}_n(x)\}_{n=0}^{\infty}\) is the monic Jacobi polynomials. Hence, we have \(a_1(x) = 1 - x\), \(a_0(x) = -\alpha\), \(b_1(x) = 1 + x\), \(b_0(x) = \beta + 1\) so that

\[
(n + x)P^{(\alpha+1, \beta+1)}_n(x) = (x - 1)(P^{(\alpha, \beta)}_n(x)') + \alpha P^{(\alpha, \beta)}_n(x)
\]

\[
= n(x - 1)P^{(\alpha+1, \beta+1)}_{n-1}(x) + \alpha P^{(\alpha, \beta)}_n(x) \tag{4.6}
\]

since \((P^{(\alpha, \beta)}_n(x))' = nP^{(\alpha+1, \beta+1)}_{n-1}(x), n \geq 0.\)

Case 4.2: \(a_1(x) = 1 + x\), \(b_1(x) = 1 - x\). Then (4.1) becomes

\[
ML[P_n(x)] = (1 - x^2)P'_n(x) + [(a_0 + b_0 + 1) - (a_0 - b_0 + 1)x]P'_n(x) + a_0b_0P_n(x) = \lambda_n P_n(x),
\]

\[
LM[Q_n(x)] = (1 - x^2)Q'_n(x) + [(a_0 + b_0 - 1) - (a_0 - b_0 + 1)x]Q'_n(x) + a_0b_0Q_n(x) = \lambda_n Q_n(x).
\]
We may also assume $a_0 + b_0 + 1 = \beta - \alpha$ and $a_0 - b_0 + 1 = \alpha + \beta + 2$ ($\alpha, \beta, \alpha + \beta + 1 \neq -1, -2, \ldots$, and $\beta \neq 0$). Then $a_0 = \beta$ and $b_0 = -\alpha - 1$ so that

\[
ML[P_n] = (1 - x^2)P''_n(x) + [(\beta - \alpha) - (\alpha + 2)x]P'_n(x) - \beta(x + 1)P_n(x) = \lambda_n P_n(x),
\]

\[
LM[Q_n] = (1 - x^2)Q''_n(x) + [(\beta - \alpha - 2) - (\alpha + 2)x]Q'_n(x) - \beta(x + 1)Q_n(x) = \lambda_n Q_n(x).
\]

Therefore, $\{P_n(x)\}_{n=0}^{\infty} = \{P_n^{(x, \beta)}(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty} = \{P_n^{(z+1, \beta-1)}\}_{n=0}^{\infty}$ so that we have

\[
(n + \beta)P_{n+1}^{(x+1, \beta-1)}(x) = (1 + x)[P_n^{(x, \beta)}(x)]' + \beta P_n^{(x, \beta)}(x)
\]

\[
= n(1 + x)D_n^{(x+1, \beta-1)}(x) + \beta P_n^{(x, \beta)}(x). \tag{4.7}
\]

We have shown that if $\{P_n(x)\}_{n=0}^{\infty}$ is either Hermite and Bessel polynomials, then $\{a_1(x)P'_n(x) + a_0P_n(x)\}_{n=0}^{\infty}$ cannot be an OPS for any polynomials $a_1(x)$ and $a_0(x)$. This fact is closely related to the absence of Hermite or Bessel polynomials in Darboux transformations [3].

4.2. $k = 1$ and $r = 1$

Let $L[\cdot] = a_1(x)D + a_0$, where $a_1(x) \neq 0$ and $a_1 + a_0 \neq 0, n \geq 0$. Define a monic PS $\{Q_n(x)\}_{n=0}^{\infty}$ by

\[
(n + 1)(a_1 + a_0)Q_n(x) = L[P'_{n+1}] = a_1P'_{n+1}(x) + a_0P'_{n+1}(x), \quad n \geq 0. \tag{4.8}
\]

We assume that $\{Q_n(x)\}_{n=0}^{\infty}$ is a monic OPS. Then there are $b_1(x) = b_{11}x + b_{10}$, $b_2(x) = b_{22}x^2 + b_{21}x + b_{20}$, $b_3(x) = b_{33}x^3 + b_{32}x^2 + b_{31}x + b_{30}$, not all zero, satisfying

\[
(b_3(x)\sigma)' - (b_2(x)\sigma)' + b_1(x)\sigma = 0,
\]

\[
a_0b_3(x)\sigma = a_1(x)[2(b_3(x)\sigma)' - b_2(x)\sigma]
\]

and $b_{33}(n-1) + b_{22}n + b_{11} \neq 0, n \geq 0$. Now (3.11) and (3.12) become

\[
MLD[P_n] = (b_3D^2 + b_2D + b_1)(a_1D + a_0)[P'_n]
\]

\[
= a_1b_3P^{(4)}_n + (2a_1' b_3 + a_0b_3 + a_1b_2)P^{(3)}_n + (a_1' b_2 + a_0b_2 + a_1b_1)P''_n + a_0 b_1 P'_n
\]

\[
= \lambda_n P_n, \tag{4.10}
\]

\[
LDM[Q_n] = (a_1D + a_0)D(b_3D^2 + b_2D + b_1)[Q_n] = \lambda_{n+1} Q_n.
\]

Therefore, $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ must be either classical or classical-type OPS. Krall and Sheffer [14] considered this case only for $\{P_n(x)\}_{n=0}^{\infty} = \{P_n^{(x, \beta)}\}_{n=0}^{\infty}$ the Gegenbauer polynomials. In case $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS, $\{P_{n+1}^{(x, \beta)}\}_{n=0}^{\infty}$ is also a classical OPS so that Case 4.2 is reduced to Case 4.1. Hence, $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ must be either Laguerre polynomials or Jacobi polynomials. We now claim that $\{P_n(x)\}_{n=0}^{\infty}$ cannot be a classical-type OPS. For example, assume that $\{P_n(x)\}_{n=0}^{\infty} = \{R_n(x)\}_{n=0}^{\infty}$ is the Laguerre-type OPS which is orthogonal relative to $\sigma = (e^{-x}H(x) + \frac{1}{x})\delta(x)$ d$x$. Then, we may assume that $a_1(x)b_3(x) = x^2$ and $a_1(x) = 1$ or $x$. If $a_1(x) = 1$ and $b_3(x) = x^2$, then we obtain from (2.3) and (4.10)

\[
2a_1'(x)b_3(x) + a_0b_3(x) + a_1(x) b_2(x) = 4x - 2x^2,
\]

\[
a_1'(x)b_2(x) + a_0b_2(x) + a_1(x) b_1(x) = x^2 - (2R + 6)x,
\]

\[
a_0b_1(x) = (2R + 2)x - 2R
\]
from which we have
\[ b_3(x) = x^2, \quad b_2(x) = -x^2 + 4x, \quad b_1(x) = -2x. \]

Then, by (3.10), \( P_n(0) = 0, \) \( n \geq 1, \) which is a contradiction. If \( a_1(x) = x \) and \( b_1(x) = x, \) then we have similarly as above either (i) \( a_0 = 2, \) \( R = 0, \) \( b_2(x) = -2x, \) \( b_3(x) = x \) or (ii) \( a_0 = -1, \) \( R = -\frac{3}{2}, \) \( b_2(x) = -2x + 3, \) \( b_3(x) = x - 3. \) In case (i), \( P_n(0) = 0, \) \( n \geq 1 \) by (3.10), which is a contradiction. In case (ii), we can see that \( (b_2\sigma)' - (b_3\sigma)' + b_1\sigma = 2\delta(x) \neq 0, \) which contradicts to (4.9). By similar arguments, we can see that \( \{P_n(x)\}_{n=0}^{\infty} \) can be neither a Legendre-type OPS nor a Jacobi-type OPS.

### 4.3. \( k = 2 \) and \( r = 0 \)

Let \( L[ \cdot ] = a_2(x)D^2 + a_1(x)D + a_0, \) where \( a_2(x) \neq 0 \) and
\[ a_n := a_{22}n(n - 1) + a_{11}n + a_0 \neq 0, \quad n \geq 0. \] (4.11)

Then, the monic PS \( \{Q_n(x)\}_{n=0}^{\infty} \) defined by
\[ a_nQ_n(x) = L[P_n](x) = a_2(x)P''_n(x) + a_1(x)P'_n(x) + a_0P_n(x), \quad n \geq 0 \]
is an OPS relative to \( \tau = a_0^{-1}\{((b_2\sigma)'' - (b_1\sigma)' + b_0\sigma) \} \) if and only if there exist \( b_0, \) \( b_1(x), \) \( b_2(x) \) (not all zero) satisfying
\[ a_2(x)\tau = b_2(x)\sigma, \]
\[ 2(a_2(x)\tau)' - a_1(x)\tau = b_1(x)\sigma, \]
\[ (a_2(x)\tau)'' - (a_1(x)\tau)' + a_0\tau = b_0\sigma. \] (4.12)

and \( b_{22}n(n - 1) + b_{11}n + b_0 \neq 0, \quad n \geq 0. \) In this case, \( b_0 \neq 0 \) and \( b_2(x) \neq 0 \) and
\[ ML[P_n] = (b_2D^2 + b_1D + b_0)(a_2D^2 + a_1D + a_0)[P_n] \]
\[ = a_2b_2P''_n + (2a'_2b_2 + a_1b_2 + a_2b_1)P'_n \]
\[ + (a'_2b_2 + 2a'_1b_2 + a_0b_2 + a'_1b_1 + a_1b_1 + a_2b_0)P'_n \]
\[ + (a'_1b_1 + a_0b_1 + a_1b_0)P'_n + a_0b_0P_n = \lambda_nP_n, \]
\[ LM[Q_n] = (a_2D^2 + a_1D + a_0)(b_2D^2 + b_1D + b_0)[Q_n] = \lambda_nQ_n. \] (4.13)

Hence, \( \{P_n(x)\}_{n=0}^{\infty} \) and \( \{Q_n(x)\}_{n=0}^{\infty} \) must be either classical or classical-type OPS. We first consider the case when \( \{P_n(x)\}_{n=0}^{\infty} \) is a classical-type OPS.

**Case 1:** \( \{P_n(x)\}_{n=0}^{\infty} = \{R_n(x)\}_{n=0}^{\infty} \) the Laguerre-type OPS. Then, \( a_2(x)b_2(x) = x^2 \) so that \( a_2(x) = 1, x, x^2. \)

**Case 1.1:** \( a_2(x) = 1 \) or \( a_2(x) = x^2. \) If \( a_2(x) = 1, \) then \( b_2(x) = x^2 \) and from (2.3) and (4.13), we obtain
\[ a_1(x)x^2 + b_1(x) = 4x - 2x^2, \]
\[ 2a'_1(x)x^2 + a_0x^2 + a_1(x)b_1(x) + b_0 = x^2 - (2R + 6)x, \]
\[ a'_1(x)b_1(x) + a_0b_1(x) + a_1(x)b_0 = (2R + 2)x - 2R, \]
from which we have
\[ a_1(x) = -2, \quad a_0 = 1, \quad b_1(x) = 4x, \quad b_0 = 0. \]
Applying (4.15) to (4.14), we obtain

\[ 2(x\tau)' - a_1(x)\tau = b_1(x)\sigma, \]

where

\[ (x\tau)'' - (a_1(x)\tau)' - a_0(x)\tau = b_0(x)\sigma. \]

Since \( \sigma = (e^{-x}H(x) + (1/R)\delta(x)) \) \( dx \),

\[ (x\tau)' = (x\sigma)' = (1-x)\sigma \quad \text{and} \quad \sigma' = -\sigma + \delta(x). \]

Applying (4.15) to (4.14), we obtain \( \tau = e^{-x}H(x) \) \( dx \) and

\[ a_1(x) = -x + 2, \ a_0(x) = -R - 1 \quad \text{and} \quad b_1(x) = -x, \ b_0(x) = -R. \]

Hence, \( \{Q_n(x)\}_{n=0}^\infty = \{I_n^{(0)}(x)\}_{n=0}^\infty \) and

\[ (-n - R - 1)L_n^{(0)}(x) = xR_n^{(0)}(x) + (2 - x)R_n^{(0)}(x) - (R + 1)R_n(x), \]

\[ (-n - R)R_n(x) = xL_n^{(0)}(x)'' - xL_n^{(0)}(x)' - RR_n(x). \]

**Case 2:** \( \{P_n(x)\}_{n=0}^\infty = \{P_n^{(x)}(x)\}_{n=0}^\infty \) the Legendre-type \( OPS \). Then \( a_2(x) = (x^2 - 1)^2 \) so that \( a_2(x) = x^2 - 1, (x^2 - 1)^2 \). If \( a_2(x) = (x + 1)^2 \) or \( a_2(x) = (x - 1)^2 \), then by the same arguments as in Case 1.1, we can derive a contradiction.

**Case 2.1:** \( a_2(x) = x^2 - 1. \) Then \( b_2(x) = x^2 - 1 \) and (4.12) becomes

\[ (x^2 - 1)\tau = (x^2 - 1)\sigma, \]

\[ 2((x^2 - 1)\tau)' - a_1(x)\tau = b_1(x)\sigma, \]

\[ ((x^2 - 1)\tau)'' - (a_1(x)\tau)' - a_0(x)\tau = b_0(x)\sigma. \]

Since \( \sigma = \sigma_L + (1/x)(\delta(x-1) + \delta(x+1)) \), where \( \sigma_L = H(1-x^2) \) \( dx \) is the Legendre moment functional, we have

\[ ((x^2 - 1)\tau)' = ((x^2 - 1)\sigma)' = 2x\sigma_L \quad \text{and} \quad \sigma'_L = \delta(x+1) - \delta(x-1). \]

Applying these to (4.18) gives \( \tau = \sigma_L \) and

\[ a_1 = 4x, \ a_0 = 2x + 2 \quad \text{and} \quad b_1 = 0, \ b_0 = 2x. \]

Hence \( \{Q_n(x)\}_{n=0}^\infty = \{P_n^{(x)}(x)\}_{n=0}^\infty \) and

\[ (n(n + 1) + 4n + 2x + 2)P_n^{(x)}(x) = (x^2 - 1)P_n^{(x)}(x)' + 4xP_n^{(x)}(x)' + (2x + 2)P_n^{(x)}, \]

\[ (n(n + 1) + 2x)P_n^{(x)}(x) = (x^2 - 1)P_n^{(0,0)}(x)' + 2xP_n^{(0,0)}(x). \]

**Case 3:** \( \{P_n(x)\}_{n=0}^\infty = \{S_n^{(x)}(x)\}_{n=0}^\infty \) the Jacobi-type \( OPS \). Then \( (a_2b_2)(x) = (x^2 - x)^2 \) and so that \( a_2(x) = x^2 - x, x^2; (x - 1)^2 \). Recall that \( \{S_n^{(x)}(x)\}_{n=0}^\infty \) is orthogonal relative to

\[ \sigma = \sigma_x + \frac{1}{M}\delta(x), \]

where \( \sigma_x = (1-x^2)H(x) \) \( dx \) is a classical moment functional satisfying the moment equation

\[ (x^2 - x)\sigma'_x = 2x\sigma_x, \quad \sigma_x \neq -1, -2, \ldots. \]
Since \( \langle \sigma_x, 1 \rangle = 1/(x+1) \), we obtain from (4.22),
\[
((x-1)\sigma_x)' = (x+1)\sigma_x - \delta(x).
\] (4.23)

**Case 3.1:** \( a_2(x) = x^2 - x \) and \( b_2(x) = x^2 - x \). Then (4.12) becomes
\[
(x^2 - x)\tau = (x^2 - x)\sigma,
\] (4.24)
\[
2((x^2 - x)\tau' - a_1(x)\tau) = b_1(x)\sigma,
\] (4.25)
\[
((x^2 - x)\tau'') - (a_1(x)\tau)' + a_0\tau = b_0\sigma.
\] (4.26)

From (4.21) and (4.24), we have
\[
\tau = \sigma_x + \lambda \delta(x) + \mu \delta(x-1)
\] (4.27)
for some constants \( \lambda \) and \( \mu \). By (4.22) and (4.27), (4.25) becomes
\[
((2\lambda + 4)x - 2 - a_1(x) - b_1(x))\sigma_x = \left( \lambda a_1(0) + \frac{1}{M} b_1(0) \right) \delta(x) + \mu a_1(1) \delta(x-1).
\]
Hence \( \mu a_1(1) = 0, a_1(x) + b_1(x) = (2\lambda + 4)x - 2, \) and
\[
\lambda a_1(0) = -\frac{1}{M} b_1(0).
\] (4.28)

Multiply (4.26) by \((x^2 - x)\) and apply (4.22). Then we have
\[
(x + 2 + a_0 - a_i(x) - b_0)(x-1) = \lambda((x+2)x - a_i(x) - 1)
\]
and \( \lambda a_1(0) = 0 \). Thus from (4.28), \( b_1(0) = 0 \) and so \( a_1(x) = Ax - 2, b_1(x) = (2\lambda + 4 - A)x \) for some constant \( A \) so that \( \lambda = 0 \) since \( a_1(0) = -2 \). There are two cases: \( x = 0 \) or \( A = x + 3 \).

**Case 3.11:** \( A = x + 3 \). Then \( a_1(x) = (x + 3)x - 2, b_1(x) = (x + 1)x \) and \( \mu = 0 \). Thus \( \tau = \sigma_x \) and (4.26) becomes
\[
((1-x)\sigma_x)' = (b_0 - a_0)\sigma_x + \frac{1}{M} b_0 \delta(x).
\] (4.29)

From (4.23) and (4.29), we obtain \( a_0(x) = x + M + 1 \) and \( b_0(x) = M \). Hence, we have
\[
(n^2 + 2n + n + x + M + 1)Q_n(x)
\]
\[
= (x^2 - x)S_n^{(x)}(x)' + ((x + 3)x - 2)S_n^{(x)}(x)' + (x + M + 1)S_n^{(x)}(x),
\] (4.30)
\[
(n^2 + 2n + M + 1)S_n^{(x)}(x) = (x^2 - x)Q_n(x)' + ((x + 1)Q_n(x)' + M Q_n(x).
\] (4.31)

Note that \( Q_n(x) = (-2)^{-n} p_n(x)(1 - 2x), n \geq 0 \).

**Case 3.12:** \( x = 0 \). Then \( \tau = \sigma_x + \mu \delta(x-1) \), \( a_1(x) = Ax - 2, \) and \( b_1(x) = (4 - A)x \). Since \( \sigma_x = H(x)H(1-x)dx, \sigma_x' = \delta(x) - \delta(x-1) \) so that we obtain from (4.26)
\[
b_0 \sigma_x + \frac{1}{M} b_0 \delta(x) = (a_0 - A + 2)\sigma_x + \delta(x) + (a_0 \mu - 3 + A) \delta(x-1).
\]
Thus, \( b_0 = M, a_0 = b_0 + A - 2, \) and \( A = -a_0 \mu + 3 \). If \( \mu = 0 \), then \( A = 3, a_1(x) = 3x - 2, a_0 = M + 1, b_1(x) = x, \) and \( b_0 = M \) so that it becomes the Case 3.11 with \( x = 0 \). If \( \mu \neq 0 \), then we have \( A = 2 \) so that \( \mu = 1/M, a_1(x) = 2(x - 1), b_1(x) = 2x, a_0 = b_0 = M \), and
\[
\tau = \left( H(x)H(1-x) + \frac{1}{M} \delta(x) \right) dx.
\]
Hence

\[(n^2 + n + M)Q_n(x) = (x^2 - x)P''_n(x) + 2(x - 1)P'_n(x) + MP_n(x),\]

\[(n^2 + n + M)P_n(x) = (x^2 - x)Q''_n(x) + 2xQ'_n(x) + MQ_n(x).\]

Note that \(Q_n(x) = (-1)^n S_n^0(1 - x), n \geq 0,\) are also Jacobi-type polynomials.

**Case 3.2:** \(a_2(x) = x^2\) or \(a_2(x) = (x - 1)^2\). Then by the same argument as in Case 1.1, we have if \(a_2(x) = x^2\), then \(a_1(x) = 0, a_0(x) = -2\) and if \(a_2(x) = (x - 1)^2\), then \(a_0(x) = 0\) and \(M = 0\). Hence, these contradict our assumptions that \(a_2 \neq 0\) in (4.11) and \(M \neq 0\).

We now consider the case when \(\{P_n(x)\}_{n=0}^\infty\) is a classical OPS. If \(\{P_n(x)\}_{n=0}^\infty\) satisfy the differential equation (2.2), then the differential operator \(M \mathcal{L}[-]\) in (4.13) must be a linear combination of \(I, \mathcal{L}, \mathcal{L}^2\) (see [20, Proposition 1]), where \(I\) is the identity operator.

Krall and Sheffer [14] considered this case only for \(\mathcal{L}\) of second order differential equations. Instead, we use moment functional relations (4.12), which is much easier to handle.

**Case 4:** \(\{P_n(x)\}_{n=0}^\infty = \{H_n(x)\}_{n=0}^\infty\) the Hermite polynomials. Then we may assume \(a_2(x) = b_2(x) = 1\). Hence \(\tau = \sigma\) by (4.12) so that \(\{Q_n(x)\}_{n=0}^\infty = \{H_n(x)\}_{n=0}^\infty\).

**Case 5:** \(\{P_n(x)\}_{n=0}^\infty = \{L_n^{(\alpha)}(x)\}_{n=0}^\infty\) the Laguerre polynomials. Then we may assume \(a_2(x)b_2(x) = x^2\) so that \(a_2(x) = x^2, b_1(x) = 1\).

**Case 5.1:** \(a_2(x) = x^2\). Then \(b_2(x) = 1\) and (4.12) becomes

\[x^2 \tau' - a_1(x) \tau = b_1(x) \sigma,\]

\[(x^2 \tau')' - (a_1(x) \tau)' + a_0 \tau = b_0 \sigma.\]

Multiplying the second equation in (4.34) by \(x^2\) and using \((x \sigma)' = (x + 1 - x) \sigma\), we have \(a_1(x) = 2x\), \(b_1(x) = -2\). Similarly from the third equation in (4.34), we have \(a_0(x) = x^2 - x\) and \(b_0 = 1\) so that \(x \neq 0, 1\). Since \(\sigma = x^0 e^{-x} dx\),

\[\tau = a_0^{-1} \{((b_2 \sigma)' - (b_1 \tau)' + b_0 \tau) = x^{2-2} e^{-x} dx \quad (x \neq 1, 0, -1, \ldots)\]

so that \(\{Q_n(x)\}_{n=0}^\infty = \{L_n^{(x-2)}(x)\}_{n=0}^\infty\) and

\[(n(n - 1) + 2x n + x^2 - 2) L_n^{(x)}(x) = x^2 L_n^{(x)}(x) + 2x L_n^{(x)}(x)' + (x^2 - x) L_n^{(x)}(x),\]

\[L_n^{(x)}(x) = L_n^{(x-2)}(x) + 2L_n^{(x-2)}(x)' + L_n^{(x-2)}(x).\]

**Case 5.2:** \(a_2(x) = x\). Then \(b_2(x) = x\) and (4.12) becomes

\[x \tau = x \sigma,\]

\[2(x \tau)' - a_1(x) \tau = b_1(x) \sigma,\]

\[(x \tau)'' - (a_1(x) \tau)' + a_0 \tau = b_0 \sigma\]

so that \(\tau = \sigma + \lambda \delta(x)\) for some constant \(\lambda\). If \(\lambda = 0\), then \(\tau = \sigma\) so that \(\{Q_n(x)\}_{n=0}^\infty = \{L_n^{(x)}(x)\}_{n=0}^\infty\). If \(\lambda \neq 0\), then we have \(a_1(x) = -x, b_1(x) = -x + 2, b_0 = a_0 - 1,\) and \(x = 0\). Then \(\tau = \sigma - (1/a_0) \delta(x)\) so that \(\{Q_n(x)\}_{n=0}^\infty\) is the Laguerre-type OPS \(\{R_n(x)\}_{n=0}^\infty\) with \(R = -a_0 \neq 0, -1, -2, \ldots\).

**Case 5.3:** \(a_2(x) = 1\). Then \(b_2(x) = x^2\) and \(\tau = x^2 \sigma = x^{x-2} e^{-x} dx\) so that \(\{Q_n(x)\}_{n=0}^\infty = \{L_n^{(x+2)}(x)\}_{n=0}^\infty\).
Case 6: \( \{P_n(x)\}_{n=0}^{\infty} = \{B_n^x(x)\}_{n=0}^{\infty} \) the Bessel polynomials. Then we may assume that \( a_2(x)b_2(x) = x^4 \) so that \( a_2(x) = x^2 \) and \( b_2(x) = x^2 \) and (4.12) becomes

\[
\begin{align*}
2 & (x^2 \tau' - a_1(x) \tau) = b_1(x) \sigma, \\
(x^2 \tau' - a_1(x) \tau) & + a_0 \tau = b_0 \sigma.
\end{align*}
\]

Hence \( \tau = \sigma + \lambda \delta(x - 1) + \mu \delta(x + 1) \) for some constants \( \lambda \) and \( \mu \). In this case, by the same arguments as in Case 5 using \( (x^2 \sigma)' = (zx + 2) \sigma \), we can obtain

\[
\begin{align*}
a_1(x) & = b_1(x) = 2x \sigma, \quad a_0 = b_0 \quad \text{and} \quad \lambda = \mu = 0 \\
\text{so that} \quad \{Q_n(x)\}_{n=0}^{\infty} & = \{B_n^x(x)\}_{n=0}^{\infty}.
\end{align*}
\]

Case 7: \( \{P_n(x)\}_{n=0}^{\infty} = \{P_n^{(\alpha, \beta)}(x)\}_{n=0}^{\infty} \) the Jacobi polynomials. Then we may assume \( a_2(x)b_2(x) = (1 - x^2)^2 \) so that \( a_2(x) = (1 - x)^2, (1 + x)^2, 1 - x^2. \)

Case 7.1: \( a_2(x) = (1 - x)^2. \) Then \( b_2(x) = (1 + x)^2 \) and (4.12) becomes

\[
\begin{align*}
(1 - x)^2 \tau & = (1 + x)^2 \sigma, \\
2 & (((1 - x)^2 \tau)' - a_1(x) \tau) = b_1(x) \sigma, \\
((1 - x)^2 \tau)' - a_1(x) \tau & + a_0 \tau = b_0 \sigma.
\end{align*}
\]

Then using \( ((1 - x^2) \sigma)' = (\beta - \alpha - (\alpha + \beta + 2) x) \sigma \), we can easily obtain from (4.37) \( a_1(x) = 2z(x - 1), b_1(x) = (2\beta + 4)(x + 1), a_0 = z(x - 1), b_0 = (\beta + 2)(\beta + 1) \) so that \( z \neq 0, 1. \) Since \( \sigma = (1 - x)_{+}^{\alpha}(1 + x)_{+}^{\beta} \) \( dx \),

\[
\tau = a_0^{-1}((b_2 \sigma)' - (b_1 \sigma)' + b_0 \sigma) = (1 - x)^{-\alpha}(1 + x)^{-\beta} \)

so that \( \{Q_n(x)\}_{n=0}^{\infty} \) and

\[
\begin{align*}
(n(n - 1) + 2zn + z(x - 1)) & P_n^{(\alpha - 2, \beta + 2)}(x) \\
& = (x - 1)^2 P_n^{(\alpha, \beta)}(x)' + 2z(x - 1) P_n^{(\alpha, \beta)}(x)' + z(x - 1) P_n^{(\alpha, \beta)}(x), \\
(n(n - 1) + (2\beta + 4)n + (\beta + 2)(\beta + 1)) & P_n^{(\alpha + 2)}(x) \\
& = (x + 1)^2 P_n^{(\alpha - 2, \beta + 2)}(x)' + (2\beta + 4)(x + 1) P_n^{(\alpha - 2, \beta + 2)}(x)' \\
& + (\beta + 2)(\beta + 1) P_n^{(\alpha - 2, \beta + 2)}(x).
\end{align*}
\]

Case 7.2: \( a_2(x) = (1 + x)^2. \) This case is reduced to Case 7.1 by replacing \( x \) by \(-x.\)

Case 7.3 \( a_2(x) = 1 - x^2. \) Then \( b_2(x) = 1 - x^2 \) and (4.12) becomes

\[
\begin{align*}
(1 - x^2) \tau & = (1 - x^2) \sigma, \\
2 & ((1 - x^2) \tau)' - a_1(x) \tau) = b_1(x) \sigma, \\
((1 - x^2) \tau)' - a_1(x) \tau & + a_0 \tau = b_0 \sigma.
\end{align*}
\]

Then we have for some constants \( \lambda \) and \( \mu \)

\[
\begin{align*}
\tau & = \sigma + \lambda \delta(x - 1) + \mu \delta(x + 1), \\
b_1(x) & = 2(\beta - \alpha - (\alpha + \beta + 2) x) - a_1(x), \\
\lambda(a_{11} + a_{10}) & = \mu(a_{11} - a_{10}) = 0.
\end{align*}
\]

Case 7.3.1: \( \lambda = \mu = 0. \) Then \( \tau = \sigma \) so that \( \{Q_n(x)\}_{n=0}^{\infty} = \{P_n^{(\alpha, \beta)}(x)\}_{n=0}^{\infty}. \)
Case 7.3.2: $a_{11} + a_{10} = a_{11} - a_{10} = 0$, that is, $a_1(x) = 0$. Then
\[ \alpha = \beta = 0, \quad b_1(x) = -4x, \quad b_0 = a_0 - 2 \]
so that $\sigma = H(1 - x)H(1 + x)\,dx$ and
\[ \tau = a_0^{-1}\{(b_2\sigma)' - (b_1\sigma)' + b_0\sigma\} = \sigma - \frac{2}{a_0}(\delta(x + 1) + \delta(x - 1)). \]
Hence, $\{Q_n(x)\}_{n=0}^\infty = \{P_n^{(-a_0/2)}(x)\}_{n=0}^\infty$ is the Legendre-type OPS.

Case 7.3.3: $\lambda = 0$ and $a_{11} - a_{10} = 0$. Then $\tau = \sigma = \mu\delta(x + 1)$ and (4.40) gives
\[ a_1(x) = -(x + 1)(x + 1), \quad b_1(x) = -(x + 3)x - x + 1, \quad b_0 = a_0 - x - 1 \]
and $\beta = 0, \mu = -2^{x+1}a_0^{-1}$ since $\sigma = H(x + 1)(1 - x)^x\,dx$. Hence $\tau = \sigma - 2^{x+1}a_0^{-1}\delta(x + 1) (a_0 \neq n(n + x), n \geq 0)$ and so $\{Q_n(x)\}_{n=0}^\infty$ is the Jacobi-type OPS satisfying
\[ LM[y] = (x^2 - 1)^2y^{(iv)} + 2(x - 1)((x + 4)x + x)y^{''} + (x + 1)((x^2 + 9x - 2a_0 + 14)x + x^2 - 3x + 2a_0 - 10)y^{'''} - 2((x + 2)(a_0 - x - 1)x - x^2 - 3x + a_0x - 2)y' + a_0(a_0 - x - 1)y = \lambda_n y. \]
In fact, $\{Q_n(x)\}_{n=0}^\infty = \{2^nS^{(\nu)}_n((x + 1)/2)\}_{n=0}^\infty (M = -\nu)$ and
\[ (n^2 + xn - a_0)Q_n(x) = (x^2 - 1)^2P_n^{(\nu,0)}(x)' + (x + 1)(x + 1)P_n^{(\nu,0)}(x)' - a_0P_n^{(\nu,0)}(x), \quad (4.41) \]
\[ (n^2 + xn + 2n + x - a_0 - 1)P_n^{(\nu,0)}(x) = (x^2 - 1)Q_n'(x) + ((x + 3)x + x - 1)Q_n(x) + (x - a_0 + 1)Q_n(x). \quad (4.42) \]

Case 7.3.4: $\mu = 0$ and $a_{11} + a_{10} = 0$. This case is reduced to Case 7.3.3 by replacing $x$ by $-x$.

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References


